ALMOST-CONVERGENT DOUBLE SEQUENCES

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(Received December 23, 1964)

1. Introduction. In this note we extend to double sequences certain results obtained by Lorentz [4] for simple sequences. We begin with the following summary of the Lorentz results. We recall the existence of Banach limits L_1 [1, p. 34] defined for each element in the Banach space $(m)_1$ of all bounded real sequences $f = \{f(k)\}$ with $\|f\| = \sup_k |f(k)|$. These limits have the following familiar properties.

(1. 1) $$L_1(af + bg) = aL_1(f) + bL_1(g), \quad (\textit{all real } a, b);$$

(1. 2) $$L_1(f) \geqq 0 \quad \textit{if all } f(k) \geqq 0;$$

(1. 3) $$L_1(f_1) = L_1(f) \quad \textit{where } f_1 = \{f(k+1)\};$$

(1. 4) $$L(e) = 1 \quad \textit{where } e(k) = 1 \textit{ for all } k.$$

If we introduce the positively homogeneous and subadditive functional,

(1. 5) $$q(f) = \inf_{n_1, n_2, \ldots n_p} \limsup_k \frac{1}{p} \sum_{i=1}^{p} f(n_i + k),$$

for $f \in (m)_1$, and set $q'(f) = -q(-f)$, then the inequality $q'(f) \leqq L_1(f) \leqq q(f)$ holds for all $f \in (m)_1$ and all Banach limits L_1. Furthermore, all Banach limits will coincide at f if and only if $q'(f) = q(f)$. Sequences f satisfying this condition are called *almost-convergent*, and we denote the class of all such sequences by $(ac)_1$. In order that $q'(f) = q(f)$ it is necessary and sufficient that the *sliding* $(C, 1)$-means of f,

(1. 6) $$\frac{1}{p} \sum_{i=1}^{p} f(n+i),$$

converge uniformly in n as $p \to \infty$. If this condition is satisfied the limit over p in (1. 6) is equal to $L_1(f)$ for every Banach limit L_1. Finally, any sequence whatever such that the means (1. 6) converge uniformly is necessarily bounded.

The problem of extending these results to double sequences $f = \{f(i, j)\}$

is of some interest since a tractable substitute for the dominating functional (1.5) must be found. It has been observed by various writers that the rather awkward expression in (1.5) can be replaced by any one of the following simpler expressions:

$$q_1(f) = \inf_p \limsup_n \frac{1}{p}\sum_{i=1}^{p} f(n+i), \qquad [3], [5];$$

$$q_2(f) = \lim_p \limsup_n \frac{1}{p}\sum_{i=1}^{p} f(n+i), \qquad [3], [5];$$

$$q_3(f) = \inf_p \sup_n \frac{1}{p}\sum_{i=1}^{p} f(n+i), \qquad [2];$$

$$q_4(f) = \lim_p \sup_n \frac{1}{p}\sum_{i=1}^{p} f(n+i), \qquad [2].$$

With these models as a guide it turns out that the functional,

$$(1.7) \qquad P_+(f) = \lim_M \inf_{p,q>M} \lim_N \sup_{m,n>N} {}_{m+p}^{\,p}Z_{n+q}^{q}(f),$$

where,

$$(1.8) \qquad {}_{m+p}^{\,p}Z_{n+q}^{q}(f) = \frac{1}{pq}\sum_{i,j=1}^{p,q} f(m+i, n+j),$$

serves adequately for the generalization of Lorentz's results to the Banach space $(m)_2$ of all bounded real sequences $f = \{f(i,j)\}$ with $\|f\| = \sup_{i,j} |f(i,j)|$. We show first that (1.7), in conjunction with the Hahn-Banach Theorem, can be used to extend the functional $\lim f = \lim_{i,j} f(i,j)$ to a Banach limit over $(m)_2$. This establishes the existence of limits L_2 satisfying (1.1)–(1.4) with the appropriate modifications. Complete analogs of the above results of Lorentz are obtained, leading to the class $(ac)_2$ of almost-convergent double sequences. In §3 we show that a certain standard double limit theorem remains true under the concept of almost-convergence. We also clarify the connection between $\{f(i,j)\}$ belonging to $(ac)_2$, and its row and column sequences belonging to $(ac)_1$.

2. Banach limits in $(m)_2$. Our results depend on the following lemma.

(2.1) LEMMA. *Corresponding to each $f \in (m)_2$, each $\varepsilon > 0$, and each pair of positive integers p and q, there exists an index $Q = Q(f, \varepsilon, p, q)$ such that,*

$$\text{(2. 2)} \qquad \lim_{N} \sup_{m,n>N} {}_{m+r}^{\;r}Z_{n+s}^{s} \leqq \lim_{N} \sup_{m,n>N} {}_{m+p}^{\;p}Z_{n+q}^{q} + \varepsilon\,,$$

$$\text{(2. 3)} \qquad \lim_{N} \inf_{m,n>N} {}_{m+r}^{\;r}Z_{n+s}^{s} \geqq \lim_{N} \inf_{m,n>N} {}_{m+p}^{\;p}Z_{n+q}^{q} - \varepsilon\,,$$

for all $r, s > Q$.

PROOF. As in (2.2) and (2.3) we omit the symbol "(f)" following the Z symbol when the dependence on f is clear. Let $r = ap+c$ $(0 \leqq c < p)$ and $s = bq+d$ $(0 \leqq d < q)$. Recalling (1.8) we have the following decomposition of the double sum involved.

$$\text{(A)} \qquad {}_{m+r}^{\;r}Z_{n+s}^{s} = \frac{pq}{rs} \sum_{\mu,\nu=1}^{a,b} \frac{1}{pq} \sum_{i,j=1}^{p,q} f[m+(\mu-1)p+i,\, n+(\nu-1)q+j]$$

$$+ \frac{1}{rs} \sum_{\mu=1}^{a} \sum_{i,j=1}^{p,d} f[m+(\mu-1)p+i,\, n+bq+j]$$

$$+ \frac{1}{rs} \sum_{\nu=1}^{b} \sum_{i,j=1}^{c,q} f[m+ap+i,\, n+(\nu-1)q+j]$$

$$+ \frac{1}{rs} \sum_{i,j=1}^{c,d} f(m+ap+i,\, n+bq+j)\,.$$

Let R stand for the final three sums on the right side of (A). The number of individual terms composing R is equal to $apd+bqc+cd$, which is less than $rq + sp + pq$. Consequently, $|R| < \|f\| \cdot (rq + sp + pq)/rs$, so that we have $|R| < \varepsilon$ for all r, s exceeding a suitably chosen $Q = Q(f, \varepsilon, p, q)$. Using this estimate in (A) we find that,

$$\sup_{m,n>N} {}_{m+r}^{\;r}Z_{n+s}^{s} \leqq \frac{pq}{rs} \sum_{\mu,\nu=1}^{a,b} \sup_{m,n>N} {}_{m+(\mu-1)p+p}^{\;p}Z_{n+(\nu-1)q+q}^{q} + \varepsilon\,,$$

$$\leqq \frac{pq}{rs}\, ab \sup_{m,n>N} {}_{m+p}^{\;p}Z_{n+q}^{q} + \varepsilon\,,$$

$$\leqq \sup_{m,n>N} {}_{m+p}^{\;p}Z_{n+q}^{q} + \varepsilon\,, \qquad (all\ r, s > Q)\,,$$

since $ab \leqq rs/pq$. The inequality (2.2) is now apparent, and (2.3) follows by applying (2.2) to $-f$.

The requisite properties of $P_{+}(f)$ are established in the next proposition.

(2. 4) PROPOSITION. (i) $P_+(tf) = tP_+(f)$ *for all* $t \geqq 0$; (ii) $P_+(f+g) \leqq P_+(f)+P_+(g)$ *for all* $f, g \in (m)_2$.

PROOF. Since (i) is obvious we proceed to the proof of (ii). Referring to (1. 7) we see that if $\varepsilon > 0$ is given there exists an index $M_0 = M_0(\varepsilon)$ such that,

$$\inf_{p,q>M_0} \lim_N \sup_{m,n>N} {}_{m+p}^{p}Z_{n+q}^{q}(f+g) > P_+(f+g) - \varepsilon .$$

Consequently,

$$\lim_N \sup_{m,n>N} {}_{m+p}^{p}Z_{n+q}^{q}(f+g) > P_+(f+g) - \varepsilon ,$$

and hence,

$$P_+(f+g) - \varepsilon < \lim_N \sup_{m,n>N} {}_{m+p}^{p}Z_{n+q}^{q}(f) + \lim_N \sup_{m,n>N} {}_{m+p}^{p}Z_{n+q}^{q}(g) ,$$

for all $p, q > M_0$. Now since,

$$\inf_{p,q>M_0} \lim_N \sup_{m,n>N} {}_{m+p}^{p}Z_{n+q}^{q}(f) \leqq P_+(f) ,$$

it follows that there exist $p', q' > M_0$ such that,

$$\lim_N \sup_{m,n>N} {}_{m+p'}^{p'}Z_{n+q'}^{q'}(f) < P_+(f) + \varepsilon .$$

Similarly, there exist $p'', q'' > M_0$ for which,

$$\lim_N \sup_{m,n>N} {}_{m+p''}^{p''}Z_{n+q''}^{q''}(g) < P_+(g) + \varepsilon .$$

By Lemma (2. 1), if $p, q > \max[M_0, Q(f, \varepsilon, p', q'), Q(g, \varepsilon, p'', q'')]$, then,

$$\lim_N \sup_{m,n>N} {}_{m+p}^{p}Z_{n+q}^{q}(f) \leqq \lim_N \sup_{m,n>N} {}_{m+p'}^{p'}Z_{n+q'}^{q'}(f) + \varepsilon ,$$

$$\lim_N \sup_{m,n>N} {}_{m+p}^{p}Z_{n+q}^{q}(g) \leqq \lim_N \sup_{m,n>N} {}_{m+p''}^{p''}Z_{n+q''}^{q''}(g) + \varepsilon .$$

An evident combination of the preceding inequalities leads to $P_+(f+g) - \varepsilon < P_+(f)+2\varepsilon+P_+(g)+2\varepsilon$ and this is equivalent to (ii).

The next proposition shows that the class of Banach limits L_2 over $(m)_2$ is nonempty.

(2. 5) PROPOSITION. *There exist continuous functionals* L_2 *over* $(m)_2$ *satisfying the conditions,*

$$L_2(af+bg)=aL_2(f)+bL_2(g), \qquad (\textit{all real } a,b); \tag{2. 6}$$

$$L_2(f)\geqq 0 \quad \textit{if all } f(i,j)\geqq 0; \tag{2. 7}$$

$$L_2(f_{uv})=L_2(f) \ \textit{where } f_{uv}(i,j)=f(i+u,j+v) \quad \textit{for all positive integers } u,v; \tag{2. 8}$$

$$L_2(e)=1 \ \textit{where } e(i,j)=1 \textit{ for all } i,j. \tag{2. 9}$$

PROOF. For f in the Banach space $(bc)_2 \subset (m)_2$ of all bounded convergent double sequences we define the continuous linear functional $\lim f$ as $\lim_{i,j} f(i,j)$. It follows immediately from (1.7) that $\lim f = P_+(f)$ for all $f\in(bc)_2$. An application of the Hahn-Banach Theorem then yields the existence of continuous functionals L_2 over $(m)_2$, satisfying (2. 6) and the condition,

$$P_-(f)\leqq L_2(f)\leqq P_+(f), \qquad (\text{all } f\in(m)_2), \tag{2. 10}$$

where $P_-(f) = -P_+(-f) = \lim_N \sup_{p,q>M} \lim_N \inf_{m,n>N} {}_{m+p}Z^q_{n+q}(f)$. If $f(i,j)\geqq 0$ then $P_-(f)\geqq 0$, and (2.7) then follows from (2.10). Since (2.9) is clear it remains to establish (2.8). In the notation of (2.8) we have,

$$\begin{aligned} {}_{m+p}^{\ p}Z^q_{n+q}(f-f_{10}) &= \frac{1}{pq}\sum_{i,j=1}^{p,q}[f(m+i,n+j)-f(m+1+i,n+j)],\\ &= \frac{1}{pq}\sum_{j=1}^{q}[f(m+1,n+j)-f(m+1+p,n+j)], \end{aligned}$$

from which we obtain

$$|{}_{m+p}^{\ p}Z^q_{n+q}(f-f_{10})|\leqq 2\|f\|/p.$$

The latter implies $P_+(f-f_{10})=P_-(f-f_{10})=0$, and (2.10) then yields $L_2(f_{10}) = L_2(f)$. A parallel argument leads to $L_2(f_{01})=L_2(f)$, and (2.8) follows by induction.

(2. 11) PROPOSITION. *In order that all Banach limits coincide at* f *it is necessary and sufficient that* $P_-(f)=P_+(f)$.

PROOF. Let Λ_2 be *any* continuous functional over $(m)_2$ satisfying the conditions (2.6)—(2.9). It follows easily that,

$$\lim_{N} \inf_{m,n>N} f(m,n) \leqq \Lambda_2(f) \leqq \lim_{N} \sup_{m,n>N} f(m,n) . \tag{2.12}$$

For fixed p, q let ${}^pZ^qf$ denote the double sequence $\{{}_{m+p}^{\ \ p}Z^q_{n+q}(f)\}$, so that ${}^pZ^qf = \Sigma_{i,j=1}^{p,q} f_{ij}/pq$, where the f_{ij} are the *shift* functions of (2.8). Replacing f in (2.12) by ${}^pZ^qf$, and observing that $\Lambda_2({}^pZ^qf) = \Lambda_2(f)$ by (2.8), we obtain,

$$\lim_{N} \sup_{m,n>N} {}_{m+p}^{\ \ p}Z^q_{n+q}(f) \leqq \Lambda_2(f) \leqq \lim_{N} \sup_{m,n>N} {}_{m+p}^{\ \ p}Z^q_{n+q}(f) ,$$

for all p, q. Consequently, $P_-(f) \leqq \Lambda_2(f) \leqq P_+(f)$, and this proves that the stated condition is sufficient.

To prove the necessity we suppose that $g \in (m)_2 - (bc)_2$ is such that $P_-(g) < P_+(g)$. According to the Hahn-Banach construction [1, p. 28] the value $\Lambda_2(g)$ of any extended functional Λ_2 at g may be chosen arbitrarily subject to the condition,

$$\sup_{f\in(bc)_2} [P_-(f+g) - \lim f] \leqq \Lambda_2(g) \leqq \inf_{f\in(bc)_2} [P_+(f+g) - \lim f] . \tag{2.13}$$

It is readily verified that $P_\pm(f+g) = \lim f + P_\pm(g)$, so that (2.13) reduces to $P_-(g) \leqq \Lambda_2(g) \leqq P_+(g)$. Hence if $P_-(g) < P_+(g)$, the value of $\Lambda_2(g)$ is not uniquely determined, and this completes the proof.

We come next to the characterization of coincident Banach limits in terms of the sliding $(C, 1)$-means of f.

(2. 14) THEOREM. *In order that $P_-(f) = P_+(f)$ it is necessary and sufficient that,*

$$\lim_{p,q} {}_{m+p}^{\ \ p}Z^q_{n+q}(f) , \tag{2.15}$$

exist uniformly for all m, n.

PROOF. The proof of sufficiency presents no difficulties and it is therefore omitted. We remark, however, that the given condition implies that the value of (2.15) is the common value of $P_-(f)$ and $P_+(f)$, and therefore the value of every Banach limit at f.

We assume now that $P_-(f) = P_+(f) = \mu$. Then for all M we have

$$\sup_{p,q>M} \lim_{N} \inf_{m,n>N} {}_{m+p}^{\ \ p}Z^q_{n+q} \geqq \mu ,$$

$$\inf_{p,q>M} \lim_{N} \sup_{m,n>N} {}_{m+p}^{\ \ p}Z^q_{n+q} \leqq \mu .$$

Hence there exist certain indices p', q', p'', q'' for which

$$\liminf_{N} \inf_{m,n>N} {}_{m+p'}^{p'}Z_{n+q'}^{q'} > \mu - \varepsilon,$$

(2. 16)

$$\limsup_{N} \sup_{m,n>N} {}_{m+p''}^{p''}Z_{n+q''}^{q''} < \mu + \varepsilon.$$

By Lemma (2.1) there exist $Q' = Q(f, \varepsilon, p', q')$, $Q'' = Q(f, \varepsilon, p'', q'')$ such that,

$$\liminf_{N} \inf_{m,n>N} {}_{m+p'}^{p'}Z_{n+q'}^{q'} - \varepsilon \leqq \liminf_{N} \inf_{m,n>N} {}_{m+r}^{r}Z_{n+s}^{s}$$

$$\leqq \limsup_{N} \sup_{m,n>N} {}_{m+r}^{r}Z_{n+s}^{s} \leqq \limsup_{N} \sup_{m,n>N} {}_{m+p''}^{p''}Z_{n+q''}^{q''} + \varepsilon,$$

for all $r, s > Q_0 = \max(Q', Q'')$. Taking (2.16) into account we obtain,

$$\mu - 2\varepsilon < \liminf_{N} \inf_{m,n>N} {}_{m+r}^{r}Z_{n+s}^{s} \leqq \limsup_{N} \sup_{m,n>N} {}_{m+r}^{r}Z_{n+s}^{s} < \mu + 2\varepsilon,$$

for all $r, s > Q_0$. Then $N_0 = N_0(\varepsilon)$ exists such that,

$$\mu - 2\varepsilon < \inf_{m,n>N_0} {}_{m+r}^{r}Z_{n+s}^{s} \leqq \sup_{m,n>N_0} {}_{m+r}^{r}Z_{n+s}^{s} < \mu + 2\varepsilon,$$

which yields,

(2. 17) $\mu - 2\varepsilon < {}_{m+r}^{r}Z_{n+s}^{s} < \mu + 2\varepsilon$, for all $r, s > Q_0$, all $m, n > N_0$.

There remains the problem of establishing this inequality, or an equivalent one, in the following cases: (i) $m, n \leqq N_0$; (ii) $m \leqq N_0$, $n > N_0$; (iii) $m > N_0$, $n \leqq N_0$. The nature of the details will be sufficiently illustrated if we confine attention to case-(iii). In this case we consider,

(2. 18)
$$ {}_{m+r}^{r}Z_{n+s}^{s} - {}_{m+r}^{r}Z_{N_0+1+s}^{s} = \frac{1}{rs}\sum_{i=1}^{r}\sum_{j=1}^{N_0+1+r} f(m+i, n+j)$$

$$- \frac{1}{rs}\sum_{i=1}^{r}\sum_{j=n+s-N_0}^{s} f(m+i, N_0+1+j).$$

Estimating the right side D we find that $|D| \leqq 2\|f\|(N_0+1)/s$, so that $|D| < \varepsilon$ for all $s > S_0(\varepsilon)$. Returning now to the left side of (2.18) and noting that the estimate (2.17) applies to the second term, we conclude that

(2. 19)
$$\mu - 3\varepsilon < {}_{m+r}^{r}Z_{n+s}^{s} < \mu + 3\varepsilon,$$

for all $r, s > \max(Q_0, S_0)$, all $m > N_0$, and all $n \leqq N_0$. A precisely similar argument in case-(ii) yields the inequality (2.19) for all $r, s > \max(Q_0, S_0)$, all $m \leqq N_0$, and all $n > N_0$. The corresponding details in case-(i) are somewhat more troublesome but the outcome is the existence of an index $R_0(\varepsilon)$ such that the inequality (2.19) holds for all $r, s > \max(Q_0, R_0)$, and all $m, n \leqq N_0$. Finally, then, we see that (2.19) is valid for all $r, s > \max(Q_0, R_0, S_0)$ and all m, n. This completes the proof.

Following Lorentz we can now define $f \in (m)_2$ to be *almost-convergent* if all Banach limits L_2 coincide at f. We use the symbol $(ac)_2$ to denote the class of all such double sequences, and mention in summary that the elements of $(ac)_2$ are completely characterized by either Proposition (2.11) or Theorem (2.14). As a final remark we call attention to the fact that any sequence satisfying the condition of Theorem (2.14) is necessarily bounded.

3. **Related topics**. Consider the following skeleton theorem.

(3. k) THEOREM. *If* $\{f(i,j)\}$ *is* __________ *to* λ, *and each of its row and column sequences*, $\{f(i,j)\}_j$ *and* $\{f(i,j)\}_i$, *is* __________ *to* λ_i^r *and* λ_j^c, *respectively, then each sequence* $\{\lambda_i^r\}$ *and* $\{\lambda_j^c\}$ *is* __________ *to* λ.

For $k=1$, read "convergent", for $k=2$, read "almost-convergent", and for $k=3$, read "summable-$(C, 1)$". Note that each concept implies the following one. Theorems (3.1) and (3.3) are standard results whose proofs are readily constructed. We are now in a position to give a

PROOF OF (3.2). The assumption of almost-convergent rows implies that given $\varepsilon > 0$ there exists an index $Q(\varepsilon, i, m)$ such that,

$$\lambda_{m+i}^r - \varepsilon < \frac{1}{q} \sum_{j=1}^{q} f(m+i, n+j) < \lambda_{m+i}^r + \varepsilon, \tag{3. 4}$$

for all $q > Q(\varepsilon, i, m)$, and all n. Summing these inequalities from $i=1$ to p, and dividing by p, we obtain,

$$\frac{1}{p} \sum_{i=1}^{p} \lambda_{m+i}^r - \varepsilon < {}_{m+p}^{\;\;p}Z_{n+q}^{q} < \frac{1}{p} \sum_{i=1}^{p} \lambda_{m+i}^r + \varepsilon, \tag{3. 5}$$

for all $q > R(\varepsilon, p, m) = \max[Q(\varepsilon, 1, m), \cdots, Q(\varepsilon, p, m)]$, and all n. On the other hand, from Theorem (2.14), we find that,

$$\lambda - \varepsilon < {}_{m+p}^{\;\;p}Z_{n+q}^{q} < \lambda + \varepsilon, \quad \text{for all } p, q > S(\varepsilon), \text{ and all } m, n.$$

Now chose in succession any $p > S(\varepsilon)$, any integer m, and then any $q > \max(R, S)$. From (3.4) and (3.5) we obtain.

$$\lambda - 2\varepsilon < \frac{1}{p}\sum_{i=1}^{p} \lambda^r_{m+i} < \lambda + 2\varepsilon,$$

which shows that the sequence $\{\lambda^r_i\}$ is almost-convergent to λ. A parallel argument proves the same fact for $\{\lambda^c_j\}$.

We discuss next the general relationship between f belonging to $(ac)_2$ and its row or column sequences belonging to $(ac)_1$. On account of (2.8) it is clear that any finite number of row and/or column sequences corresponding to an $f \in (ac)_2$ may be arbitrary bounded sequences. Going further we now outline an example which shows that $\{f(i,j)\}$ may belong to $(ac)_2$ while *none* of its row or column sequences belongs to $(ac)_1$. For $k = 1, 2, \cdots, v$ and $v = 1, 2, 3, \cdots$, we define $f(1,j) = 1$ if $j = v^5 + k$; $f(2,j) = 1$ if $j = (v+1)^5 + (v+1) + k$; $f(3,j) = 1$ if $j = (v+2)^5 + (v+2) + (v+1) + k$; and so on. Make the parallel definition for $f(i,1), f(i,2), (i,3), \cdots$, and let $f(i,j) = 0$ otherwise. Each row and column sequence is summable-$(C,1)$ to 0 since the 1's are sufficiently sparse, but none of them belongs to $(ac)_1$ since each contains arbitrarily long runs of 0's and arbitrarily long runs of 1's. However, it can be shown that $\{f(i,j)\}$ is almost-convergent to 0, although we shall not go into the details.

At the other extreme the following example shows that each row and column sequence may belong to $(ac)_1$ while $\{f(i,j)\}$ does not belong to $(ac)_2$. Along the main diagonal place any sequence of 0's and 1's that is *not* summable-$(C,1)$. For $k = 1, 2, 3, \cdots$ let the kth row to the right of the main diagonal and the kth column below the main diagonal consist entirely of the symbol, 0 or 1, on the main diagonal at their intersection. Each row and column sequence is convergent, and therefore almost-convergent. However, neither the sequence of row limits nor the sequence of column limits is summable-$(C,1)$, and hence neither belongs to $(ac)_1$. Therefore, by Theorem (3.2), f does not belong to $(ac)_2$.

A certain amount of order is restored by the following result.

(3.6) THEOREM. *For arbitrary* $\{f(i,j)\}$, *in order that each row sequence* $\{f(i,j)\}_j$ *belong to* $(ac)_1$, *it is necessary and sufficient that for each* $R > 0$ *we have* $\lim_q {}^p_pZ^q_{n+q}(f)$ *exist uniformly for all* n, *and all* $p \leqq R$. *The corresponding fact is true for the column sequences.*

PROOF. Suppose for each i that,

$$\lim_q \frac{1}{q}\sum_{j=1}^{q} f(i, n+j) = \lambda^r_i, \quad \text{uniformly in } n.$$

Then given i and $\varepsilon > 0$ there exists an index $Q(i,\varepsilon)$ such that,

$$\lambda_i^r - \varepsilon < \frac{1}{q}\sum_{j=1}^{q} f(i, n+j) < \lambda_i^r + \varepsilon,$$

for all $q > Q(i,\varepsilon)$ and all n. Let $R > 0$ be specified and set $Q_0(\varepsilon, R) = \max[Q(1,\varepsilon), \cdots, Q(R,\varepsilon)]$. Select p, $1 \leqq p \leqq R$, and sum the preceding inequalities from $i=1$ to p. After division by p we obtain,

$$\frac{1}{p}\sum_{i=1}^{p}\lambda_i^r - \varepsilon < {}_p^pZ_{n+q}^q < \frac{1}{p}\sum_{i=1}^{p}\lambda_i^r + \varepsilon, \tag{3. 7}$$

for all $q > Q_0(R,\varepsilon)$, all $p \leqq R$, and all n. This establishes the stated condition as necessary.

To show that it is also sufficient, fix an arbitrary row index $k > 1$ and choose $R \geqq k$. Setting $p = k$ and $p = k-1$ in (3.7) in turn, eliminating the factors $1/k$ and $1/(k-1)$, and then subtracting the second inequality from the first, we arrive at,

$$\lambda_k^r - (2k-1)\varepsilon < \frac{1}{q}\sum_{i=1}^{q} f(k, n+j) < \lambda_k^r + (2k-1)\varepsilon,$$

for all $q > Q_0(R,\varepsilon)$ and all n, with k independent of ε. The inequality is also true if $k=1$, and this completes the proof.

We assemble certain of the preceding results in the form of the following theorem.

(3. 8) THEOREM. *In order that* $\{f(i,j)\}$ *belong to* $(ac)_2$, *with all row and column sequences in* $(ac)_1$, *the conditions of Theorems* (2.14) and (3.6) *are necessary and sufficient. Moreover, if these conditions are satisfied then the conclusion of Theorem* (3.2) *holds.*

Another familiar double limit theorem has its analog in the following result.

(3. 9) THEOREM. *If the row and column sequences of* $\{f(i,j)\}$ *belong to* $(ac)_1$, *and the row* (*or column*) *sequences belong uniformly, then* f *belongs to* $(ac)_2$.

PROOF. Let us suppose that the row sequences belong to $(ac)_1$ uniformly, that is to say, that

$$\lim_q \frac{1}{q} \sum_{j=1}^{q} f(m+i, n+j) = \lambda_{m+i}^r, \text{ uniformly in } n, m, \text{ and } i.$$

Then given $\varepsilon > 0$ there is an index $Q_0(\varepsilon)$ such that

$$\lambda_{m+i}^r - \varepsilon < \frac{1}{q} \sum_{j=1}^{q} f(m+i, n+j) < \lambda_{m+i}^r + \varepsilon,$$

for all $q > Q_0$ and all n, m, i. Summing from $i=1$ to p, and dividing by p, we obtain,

$$\frac{1}{p} \sum_{i=1}^{p} \lambda_{m+i}^r - \varepsilon < {}_{m+p}^{\,p}Z_{n+q}^{q} < \frac{1}{p} \sum_{i=1}^{p} \lambda_{m+i}^r + \varepsilon, \tag{3.10}$$

for all $q > Q_0$ and all m, n, p. Since the column sequences also belong to $(ac)_1$, but not necessarily uniformly, similar considerations lead us to,

$$\frac{1}{q} \sum_{j=1}^{q} \lambda_{n+j}^c - \varepsilon < {}_{m+p}^{\,p}Z_{n+q}^{q} < \frac{1}{q} \sum_{j=1}^{q} \lambda_{n+j}^c + \varepsilon, \tag{3.11}$$

for all $p > R_0(\varepsilon, n, q)$ and all m. We now fix $q_0 > Q_0$, fix n_0 arbitrarily, and consider $p > R_0(\varepsilon, n_0, q_0)$. Then we find from (3.10) and (3.11) that,

$$\frac{1}{q_0} \sum_{j=1}^{q_0} \lambda_{n_0+j}^c - 2\varepsilon < \frac{1}{p} \sum_{i=1}^{p} \lambda_{m+i}^r < \frac{1}{q_0} \sum_{j=1}^{q_0} \lambda_{n_0+j}^c + 2\varepsilon,$$

for all m. An immediate consequence is the Cauchy condition,

$$\left| \frac{1}{p} \sum_{i=1}^{p} \lambda_{m+i}^r - \frac{1}{p'} \sum_{i=1}^{p'} \lambda_{m+i}^r \right| < 4\varepsilon, \text{ all } p, p' > R_0(\varepsilon, n_0, q_0), \text{ all } m,$$

for the existence of the limit,

$$\lim_p \frac{1}{p} \sum_{i=1}^{p} \lambda_{m+i}^r \equiv \lambda, \text{ uniformly in } m. \tag{3.12}$$

(This, of course, shows that the sequence $\{\lambda_i^r\}$ of row limits belongs to $(ac)_1$.) Using (3.12) in (3.10) we find that,

$$\lim_{p,q} {}_{m+p}^{\,p}Z_{n+q}^{q} = \lambda, \text{ uniformly in } m, n,$$

and so $f \in (ac)_2$ by Theorem (2.14).

In the converse direction, it is not difficult to show that under the assumptions of Theorem (3.2), the row and column sequences belong to $(ac)_1$ *uniformly*.

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