ON THE SPECTRUM OF A HYPONORMAL OPERATOR

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1. In this note we show several results on the spectrum of a hyponormal operator T (i.e. $T^*T \geqq TT^*$). Throughout this paper, an operator means a bounded linear operator on a Hilbert space. $\sigma(T)$, $P\sigma(T)$, $A\sigma(T)$. $C\sigma(T)$ and $R\sigma(T)$ denote the spectrum, the point spectrum, the approximate point spectrum, the continuous spectrum and the residual spectrum of an operator T respectively. It is known that $P\sigma(T) \cup C\sigma(T) \subset A\sigma(T)$. And $\Sigma(T)$ means the convex hull of $\sigma(T)$. The numerical range of an operator T, denoted by $W(T)$, is the set $W(T) = \{(Tx, x) : \|x\| = 1\}$. We write $\overline{W(T)}$ for the closure of $W(T)$. An operator T is normaloid if $\|T\| = \sup\{|\lambda| : \lambda \in W(T)\}$, or equivalently, if $\|T\| = \max\{|\lambda| : \lambda \in \sigma(T)\}$. It is known that a hyponormal operator is normaloid [7 ; Theorem 1]

Following G. H. Orland [4], for a compact convex subset X of the plane, a point $p \in X$ is bare if there is a circle through p such that no points of X lie outside this circle. In this paper, we also consider the "semi-bare point" defined as follow: For a bounded closed subset Y of the plane, a point $p \in Y$ is semi-bare if there is a circle through p such that no points of Y lie inside this circle.

$B_s(Y)$ denotes the set of all semi-bare points of Y. It is clear that if Y is a compact convex set of the plane, then all bare points of Y are included in $B_s(Y)$. Some results on hyponormal operators,

2. In this section, we shall prove some results on hyponormal operators.

THEOREM 1. *If T is an invertible, hyponormal operator, then T^{-1} is normaloid and* $\|T^{-1}\| = [\min\{|\mu| : \mu \in \sigma(T)\}]^{-1}$.

The proof is based on the following fact. (see [1]).

LEMMA 1. *If* $\|T^2\| = \|T\|^2$, *then T is normaloid.*

PROOF OF THEOREM 1. Let $y \in H$ and $\|y\| = 1$, then there exists a non-zero element $x \in H$ such that $y = T^2x$ by the invertibility of T. By the hyponormality of T, we have $\|Tx\|^2 = (Tx, Tx) = (T^*Tx, x) \leqq \|T^*Tx\| \|x\| \leqq \|T^2x\| \|x\|$; therefore $\|Tx\|^2 \leqq \|x\|$ because $\|T^2x\| = \|y\| = 1$; hence $\|T^{-1}y\|^2 \leqq \|T^{-2}y\|$ and $\|T^{-1}\|^2 \leqq \|T^{-2}\|$. Therefore T^{-1} is normaloid by Lemma 1. By [3; §33-Theorem 2.],

$$\sigma(T^{-1}) = \{\lambda^{-1} : \lambda \in \sigma(T)\}; \text{ hence } \|T^{-1}\| = [\min\{|\mu| : \mu \in \sigma(T)\}]^{-1}.$$

As an application of Theorem 1, we have the following results which are announced in [8].

COROLLARY 1. *If T is a hyponormal operator with its spectrum on the unit circle, then T is unitary.*

PROOF. By the assumption of T, T is invertible and hence, by Theorem 1, we have $\|T^{-1}\| = 1$. On the other hand, T is normaloid and so $\|T\| = 1$; therefore $\|Tx\| = \|x\|$ for all $x \in H$. Since T is invertible, T is unitary.

The following Corollary is proved in [5], by an another method:

COROLLARY 2. *If T is a hyponormal operator, then $\Sigma(T) = \overline{W(T)}$.*

PROOF. If $\lambda \notin \Sigma(T)$, then $T - \lambda I$ is invertible. And, by the simple calculation, it is easy to show that if T is a hyponormal operator, then $T - \mu I$ is also a hyponormal operator for any complex number μ. And hence, by Theorem 1, we have

$$\begin{aligned} \|(T-\lambda I)^{-1}\| &= [\min\{|\mu-\lambda| : \mu \in \sigma(T)\}]^{-1} \\ &\leqq [\min\{|\mu-\lambda| : \mu \in \Sigma(T)\}]^{-1} \quad \text{for all } \lambda \notin \Sigma(T). \end{aligned}$$

By [4: Theorem 2.], we have $\Sigma(T) = \overline{W(T)}$.

COROLLARY 3. *If T is a hyponormal operator and if $\sigma(T)$ is real, then T is self-adjoint.*

PROOF. By Theorem 1, $\|(T-\lambda I)^{-1}\| \leqq |\lambda|^{-1}$ for all non-zero, purely imaginary λ; hence, by [4; Theorem 1.], T is self-adjoint.

It is known that $\|(T-\lambda I)^{-1}\| \leqq \frac{1}{\lambda}$ for all $\lambda > 0$ if and only if $Re\,\overline{W(T)} \leqq 0$ (see [4; Lemma 2.]). Therefore we have the following lemma, because $\overline{W(\alpha T + \beta I)} = \alpha \overline{W(T)} + \beta$.

LEMMA 2. *For any operator T,*

$$\|(T-\lambda I)^{-1}\| \leqq [\min\{|\mu-\lambda|: \mu\in\overline{W(T)}\}]^{-1} \quad \textit{for all} \quad \lambda\notin\overline{W(T)}.$$

In [2], Donoghue gave a simple alternative proof of the following well-kown fact;

If $W(T)$ is a subset of the real axis, then T is self-adjoint.

We can also give an alternative proof of this fact by using Lemma 2, and Corollary 3 is an immediate consequence of this fact since if a hyponormal operator T has real spectrum, $W(T)$ is a subset of the real axis by Corollary 2.

3. It is known that the approximate proper vectors of a hyponormal operator T belonging to λ is also those of T^* belonging to $\bar{\lambda}$. However, the converse assertion is not always true.

The following lemma is proved in [6].

LEMMA 3. *For any operator T, if $\lambda\in\sigma(T)$, $|\lambda|=\|T\|$, then $\lambda\in A\sigma(T)$ and the approximate proper vectors of T belonging to λ are those of T^* belonging to $\bar{\lambda}$.*

For convenience we shall state the following lemma without proof.

LEMMA 4. *If T is invertible, then for any sequence of unit vectors $\{x_n\}$, $\|Tx_n-\lambda x_n\|\to 0$ as $n\to\infty$ if and only if $\|T^{-1}x_n-\lambda^{-1}x_n\|\to 0$ as $n\to\infty$.*

By the definition of the semi-bare point of $\sigma(T)$, it is clear that $B_s(\sigma(T))\subset\sigma(T)$. And, as a consequence of Theorem 1, we have

THEOREM 2. *If T is a hyponormal operetor, then $B_s(\sigma(T))\cap R\sigma(T)=\emptyset$.*

PROOF. If $\lambda\in B_s(\sigma(T))$, then there exists a complex number $z_0\notin\sigma(T)$ such that $|\lambda-z_0|=\min\{|\mu-z_0| : \mu\in\sigma(T)\}$ by the definition of the semi-bare point of $\sigma(T)$. It is clear that $T-z_0I$ is invertible and that $T-z_0I$ is hyponormal; and hence, by Theorem 1,

$$\lambda - z_0 \in \sigma(T - z_0 I), \ |\lambda - z_0|^{-1} = \|(T - z_0 I)^{-1}\| \cdots (*).$$

If we assume $\lambda \in R\sigma(T)$, then $\bar{\lambda} \in P\sigma(T^*)$ and $\bar{\lambda} - \bar{z}_0 \in P\sigma(T^* - \bar{z}_0 I)$; therefore $(\bar{\lambda} - \bar{z}_0)^{-1} \in P\sigma((T^* - \bar{z}_0 I)^{-1})$. (*) implies that

$$|(\bar{\lambda} - \bar{z}_0)^{-1}| = \|(T - z_0 I)^{-1}\| \text{ ; and } (\lambda - z_0)^{-1} \in P\sigma((T - z_0 I)^{-1})$$

by [9; Théorème 1.]; hence $\lambda - z_0 \in R\sigma(T - z_0 I)$ and $\lambda \in P\sigma(T)$. This is a contradiction.

In the proof of Theorem 3, $(\lambda - z_0)^{-1}$ and $(\bar{\lambda} - \bar{z}_0)^{-1}$ satisfy the condition of Lemma 3 by (*). And by Lemma 4, we have the following thorem.

THEOREM 3. *If T is a hyponormal operator, then for any $\lambda \in B_s(\sigma(T))$, the approximate proper vectors of T^* belonging to $\bar{\lambda}$ are also those of T belonging to λ.*

And hence, if μ and ν are distinct points of $B_s(\sigma(T))$ and if $\{x_n\}$ and $\{y_n\}$ are the corresponding approximate proper vectors of T belonging to $\bar{\mu}$ and $\bar{\nu}$ resepectively, then $(x_n, y_n) \to 0$ as $n \to \infty$ because

$$\begin{aligned} |(\mu - \nu)(x_n, y_n)| &= |(\mu x_n - T x_n, y_n) + (x_n, T^* y_n - \bar{\nu} y_n)| \\ &\leqq \|\mu x_n - T x_n\| + \|T^* y_n - \bar{\nu} y_n\| . \end{aligned}$$

Added in proof. After we wrote this note, we knew that J. G. Stampfli also proved Theorem 1 by an another method in his manuscript "Hyponormal operators and spectral density".

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