ON FOCAL ELEMENTS AND THE SPHERES

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For a point p of a two dimensional Riemannian manifold M, let $Q(p)$ be the locus of the first conjugate points of p. M is called a Wiedersehensfläche, if for any point p, $Q(p)$ is a single point.

The following theorem, due to L. W. Green [5], is interesting in the sense that a Riemannian manifold isometric to a two-sphere is characterized in terms of conjugate points only:

An orientable Wiedersehensfläche is a sphere of constant curvature.

It is the aim of the present author to generalize the above theorem for higher dimensional Riemannian manifold, making use of the concept of focal elements which is firstly introduced in this paper and contains the ones of conjugate points or focal points in the theory of geodesics.

1. Focal elements. Let M be a complete n-dimensional C^∞ Riemannian manifold. Denote the unit tangent bundle of M by $S(M)$ and its projection mapping by τ. Furthermore, we denote the bundle of all p-dimensional tangent subspaces to M by $T^p(M)$ whose fibre is the Grassmann manifold $G_{p,n}$ of all p-dimensional subspaces through the origin of R^n and its projection mapping by π_p.

A pair (ξ, u), $\xi \in T^p(M)$, $u \in S(M)$ such that $\pi_p(\xi) = \tau(u)$ and $u \perp \xi$ will be called a *p-element of M.* For any p-element (ξ, u) at a point $x \in M$, let $\sigma(s)$ be the unique geodesic in M, parameterized by arc length, with the initial conditions $\sigma(0) = x$, $\sigma'(0) = u$ and $J(\xi, u)^*$ be the set of Jacobi fields X along σ such that

$$X(0) \in \xi, \quad X'(0) \perp \xi \quad \text{and} \quad X'(0) \perp u .$$

$J(\xi, u)^*$ is J_S^* in the notation of W. Ambrose [2] putting $S = (S_1, S_2)$, where $S_1 = \xi$ and $S_2 | \xi = 0$, in other words, which is the boundary condition corresponding to a p-dimensional submanifold with the tangent subspace ξ and totally geodesic at x and the geodesic σ. We introduce the notations:

$$J(\xi, u) = \{X | X \in J(\xi, u)^*, \ X'(0) = 0\} ,$$

$$J(\xi, u)_0 = \{X | X \in J(\xi, u)^*, \ X(0) = 0\} ,$$

then we have clearly the direct sum decomposition of $J(\xi, u)^*$:

$$J(\xi, u)^* = J(\xi, u) + J(\xi, u)_0 .$$

DEFINITION. *A p-element (ξ, u) at x is a focal p-element of a q-element (η, v) at y, if the following hold:*

(i) *The geodesic σ, parameterized by arc length, with the initial conditions $\sigma(0)=y$, $\sigma'(0)=v$ passes through x and its tangent vector at x is u.*

(ii) $$J(\eta, v)^* \cap J(x, u)^* \neq 0 \quad \textit{and} \quad \subset J(\xi, u)_0 .$$

Furthermore, (ξ, u) is called *the first focal p-element of* (η, v), if there exist no focal p-elements along σ between y and x.

By means of (ii), it must be $0 \leqq p < n-1$ and a focal 0-element is a focal point $(q > 0)$ or a conjugate point $(q = 0)$ in the ordinary sense along σ for the boundary condition corresponding to (η, v).

EXAMPLE. Denoting an i-dimensional Euclidean space by E^i for any integer i. Let S^n be an n-dimensional sphere in E^{n+1}. Take E^{p+1} and E^{q+1} in E^{n+1} through the center of S^n and orthogonal to each other, and put $S^p = S^n \cap E^{p+1}$ and $S^q = S^n \cap E^{q+1}$, where $p > 0$, $q > 0$ and $p+q+1=n$. For $y \in S^q$ and $x \in S^p$, let σ be the great circle joining y and x. Let η and ξ be the tangent spaces to S^q at y and S^p at x respectively and v and u the tangent unit vectors to σ at y and x respectively. Then the p-element (ξ, u) is a focal p-element of the q-element (η, v).

For any point $x \in M$, we denote the tangent space to M at x by M_x. At each point $\sigma(s)$ of a geodesic σ in M we have the Ricci transformation $R(s)$ of the orthogonal complement M_s of $\boldsymbol{R}\sigma'(s)$ in $M_{\sigma(s)}$ into itself defined by

$$R(s)\, w = R_{w\sigma'(s)}(\sigma'(s)) ,$$

where R_{wz} is the curvature transformation (of the Riemannian connection) on $M_{\sigma(s)}$, depending on w and z in $M_{\sigma(s)}$.

Let X be a Jacobi field along σ orthogonal to σ, then we have

$$X''(s) = R(s)\, X(x) ,$$

and so

$$\frac{d}{ds} \langle X, X' \rangle = \langle X', X' \rangle + \langle X, RX \rangle .$$

If the sectional curvature of M is non positive at each point of σ, $\langle X, X' \rangle$

is non-decreasing. If $X(0)=0$ and $X'(0)\neq 0$, then $<X, X'>\neq 0$ for any $s>0$. Hence we have

PROPOSITION 1.1. *For any q-element of a Riemannian manifold with non positive sectional curvature, there exists no focal p-element.*

2. $W_{p,q}$-condition. For any $\xi\in T^p(M)$, we denote the set of all q-elements (η, v) such that $(\eta, -v)$ has a first focal p-element of the form (ξ, u) by $\Theta_q(\xi)$ and put

$$Q_q(\xi)=\{\pi_q(\eta)\,|\,(\eta, v)\in\Theta_q(\xi)\}\,.$$

DEFINITION. *A complete n-dimensional Riemannian manifold M satisfies the $W_{p,q}$-condition if the following hold*:

(i) *For any q-element (η, v), there exists a uniquely determined first focal p-element (ξ, u) of (η, v).*

(ii) *Any p-element (ξ, u) is the first focal p-element of a q-elements (η, v) which is uniquely determined.*

(iii) *For any $\xi\in T^p(M)$, $Q_q(\xi)$ is a q-dimensional submanifold, η is the tangent space to $Q_q(\xi)$ at the point $\pi_q(\eta)$ for any $(\eta, v)\in\Theta_q(\xi)$, and the mapping φ: $S^{n-p-1}(\xi)=\{u\,|\,u\perp\xi, u\in S(M)\}\rightarrow Q_q(\xi)$ by $\varphi(u)=\pi_q(\eta)$ is differentiable, where $(\xi, -u)$ and $(\eta, -v)$ are in* (ii).

For such M, ξ is called *a center element of $Q_q(\xi)$*.

By virtue of (i) of the $W_{p,q}$-condition, it must be $p<n-1$. When $p=q=0$, $Q_0(x)$ is identical with $Q(x)$ in L. W. Green [5].

If M satisfies (ii) of the $W_{p,q}$-condition, for any p-element (ξ, u), let σ be the geodesic, parameterized by arc length, with the initial conditions $\sigma(0)=\pi_p(\xi)$, $\sigma'(0)=u$, let (η, v) be the q-element such that $(\xi, -u)$ is the first focal p-element of $(\eta, -v)$ and put $l=f(\xi, u)$ be the least value such that $\pi_q(\eta)=\sigma(l)$.

LEMMA 2.1. *Let M be an n-dimensional complete Riemannian manifold satisfying the $W_{p,q}$-condition, then $q\leqq n-p-1$ and for any $\xi\in T^p(M)$, the function $f(\xi, u)$ depends only on ξ.*

PROOF. By (iii) of the $W_{p,q}$-condition and the Gauss' lemma, we get easily this lemma.

For M as in Lemma 2.1, we put $f(\xi, u)=f(\xi)$.

COROLLARY 2.2. *The diameter of $Q_q(\xi)\leqq 2f(\xi)$.*

For a submanifold N of M and a normal unit vector u to N at a point $x \in N$, we denote the boundary condition corresponding to N and u in the sense of W. Ambrose [2] by $(N; x, u)$.

LEMMA 2.3. *Let M be a Riemannian manifold as in Lemma 2.1 and let a p-element $(\xi, -u)$ at $x \in M$ be the first focal p-element of a q-element $(\eta, -v)$ at $y \in M$. Then we have*

$$J(\xi, u)_0 \subset J(Q_q(\xi); y, v)^* .$$

PROOF. By Lemma 2.1, all geodesics emanating from x and orthogonal to ξ are orthogonal to $Q_q(\xi)$ at the points of length $f(\xi)$ measured from x along them. Hence, denoting the boundary condition $(Q_q(\xi); y, v)$ by (T_1, T_2) according to Ambrose [2], for any $X \in J(\xi, u)_0$ we have $X(l) \in \eta$, $X'(l) - T_2 X(l) \in \eta^\perp$, where $l = f(\xi)$, that is $X \in J(Q_q(\xi); y, v)^*$. q.e.d.

COROLLARY 2.4. *For any $X \in J(x, u)^* \cap J(\eta, v)^*$, $T_2 X(l) = 0$.*

PROOF. Since $J(x, u)^* \cap J(\eta, v)^* \subset J(\xi, u)_0$, for $X \in J(x, u)^* \cap J(\eta, v)^*$ we have $X'(l) - T_2 X(l) \in \eta^\perp$ and $X'(l) \in \eta^\perp$ hence we get $T_2 X(l) \in \eta^\perp$. On the other hand $X(l) \in \eta$ and $T_2 X(l) \in \eta$, it follows $T_2 X(l) = 0$.

COROLLARY 2.5. *$\{X \mid X(l) = 0,\ X \in J(\xi, u)_0\} \subset J(\eta, v)_0$ and the dimension of the left hand side is equal to $n-p-q-1$.*

PROOF. The first part is clear from the lemma. Since $\{X(l) \mid X \in J(\xi, u)_0\} = \eta$ and $\dim \eta = q$, we have

$$\dim \{X \mid X(l) = 0,\ X \in J(\xi, u)_0\} = n-p-q-1 .$$

3. W_p-condition.

DEFINITION. *A complete n-dimensional Riemannian manifold M satisfies the W_p^*-condition if the following hold:*

(i) *M satisfies the $W_{p,q}$-condition, where $q = n-p-1$.*
(ii) *Let $(\xi, -u)$ be the first focal p-element of a q-element $(\eta, -v)$. Then*

$$J(x, u)^* \cap J(\eta, v)^* = J(\xi, u)_0 .$$

By means of (i), it must be $0 \leqq p < n-1$.

THEOREM 3.1. *Let M be an n-dimensional Riemannian manifold satisfying the W_p^*-condition. For any $\xi \in T^p(M)$, the second fundamental form of $Q_{n-p-1}(\xi)$ at y with respect to v, where $y=\pi_{n-p-1}(\eta)$, $(\eta, v) \in \Theta_{n-p-1}(\xi)$, vanishes.*

PROOF. Let $(\xi, -u)$ be the first focal p-element of $(\eta, -v)$, $q = n-p-1$, and (T_1, T_2) be the boundary condition corresponding to $Q_q(\xi)$ and v. For any $X \in J(\xi, u)_0$, parameterized by arc length measured from $x = \pi_p(\xi)$, along the geodesic σ such that $\sigma(0)=x$, $\sigma'(0)=u$, we have

$$X(l) \in \eta\,, \quad T_2X(l) = 0$$

by Corollary 2.4 and (ii) of the W_p^*-condition. By Corollary 2.5, we have $\{X(l) | X \in J(\xi, u)_0\} = \eta$. Hence we have $T_2|\eta=0$, in other words, the second fundamental form of $Q_q(\xi)$ at $y=\pi_q(\eta)$ with respect to the normal unit vector v vanishes. q.e.d.

THEOREM 3.2. *Let M be an n-dimensional Riemannian manifold saitsfying the W_0^*-condition. For any point $x \in M, Q_{n-1}(x)$ is totally geodesic.*

DEFINITION. *An n-dimensional Riemannian manifold M satisfying the W_p^*-condition is said to satisfy the W_p-condition if for any $\xi \in T^p(M)$, $Q_{n-p-1}(\xi)$ is totally geodesic.*

When $p=0$, by Theorem 3.2 the W_0^*-condition implies the W_0-condition.

LEMMA 3.3. *Let M satisfy the W_p-condition. For any $\xi \in T^p(M)$, take a q-element $(\eta, v) \in \Theta_q(\xi)$, $q=n-p-1$, and a unit normal vector v to η. Let $(\bar{\xi}, -\bar{u})$ be the first focal p-element of $(\eta, -v)$, then $Q_q(\xi) = Q_q(\bar{\xi})$.*

PROOF. η is the common tangent space to $Q_q(\xi)$ and $Q_q(\bar{\xi})$ at $y=\pi_q(\eta)$. By the definition, $Q_q(\xi)$ and $Q_q(\bar{\xi})$ are totally geodesic. Since M is complete, hence $Q_q(\xi) = Q_q(\bar{\xi})$.

LEMMA 3.4. *Let M satisfy the W_p-condition $(0 < p < n-1)$. For $\xi, \bar{\xi}$ such that $Q_q(\xi) = Q_q(\bar{\xi})$, we have $f(\xi) = f(\bar{\xi})$ and the diameter of $Q_q(\xi)$ is $2f(\xi)$.*

PROOF. Let $(\xi, -u)$ be the first focal p-element of $(\eta, -v) \in \Theta_q(\xi)$ and put $\pi_q(\eta) = y$, $\pi_p(\xi) = x$. Let v_α be a unit normal vector to η depending on a parameter α, $(\xi_\alpha, -u_\alpha)$ be the first focal p-element of $(\eta, -v_\alpha)$ and $(\bar{\eta}_\alpha, \bar{v}_\alpha)$ be the q-element of $\Theta_q(\xi)=\Theta_q(\xi_\alpha)$ (by Lemma 3.3) corresponding to $(\xi_\alpha, -u_\alpha)$. Let

$\sigma_\alpha : [-l_\alpha, l_\alpha] \to M$, $l_\alpha = f(\xi_\alpha)$, be the geodesic such that $\sigma_\alpha(0) = x_\alpha = \pi_p(\xi_\alpha)$, $\sigma'_\alpha(0) = u_\alpha$. If v_α is differentiable on α, x_α and u_α are also differentiable on α (I. N. Patterson [3]). Since σ_α cutting $Q_q(\xi)$ orthogonally at $y_\alpha = \pi_q(\bar{\eta}_\alpha)$, we have $f(\xi_\alpha) = f(\xi)$ by Gauss' lemma.

Now, let $y, \bar{y} \in Q_q(\xi)$ be the two points such that

$$\operatorname{dis}(y, \bar{y}) = \operatorname{diam} Q_q(\xi) .$$

By the completeness of M, let σ be a geodesic segment joining y and $\bar{y}$. Let v and η be the tangent unit vector to σ and the tangent space to $Q_q(\xi)$ at y. Then, $v \perp \eta$. We may put $(\xi, -u)$ be the first focal p-element of $(\eta, -v)$. We have clearly $\operatorname{dis}(y, \bar{y}) = \operatorname{length} \sigma \geqq 2f(\xi)$. Hence making use of Corollary 2.2, we have diam $Q_q(\xi) = 2f(\xi)$. q.e.d.

Now, we denote the locus of the supporting points of center element $\bar{\xi}$ of $Q_q(\xi)$, for a fixed $\xi \in T^p(M)$, by $\bar{Q}_p(\xi)$.

THEOREM 3.5. *Let M be an n-dimensional Riemannian manifold satisfying the W_p-condition $(0 < p < n-1)$. For $\xi \in T^p(M)$ at $x \in M$ and $(\eta, v) \in \Theta_q(\xi)$, $q = n-p-1$, we have the following*:

(i) *Putting $\eta_l^\perp = \{w \mid w \in M_y, w \perp \eta, \| w \| \leqq l\}$, where $y = \pi_q(\eta)$ and $l = f(\xi)$ and denoting the boundary of $\eta_l^\perp$ by $\partial \eta_l^\perp$, the exponential mapping* Exp_y *is locally regular on $\partial \eta_l^\perp$ and* $\operatorname{Exp}_y(\partial \eta_l^\perp) = \bar{Q}_p(\xi)$.

(ii) $$J(\eta, v)_0 \subset J(\bar{Q}_p(\xi); x, u)^* .$$

PROOF. By virtue of Lemma 3.4. we have easily

$$\bar{Q}_p(\xi) = \operatorname{Exp}_y \partial \eta_l^\perp .$$

Let v_α be a one-parameter family of unit normal vectors to η at $y = \pi_q(\eta)$ and $\sigma(\alpha, t)$ be the corresponding geodesic such that $\sigma(\alpha, l) = y$ and $\sigma'(\alpha, l) = v_\alpha$. Putting $X = \partial\sigma/\partial\alpha \,|_{\alpha=0}$, we have

$$X(l) = 0 , \quad X'(l) \in \eta^\perp ,$$

hence $X \in J(\eta, v)_0$. We may put $X'(l) \neq 0$. If $X(0) = 0$, we have $X \in J(x, u)^* \cap J(\eta, v)^*$. By (ii) of the W_p^*-condition, we have $X \in J(\xi, u)_0$. By Corollary 2.5, it must be $X = 0$ which contradicts to $X \neq 0$. Hence we have $X(0) \neq 0$. This shows that the exponential mapping Exp_y is locally regular on $\partial \eta_l^\perp$. (ii) is clear from (i). q.e.d.

COROLLARY 3.6. *$J(\overline{Q}_p(\xi)\,;x,u)^*=J(\eta,v)_0+J(\xi,u)_0=J(\overline{Q}_p(\xi)\,;x,u)+J(\xi,u)_0$ $=J(\eta,v)^*$, where "+" denotes the direct sum.*

From Theorem 3.5 and $J(\eta,v)^*\supset J(\xi,u)_0$, these equalities are clear.

LEMMA 3.7. *Let M be an n-dimensional Rimannian manifold satisfying the W_p-condition. For any $\xi\in T^p(M)$ and any point $z\in M$, we have* $\mathrm{dis}\,(z,Q_q(\xi))\leqq f(\xi)$, $p=n-p-1$.

PROOF. There exists a point y of $Q_q(\xi)$ such that $\mathrm{dis}(z,y)=\mathrm{dis}(z,Q_q(\xi))$. Let g be a geodesic segment joining z and y with length $\mathrm{dis}(z,y)$ and η be the tangent space to $Q_p(\xi)$ at y. The tangent unit vector v to g at y is orthogonal to η. Let $(\xi,-u)$ be the first focal p-element of $(\eta,-v)$ and put $x=\pi_p(\xi)$. Let σ be the geodesic parameterized by arc length with the initial conditions $\sigma(0)=x$, $\sigma'(0)=u$. It is clear that z is on σ between x and y by the well known property of focal points. Hence we get

$$\mathrm{dis}\,(z,Q_q(\xi))\leqq f(\xi)\,. \qquad \text{q.e.d.}$$

This lemma and Lemma 3.4 imply immediately

COROLLARY 3.8. *Let M be an n-dimensional Riemannian manifold satisfying the W_p-condition, we have* $\mathrm{diam}\,M\leqq 4f(\xi)$, *for any $\xi\in T^p(M)$.*

4. Manifolds satisfying $W_{0,0}$-condition. For a complete Riemannian manifold M and any $u\in S(M)$, denote by $\sigma(s,u)$ and $\psi(u)$ the geodesic parameterized with respect to arc length with the initial conditions $\sigma(0,u)=\tau(u)$, $\sigma'(0,u)=u$ and the length from $\tau(u)$ to its first conjugate point on this geodesic. In the following, using the notations in §2, let M be an n-dimensional Riemannian manifold satisfying $W_{0,0}$-condition. A 0-element (x,u) is a focal 0-elemant of a 0-element (y,v) if x and y are conjugate points each other on the corresponding geodesic σ with the tangent vectors u and v at x and y respectively. For any point $x\in M$, $Q_0(x)$ is a point because it is 0-dimensional and connected by (iii) of Definition in §2. Hence M is a generalization of Wiedersehensfläche of L. W. Green [5].

The following Lemmas 4.1～4.4 are included in [5] for Wiedersehensfläche.

LEMMA 4.1. *Every geodesic of M is closed.*

PROOF. Let σ is a geodesic parameterized with respect to arc length. Put $\sigma(0)=x$, $f(x)=l$ and $y=Q_0(x)$, $f(y)=m$. $Q_0:M\to M$ is involutive and differentiable by the property of conjugate points. Since $l=m$, for any $\varepsilon>0$, $Q_0(\sigma(\varepsilon))=\sigma(l+\varepsilon_1)$ and $Q_0(\sigma(l+\varepsilon))=\sigma(2l+\varepsilon_2)$, for some positive $\varepsilon_1=\varepsilon_1(\varepsilon)$ and $\varepsilon_2=\varepsilon_2(\varepsilon)$. $\varepsilon_1(\varepsilon)$ and $\varepsilon_2(\varepsilon)$ are monotone increasing continuous functions of ε. Since $Q_0(Q_0(\sigma(\varepsilon)))=\sigma(\varepsilon)=\sigma(2l+\varepsilon_2(\varepsilon_1(\varepsilon)))$, for sufficiently small ε, the arcs corresponding to $[0,\varepsilon]$ and $[2l, 2l+\varepsilon_2(\varepsilon_1(\varepsilon))]$ of σ coincide. This shows that σ is closed and $f(x)$ is constant on σ. q.e.d.

COROLLARY 4.2. *M is compact.*

COROLLARY 4.3. *$\psi(u)$ is finite and constant on $S(M)$.*

PROOF. Suppose M satisfy the $W_{0,0}$-condition. For any $u\in S(M)$, we have $\psi(u)=f(x)$, $x=\tau(u)$, by Lemma 2.1. Since M is complete, any two points x and y lie on a geodesic σ. Hence $f(x)=f(y)$ and so $\psi(v)$ is constant on $S(M)$.

LEMMA 4.4. *When $Q_0\neq 1$, Q_0 is an isometric involution.*

LEMMA 4.5. *If M is simply connected, M is homeomorphic to the n-sphere.*

The proofs of these lemmas are evident and analogous to the ones of Corollaries 2.3 and 2.4 of [2].

For a fixed point $x\in M$ and positive r, let $N(r)=N(r,x)=\{\sigma(r,e),\ e\in S(M),\ \tau(e)=x\}$ and put $l=f(x)$.

LEMMA 4.6. *If M is simply connected, any closed geodesic of M intersects at least two points with $N\left(\frac{l}{2}\right)$ as point sets.*

PROOF. A homeomorphism of the n-sphere onto M is given by means of the polar coordinates of an n-sphere of radius l/π and the exponential mapping $\mathrm{Exp}_x: M_x\to M$. Hence $N\left(\frac{l}{2}\right)$ divides M into the two regions containing x and $y=Q_0(x)$ respectively. Q_0 transforms the regions each other onto and any closed geodesic of M is invariant under Q_0. This yields the lemma. q.e.d.

In the following, we denote by V_m the measure on Borel sets of m-dimensional Riemannian manifolds and by c_m the total measure of the unit m-sphere. Let $\tilde{\mu}$ be the $2n-1$ dimensional measure on $S(M)$ induced naturally from the metric of M.

THEOREM 4.7. *If M is a simply connected n-dimensional $W_{0,0}$-manifold and any closed geodesic of M intersects $N\left(\frac{l}{2}\right)$ at two points or tangent to it without intersections, then*

$$V_{n-1}\left(N\left(\frac{l}{2}\right)\right) = \frac{n-1}{2l} \cdot \frac{c_{n-1}}{c_{n-2}} V_n(M).$$

PROOF. Put $N = N\left(\frac{l}{2}\right)$ and denote the regions of M divided by N by $G_1 \ni x$ and $G_2 \ni y = Q_0(x)$, then $M = G_1 + N + G_2$ (disjoint sum). Putting $A = \{\sigma'(s, v) \mid v \in S(N),\ 0 \leqq s < 2l\}$, since

$$\dim A \leqq \{2(n-1)-1\} + 1 = 2(n-1) < \dim S(M) = 2n-1\,,$$

the measure of the set of all unit tangent vectors of closed geodesics of M intersecting N is identical with the total measure of $S(M)$.

Putting $E = \{v \mid v \in \tau^{-1}(N), \sigma(s, v) \in G_1 \text{ for } 0 < s < l\}$, we can apply the formula (8) in Appendix for this E, we have

$$\tilde{\mu}\left(\bigcup_{0<r<l} E_r\right) = \frac{c_{n-2}}{n-1} V_{n-1}(N) \cdot l\,.$$

Making use of the involutive isometry Q_0, we get

$$\tilde{\mu}(S(M)) = \frac{c_{n-2}}{n-1} V_{n-1}(N) \cdot l \times 2 = V_n(M) \times c_{n-1}\,,$$

hence

$$V_{n-1}(N) = \frac{n-1}{2l} \cdot \frac{c_{n-1}}{c_{n-2}} V_n(M)\,. \qquad \text{q.e.d.}$$

THEOREM 4.8. *Let M be a simply connected n-dimensional $W_{0,0}$-manifold. If any closed geodesic of M intersects $N(r)$, $0 < r < l$, at two points or tangent to it without intersections, then*

$$V_{n-1}(N(r_1)) \leqq V_{n-1}(N(r_2)), \quad 0 < r_1 < r_2 \leqq \frac{l}{2}\,.$$

PROOF. For a fixed positive $r, 0 < r \leqq \frac{l}{2}$, denote the regions of M divided by $N(r)$ by $G_1(r) \ni x$ and $G_2(r) \ni y = Q_0(x)$. Putting

$$E(r) = \{v \mid v \in \tau^{-1}(N(r)),\ \sigma(s, v) \in G_2(r), \text{ for } 0 < s < l\}\,,$$

we can apply the formula (8) in Appendix for this $E(r)$ from the assumption of the theorem, hence

$$\tilde{\mu}\Big(\bigcup_{0<t<l}(E(r))_t\Big)=\frac{c_{n-2}}{n-1}V_{n-1}(N(r))\cdot l. \tag{1}$$

Let $0<r_1<r_2\leqq\frac{l}{2}$. For any $v_1\in E(r_1)$, let g be the closed geodesic $\sigma(s,v_1)$, $0\leqq s\leqq 2l$. g has common points with $N(r_2)$ and $N(l-r_2)$. Let $z_2=\sigma(s_2,v_1)$ be the first common point with $N(r_2)$ along g starting at $z_1=\tau(v_1)$ and v_2 be the tangent vector to g at z_2. Putting $v_2=\varphi(v_1)$ and $s_2=\lambda(v_1)$, φ and λ are clearly continuous on $E'(r_1)=E(r_1)-\varphi^{-1}(S(N(r_2)))$. Since

$$\dim\{\sigma'(s,v)\,|\,v\in S(N(r_2)),\ 0<s<l\}=2n-2\,,$$

we have

$$\tilde{\mu}\Big(\bigcup_{0<t<l}(E(r_1))_t\Big)=\tilde{\mu}\Big(\bigcup_{0<t<l}(E'(r_1))_t\Big).$$

For any $v_1\in E'(r_1)$, we have

$$dQ_0(\sigma'(s,v_1))=\sigma'(l+s,v_1)\,.$$

Hence, putting $\Omega=\bigcup_{0<t<l}(E'(r_1))_t$, $\Omega_1=\{\sigma'(s,v_1)\,|\,v_1\in E'(r_1),\ 0<s<\lambda(v_1)\}$, we have

$$dQ_0(\Omega_1)+(\Omega-\Omega_1)=\bigcup_{0<t<l}(\varphi(E'(r_1)))_t\,.$$

Since Q_0 is an isometry on M, we have

$$\tilde{\mu}(\Omega)=\tilde{\mu}(\Omega_1+(\Omega-\Omega_1))=\tilde{\mu}(dQ_0(\Omega_1)+(\Omega-\Omega_1))$$

and so from these relations

$$\begin{aligned}\tilde{\mu}\Big(\bigcup_{0<t<l}(E(r_1))_t\Big)&=\tilde{\mu}\Big(\bigcup_{0<t<l}(\varphi(E'(r_1)))_t\Big)\\&\leqq\tilde{\mu}\Big(\bigcup_{0<t<l}(E(r_2))_t\Big).\end{aligned}$$

By (1) and this we get

$$V_{n-1}(N(r_1))\leqq V_{n-2}(N(r_2))\,. \qquad \text{q.e.d.}$$

THEOREM 4.9. *Let M be a simply connected n-dimensional $W_{0,0}$-manifold. If the hypersurface* $N\left(\frac{l}{2}, x\right)=\left\{\mathrm{Exp}_x \frac{l}{2} e \,\middle|\, e \in \tau^{-1}(x)\right\}$, $x \in M$, *is always totally geodesic, then M is isometric to an n-sphere with radius l/π.*

PROOF. When $n=2$, this is true according to L. W. Green [5]. Assume that this theorem is true for the dimension $n-1$. $N\left(\frac{l}{2}, x\right)$ is simply connected and satisfies the $W_{0,0}$-condition from the assumption. Hence, it is isometric with an $(n-1)$-sphere with radius l/π. Its sectional curvature is $\frac{\pi^2}{l^2}$ and equal to the one of M, since $N\left(\frac{l}{2}, x\right)$ is totally geodesic. This holds for any point $x \in M$. Hence, M has constant sectional curvature π^2/l^2. Accordingly, M is isometric to an n-sphere with radius l/π. q.e.d.

THEOREM 4.10. *If M satisfies the condition in Theorem 4.8 for $N(r)$ $=N(r, x)$, $0<r<l$, with any center $x \in M$, then $N\left(\frac{l}{2}\right)$ is totally geodesic.*

PROOF. Let g be any closed geodesic of M such that $x \bar{\in} g$. Let $z_0 \in g$ be a point such that $r_0 = \mathrm{dis}(x, g)$. We may assume that g is given by $\sigma(s)$, $-l \leqq s \leqq l$ and $z_0=\sigma(0)$. Let g_0 be the geodesic through x and z_0. We have

$$\max \mathrm{dis}(x, \sigma(s)) = l - r_0 .$$

If $r_0 = \frac{l}{2}$, $g \subset N\left(\frac{l}{2}, x\right)$. Suppose $0<r_0<\frac{l}{2}$. Since g is invariant under Q_0, $y = Q_0(x) \bar{\in} g$. Putting $\sigma(s) = \sigma(r(s), e(s))$, $e(s) \in \tau^{-1}(x)$, $r_0 \leqq r(s) \leqq l - r_0 < l$. Let g_0 be given by $\sigma(s, e_0)$, $e_0 = e(0)$.

For any s, $-l \leqq s \leqq l$, there exists a point on g_0 with the same distance to z_0 and $\sigma(s)$, because $l-r_0 \geqq l-r(s)$. Let $\sigma(r_0+p(s), e_0)$ be the first such point along g_0 starting from z_0 for $s \neq 0$. $p(s)$ is differentiable and $0<p(s) \leqq l-r_0$. From the assumption, $N(p(s), \sigma(r_0+p(s), e_0))$ can not intersect g at $\sigma(s)$. Hence the geodesic joining $\sigma(s)$ and $\sigma(r_0+p(s), e_0)$ is orthogonal to g. If $p(s)$ is not constant, there exists s_0 such that $p'(s_0) \neq 0$. Then, for s sufficiently near to s_0, the family of geodesics joining $\sigma(s)$ and $\sigma(r_0+p(s), e_0)$ envelops a subarc of g_0, this implies $z_0=\sigma(s)$. Hence $p(s)$ is a constant. By Lemma 4.6, it must be $\frac{l}{2}$.

Now, let g be tangent to $N\left(\frac{l}{2}, x_0\right)$ at z_0 and g_0 be the geodesic joining

x_0 and z_0. If we take a point x on g_0 near to z_0 such that $\mathrm{dis}(x, z_0) = \mathrm{dis}(x, g)$, the above consideration can be applied. Hence $g \subset N\left(\frac{l}{2}, x_0\right)$. $N\left(\frac{l}{2}, x_0\right)$ is totally geodesic at z_0. q.e.d.

This theorem and Theorem 4.9 give immediately

COROLLARY 4.11. *Let M be a simply connected n-dimensional $W_{0,0}$-manifold. If any closed geodesic of M intersects $N(r, x)$, $0 < r < l$, at two points or tangents to it without intersections, then M is isometric to an n-sphere with radius l/π.*

5. Manifold satisfying the W_p-condition.

DEFINITION. *An n-dimensinal Riemannian manifold M satisfies strongly the W_p-condition if the following hold:*

(i) *M is a W_p-manifold.*

(ii) *For any $(n-p-1)$-element (η, v) and the geodesic σ parameterized by arc length such that $\sigma(0) = \tau(v), \sigma'(0) = v$, take any $X \in J(\eta, v)^*, X \neq 0$, then $X(s) \neq 0$ for $0 < s < l$, where $\sigma(l)$ is the supporting point of the first focal p-element of (η, v).*

By virtue of the definition for the W_0-condition in §3 and Theorem 3.2, an n-dimensional W_0-manifold M has the following property: For any $(n-1)$-element (η, v) and its first focal 0-element (x, v), it must be

$$J(x, u)^* = J(\eta, v).$$

LEMMA 5.1. *For a manifold satisfying strongly the W_0-condition, the $W_{0,0}$-condition holds.*

PROOF. Let M be an n-dimensional manifold satisfying strongly the W_0-condition. For any $u \in \tau^{-1}(x)$, $x \in M$, let (η, v) be the $(n-1)$-element such that $(x, -u)$ is the first focal 0-element of $(\eta, -v)$ and let $(\bar{x}, \bar{u})$ be the one of (η, v). By Theorem 3.2, $Q_{n-1}(x)$ and $Q_{n-1}(\bar{x})$ are totally geodesic and tangent at $y = \pi_{n-1}(\eta)$ each other. Hence $Q_{n-1}(x) = Q_{n-1}(\bar{x})$. Accordingly, all geodesic rays emanating from x meet again at $\bar{x}$. The point $\bar{x}$ is the first conjugate point of x along any geodesic through x, since M satisfies strongly the W_0-condition. q.e.d.

THEOREM 5.2. *A simply connected n-dimensional manifold satisfying strongly the W_0-condition is isometric to an n-sphere.*

PROOF. Let M be a simply connected n-dimensional manifold satisfying strongly the W_0-condition. By Lemma 5.1 and Lemma 4.5, M is homeomorphic to an n-sphere. For any point $x \in M$, using the notations in §4, $Q_{n-1}(x)$ is also homeomorphic to an $(n-1)$-sphere. $Q_{n-1}(x)$ is totally geodesic by Theorem 3.2. As an $(n-1)$-dimensional Riemannian manifold, $Q_{n-1}(x)$ satisfies the same conditions as M. This theorem holds for $n=2$, by virtue of Green's theorem [5]. For $n>2$, inductively, we assume that this theorem is true for $(n-1)$-dimensional manifolds. $Q_{n-1}(x)$ is isometric to an $(n-1)$-sphere for any $x \in M$. Accordingly, every geodesic on $Q_{n-1}(x)$ is closed and has the same length.

For any two closed geodesics passing through any point $y \in M$, there exists a point x such that $Q_{n-1}(x)$ contains these geodesics. Hence, they have the same length. Since M is complete, every closed geodesic on M has the same length, i.e. $2l$. Accordingly, the sectional curvature of M is constant and equal to π^2/l^2. M is isometric to an n-sphere. q.e.d.

LEMMA 5.3. *Let M be an n-dimensional Riemannian manifold satisfying strongly the W_p-condition* $(0 < p < n-1)$, *then for any* $\xi \in T^p(M)$, $Q_q(\xi)$ *is a q-dimensional Riemannian manifold satisfying strongly the W_0-condition,* $q = n-p-1$.

PROOF. Let η be the tangent space to $Q_q(\xi)$ at a point y and v be a normal unit vector to η. Let (ξ, u) be the p-element such that $(\xi, -u)$ is the first focal p-element of $(\eta, -v)$. Take any $(q-1)$-dimensional subspace ζ of η and let v_1 be a normal unit vector to ζ in η. (See Fig. 1.) Putting $\eta_1 = \zeta \cup v$,

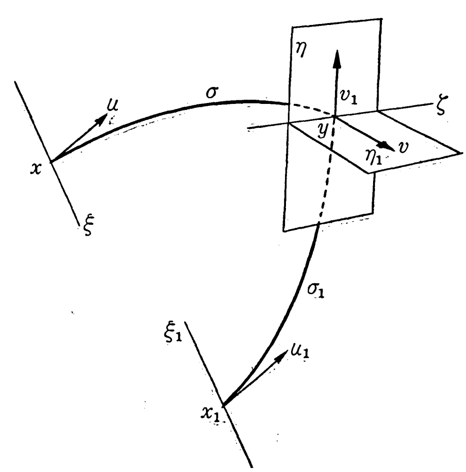

Fig. 1

let $(\xi_1, -u_1)$ be the first focal p-element of $(\eta_1, -v_1)$. Let σ and σ_1 be the geodesic segments joining $x = \tau(u)$ and $x_1 = \tau(u_1)$ to y given by $\sigma(s, u)$ and

$\sigma(s, u_1)$ respectively. Since $Q_q(\xi)$ is totally geodesic, $\sigma_1 \subset Q_q(\xi)$.

Since M satisfies the W_p^*-condition, we have

$$J(x_1, u_1)^* \cap J(\eta_1, v_1)^* = J(\xi_1, u_1)_0 .$$

Putting $L(\sigma)=l$ and $L(\sigma_1)=l_1$, for any $X \in J(\xi_1, u_1)_0$, $X \neq 0$, we have

$$X(0) = 0,\ X'(0) \in \xi_1^\perp,\ X(l_1) \in \eta_1,\ X'(l_1) \in \eta_1^\perp$$

and $X(l_1) \neq 0$, for otherwise $X \in J(\eta_1, v_1)_0$ which implies $X = 0$ by means of Corollary 3.6, where L denotes the arc length. Since M satisfies strongly the W_p-condition, $X(s) \neq 0$ for $0 < s < l_1$.

Now, if $X(l_1) \in \zeta$, the Jacobi field X can be constructed from a family of geodesics of M emanating from x_1, which are also geodesics of $Q_q(\xi)$. Since $Q_q(\xi)$ is totally geodesic, we may regard X as a Jacobi field along the geodesic σ_1 in the q-dimensional Riemannian manifold $Q_q(\xi)$. Hence, putting $v_1 = \sigma_1'(l_1, u_1)$, $(x_1, -u_1)$ is the first focal 0-element of the $(q-1)$-element $(\zeta, -v_1)$ in $Q_q(\xi)$. From these considerations, $Q_q(\xi)$ satisfies strongly the W_0-condition. q.e.d.

LEMMA 5.4. *Assume that M satisfies strongly the conditions in Lemma 5.3, then all geodesic of M are closed.*

PROOF. When $q=1$, $Q_1(\xi)$ is regular at each point and closed. From the definition of the W_p-condition, $Q_1(\xi)$ is geodesic.

When $q>1$, the universal covering Riemannian manifold $Q_q^*(\xi)$ of $Q_q(\xi)$ satisfies strongly the W_0-condition and is homeomorphic to a q-sphere. Hence, by means of Theorem 5.2, $Q_q^*(\xi)$ is isometric to a q-sphere. Accordingly, any geodesic of $Q_q^*(\xi)$ is closed and the same fact holds for $Q_q(\xi)$. For any geodesic g in M, there exists a $Q_q(\xi)$ containing g. Hence g is a closed geodesic. q.e.d.

THEOREM 5.5. *Let M be a simply connected n-dimensional Riemannian manifold strongly satisfying the W_p-condition $(0 < p < n-1)$, then M is isometric to an n-sphere.*

PROOF. *Case* $0 < p < n-2$. For any $\xi \in T^p(M)$, take two independent unit tangent vectors $v_1, v_2 \in \eta = (Q_q(\xi))_y$ at any point $y \in Q_q(\xi)$, where $q = n-p-1$. Denoting the sectional curvatures for the tangent space spanned by v_1 and v_2 of M and $Q_q(\xi)$ by $R(v_1, v_2)$ and $\bar{R}_\xi(v_1, v_2)$ respectively, we have $R(v_1, v_2) = \bar{R}_\xi(v_1, v_2)$, since $Q_q(\xi)$ is totally geodesic. By the definition of the $W_{p,q}$-condition in §3, the universal covering Riemannian manifold $Q_q^*(\xi)$ of $Q_q(\xi)$

is homeomorphic to a q-sphere. By Lemma 5.3 and Theorem 5.2, $Q_q{}^*(\xi)$ is isometric to a q-sphere. Hence, denoting the measure of the radius of $Q_q{}^*(\xi)$ by r_ξ, we have

$$\bar{R}_\xi(v_1, v_2) = (1/r_\xi)^2 .$$

On the other hand, the length of any closed geodesic of $Q_q{}^*(\xi)$ is equal to $2\pi r_\xi$. According to Lemma 5.4, let g_1 be the closed geodesic in M tangent to v_1 and g_1^* be a lift of g_1 in $Q_q{}^*(\xi)$, then $L(g_1^*) = mL(g_1)$, $m =$ an integer. Hence, from these considerations, we have

$$R(v_1, v_2) = \left(\frac{2\pi}{mL(g_1)}\right)^2 .$$

This shows that for all tangent 2-planes of $Q_q(\xi)$ containing tangent vectors of g_1 the sectional curvature of M is constant. We may regard η as any q-plane at $y \in M$ and join any two points in M by a geodesic segment, hence the sectional curvature of M is everywhere constant. Since M is simply connected, M is isometric to an n-sphere.

Case $p = n-2$ $(n > 3)$. For any $\xi \in T^p(M)$, $Q_1(\xi)$ is a closed geodesic by Lemma 5.4. Let (η, v) and $(\bar{\eta}, \bar{v})$ be 1-elements such that $(\xi, -u)$ is the first focal $(n-2)$-element of $(\eta, -v)$ and (ξ, u) is the one of $(\bar{\eta}, -\bar{v})$. Put $y = \tau(v)$ and $\bar{y} = \tau(\bar{v})$. By Corollary 3.6, we have

$$J(\eta, v)_0 \subset J(\bar{Q}_p(\xi)\,;\, x, u)^* = J(\eta, v)^* = J(\bar{\eta}, \bar{v})^* .$$

Putting $f(\xi) = l$, this yields that : For any $X \in J(\eta, v)_0$ (that is $X(l) = 0$, $X'(l) \in \eta^\perp \cap v^\perp$), it must be $X(-l) \in \bar{\eta}$. Since M satisfies strongly the W_p-conditoin, for the 1-element (η, v) and $(\bar{\eta}, \bar{v})$ it must be

$$X(s) \neq 0, \quad \text{for} \quad 0 < |s| < l .$$

Furthermore $X(0) \neq 0$, for otherwise $X \in J(x, u)^*$ and so $X \in J(\xi, u)_0$ by (ii) of the first definition in §3, which implies $X \equiv 0$ by Corollary 3.6.

On the other hand, we have easily

$$\dim \{X \,|\, X(-l) = 0,\ X \in J(\eta, v)_0\} \geqq n - 2 - \dim \eta = n - 3 > 0 ,$$

hence there exists a Jacobi field $X \in J(\eta, v)_0$ such that $X(-l) = 0$ and $X \neq 0$. Accordingly, $\bar{y}$ is the first conjugate point of y on the geodesic $\sigma(s, u)$ towards the direction of $-v$ at y. For, as is well known, on the geodesic ray $\sigma(s, -v)$, the conjugate points of y are isolated. The points $\bar{y}$ as the first common point of the fixed geodesic ray $\sigma(s, -v)$ and the variable geodesic ray $\sigma(s, w)$, $<v, w> = 0$, depend continuously on w. Hence, $\bar{y}$ is a fixed point for y, and so this shows that $\bar{y}$ is the first conjugate point of y on $\sigma(s, -v)$.

The length $2l$ of its subarc between y and $\bar{y}$ is equal to the diameter of $Q_1(\xi)$ by Lemma 3.4. By Lemma 5.4, all geodesics of M are closed. We may consider any closed geodesic g as $Q_1(\xi)$ in the above consideration. Making use of the function $\psi: S(M) \rightarrow R$ defined in §4, ψ is constant for unit tangent vectors of M normal to g. Since $n > 2$, moving η, we see that ψ depends only on the supporting points of unit tangent vectors to M. Since any two points of M can be joined by a geodesic segment by the completeness of M, hence ψ must be a constant on $S(M)$, which implies $f: T^p(M) \rightarrow R$ is also a constant l and $\psi = 2l$. From those, we see that at any point $x \in M$, for any two $u_1, u_2 \in \tau^{-1}(x)$ such that $<u_1, u_2> = 0$, it must be $\sigma(2l, u_1) = \sigma(2l, u_2)$. Furthermore, since $n > 2$, this is true without the condition: $<u_1, u_2> = 0$. Hence, M satisfies the $W_{0,0}$-condition. From the fact that $Q_1(\xi) = \{\sigma(l, u) | u \in S(M), u \perp \xi\}$ is a closed geodesic of M, all $N(l, x)$ are totally geodesic. By means of Theorem 4.9, M is isometric to an n-sphere.

Case $n = 3$, $p = 1$. Using the notations in the previous case, for $\xi \in T^1(M)$, $Q_1(\xi)$ is a simple closed geodesic. For the geodesic rays $\sigma(s, -v)$ and $\sigma(s, w)$,

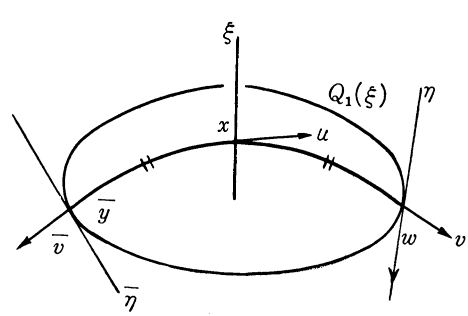

Fig. 2

$0 \leqq s$, $w \in \eta$, the point $\bar{y}$ is the first common point. The length of the subarc of the ray $\sigma(s, -v)$ between y and $\bar{y}$ is $2f(\xi)$ which is the same value for v orthogonal to η at y by Lemma 3.4. Accordingly, considering them in exchange of their stand-points, it must be that the points y and $\bar{y}$ divide $Q_1(\xi)$ into two geodesic segments with the same length. Hence, all geodesic rays emanating from y orthogonal to $Q_1(\xi)$ pass through the same point $\bar{y}$ and the length of their subarc between y and $\bar{y}$ is the common value $2f(\xi)$. By means of the same consideration of the previous case, we can show that M is isometric to a 3-sphere. q.e.d.

Appendixes

Let M be an n-dimensional manifold with line element $ds^2=g_{ij}(u)\,du^i du^j$. Geodesics in M are given by the equations:

$$\frac{d^2u^j}{ds^2}+\Gamma^j_{ih}(u)\frac{du^i}{ds}\frac{du^h}{ds}=0\,, \tag{1}$$

where Γ^j_{ih} are the components of the Levi-Civita's connection of M. Let (u^i, v^i) be the local coordinates of $T(M)$ as $v^i\partial/\partial u^i$ represents a tangent vector of M. In terms of the local coordinates (u^i, v^i), let $\tilde{X}$ be a tangent vector field on $T(M)$ given by

$$\tilde{X}(u^j)=v^j,\quad \tilde{X}(v^j)=-\Gamma^j_{ih}(u)\,v^i v^h\,. \tag{2}$$

Since we have

$$\begin{aligned}\tilde{X}(g_{ij}(u)\,v^i v^j)&=\tilde{X}(g_{ij})\,v^i v^j+2g_{ij}\,v^i\,\tilde{X}(v^j)\\&=\frac{\partial g_{ij}}{\partial u^h}v^i v^j v^h-2g_{kj}\Gamma^j_{ih}v^k v^i v^h=0\,,\end{aligned}$$

$\tilde{X}$ is tangent everywhere to the hypersurface $g_{ij}(u)v^i v^j=$ constant, especially the unit tangent bundle $S(M)$. Hence, the integral curves of the field $\tilde{X}$ lie in these hypersurfaces. When M is orientable, by means of the volume element of M:

$$dV_M=\sqrt{g}\;du^1\wedge\cdots\wedge du^n,\quad g=\det((g_{ij})) \tag{3}$$

and the $(n-1)$-form giving the $(n-1)$-dimensional angular volume on fibres of $S(M)$:

$$d\omega_{n-1}=\sqrt{g}\sum_{i=1}^{n}(-1)^{i-1}v^i\,dv^1\wedge\cdots\wedge dv^{i-1}\wedge dv^{i+1}\wedge\cdots\wedge dv^n\,, \tag{4}$$

define an $(2n-1)$-form on $T(M)$ by

$$\begin{aligned}d\tilde{\mu}&=dV_m\wedge d\omega_{n-1}\\&=g\sum_{i=1}^{n}(-1)^{i-1}v^i du^1\wedge\cdots\wedge du^n\wedge dv^1\cdots\wedge dv^{i-1}\wedge dv^{i+1}\wedge\cdots\wedge dv^n\,.\end{aligned} \tag{5}$$

In Appendixes, $T(M)$ represents the tangent bundle of M.

Making use of (2), (5) the Lie derivative of $d\tilde{\mu}$ with respect to $\tilde{\tilde{X}}$ is given by

$$L_{\tilde{X}}(d\tilde{\mu}) = \frac{\partial \log g}{\partial u^h} v^h d\tilde{\mu}$$

$$- dV_M \wedge \left(\sqrt{g} \sum_{i=1}^{n} (-1)^{i-1} \Gamma^i_{hk} v^h v^k dv^1 \wedge \cdots \wedge \widehat{dv^i} \wedge \cdots \wedge dv^n\right)$$

$$+ (-1)^{n-1}\left(\sqrt{g} \sum_{i=1}^{n} (-1)^{i-1} v^i du^1 \wedge \cdots \wedge \widehat{du^i} \wedge \cdots \wedge du^n\right) \wedge \sqrt{g}(dv^1 \wedge \cdots \wedge dv^n)$$

$$- 2\sqrt{g}\, dV_M \wedge \left\{\sum_{i=1}^{n} (-1)^{i-1} v^i \sum_{j=1}^{i-1} dv^1 \wedge \cdots \wedge \Gamma^j_{hk} v^h dv^k \wedge \cdots \wedge \widehat{dv^i} \wedge \cdots \wedge dv^n\right.$$

$$\left. + \sum_{i=1}^{n} (-1)^{i-1} v^i \sum_{j=i+1}^{n} dv^1 \wedge \cdots \wedge \widehat{dv^i} \wedge \cdots \wedge \Gamma^j_{hk} v^h dv^k \wedge \cdots \wedge dv^n \right\},$$

where "$\wedge$" denotes the omission of the symbols under it. By some calculations, the expression in the braces of the last equation can be written as

$$\Gamma^j_{hj} v^h \sum_{i=1}^{n} (-1)^{i-1} v^i dv^1 \wedge \cdots \wedge dv^{i-1} \wedge dv^{i+1} \wedge \cdots \wedge dv^n$$

$$- \sum_{j=1}^{n} (-1)^{j-1} \Gamma^j_{hk} v^h v^k dv^1 \wedge \cdots \wedge dv^{j-1} \wedge dv^{j+1} \wedge \cdots dv^n$$

and, as is well known, we have $\Gamma^j_{hj} = \frac{\partial}{\partial u^h} \log\sqrt{g}$. Hence it must be

$$(6) \quad L_{\tilde{X}}(d\tilde{\mu}) = dV_M \wedge \left(\sqrt{g} \sum_{i=1}^{n} (-1)^{i-1} \Gamma^i_{hk} v^h v^k dv^1 \wedge \cdots \wedge \widehat{dv^i} \wedge \cdots \wedge dv^n\right)$$

$$+ (-1)^{n-1}\left(\sqrt{g} \sum_{i=1}^{n} (-1)^{i-1} v^i du^1 \wedge \cdots \wedge \widehat{du^i} \wedge \cdots \wedge du^n\right)$$

$$\wedge (\sqrt{g}\, dv^1 \wedge \cdots \wedge dv^n).$$

LEMMA 1. *Let X and θ be a vector field and a 1-form on an n-dimensional differentiable manifold such that $\theta(X) = \langle\theta, X\rangle = 0$. Then, the $(n-1)$-form:*

$$(7) \qquad \omega = \sum_{i=1}^{n} (-1)^{i-1} X(u^i) du^1 \wedge \cdots \wedge du^{i-1} \wedge du^{i+1} \wedge \cdots \wedge du^n$$

vanishes under the condition: $\theta = 0$.

PROOF. Putting $\theta = \sum f_i du^i$, $f_n \neq 0$, we substitute $du^n = -\sum_{i=1}^{n-1} \frac{f_i du^i}{f_n}$ into ω, then

$$\begin{aligned}\omega &= \sum_{i=1}^{n-1} (-1)^{i-1} X(u^i) du^1 \wedge \cdots \wedge \widehat{du^i} \wedge \cdots \wedge du^{n-1} \wedge \left(-\sum_{j=1}^{n-1} \frac{f_j du^j}{f_n}\right) \\ &\quad + (-1)^{n-1} X(u^n) du^1 \wedge \cdots \wedge du^{n-1} \\ &= (-1)^{n-1} \left\{ \sum_{j=1}^{n-1} \frac{f_j X(u^j)}{f_n} + X(u^n) \right\} du^1 \wedge \cdots \wedge du^{n-1} = 0 . \end{aligned}$$

q.e.d.

Now, the right hand side of (6) can be written as

$$\begin{aligned} L_{\tilde{X}}(d\tilde{\mu}) = (-1)^{n-1} g \Big\{ &\sum_{i=1}^{n} (-1)^{i-1} \tilde{X}(u^i) du^1 \wedge \cdots \wedge \widehat{du^i} \wedge \cdots \wedge du^n \wedge dv^1 \wedge \cdots \wedge dv^n \\ &+ \sum_{i=1}^{n} (-1)^{n+i-1} \tilde{X}(v^i) du^1 \wedge \cdots \wedge du^n \wedge dv^1 \wedge \cdots \wedge \widehat{dv^i} \wedge \cdots \wedge dv^n \Big\} \end{aligned}$$

and for the 1-form $d(g_{ij} v^i v^j)$ on $T(M)$ we have

$$\tilde{X}(g_{ij} v^i v^i) = \langle d(g_{ij} v^i v^j), \tilde{X} \rangle = 0 .$$

Hence, by Lemma 1, we get the following

LEMMA 2. *When M is orientable, on each hypersurface $g_{ij}v^iv^j = constant$ of $T(M)$, we have $L_{\tilde{X}}(d\tilde{\mu}) = 0$, that is the $(2n-1)$-dimensional volume in the hypersurface is invariant under the flow of the integral curves of $\tilde{X}$.*

On $S(M)$, $d\tilde{\mu}$ is called the kinematic density and the flow of the integral curves of $\tilde{X}$ is the geodesic flow of M.

The following result is a generalization of Liouville's theorem.

LEMMA 3. *Let N be an orientable hypersurface in an n-dimensional orientable Riemannian manifold M. Let $\varphi_t: S(M) \to S(M)$ be the one-parameter group defined by the geodesic flow. If E is the set of unit vectors on N and pointing toward one side of N in M, then*

$$\tilde{\mu}\left(\bigcup_{0<t<T} E_t\right) = \int_{\bigcup_{0<t<T} E_t} d\tilde{\mu} = \frac{c_{n-2}}{n-1} V_{n-1}(N) \cdot T , \tag{8}$$

provided $E_t \cap E_{t'} \neq \emptyset$ for $0 < t < t' < T$, where c_{n-2} is the volume of the unit

$(n-2)$-sphere, $V_{n-1}(N)$ is the $(n-1)$-dimensional volume of N and $E_t=\varphi_t(E)$.

PROOF. Introduce local coordinates in $\bigcup_{0<t<T} E_t = B$ as: To $e \in B, e=\varphi_t(\xi)$ $= \operatorname{Exp} t\xi, \xi \in E, \tau(\xi) = x$, let correspond the triple (t, x, ξ). Let η be the field of normal unit vectors to N contained in E, then the kinematic density $d\tilde{\mu}$ can be written on E as

$$d\tilde{\mu} = <\eta, \xi> dt \wedge dV_N \wedge d\omega_{n-1} ,$$

where dV_N denotes the volume element of N. Since $\varphi_s(t, x, \xi) = (t+s, x, \xi)$ and $d\tilde{\mu}$ is invariant under φ_s by Lemma 2, the above expression for $d\tilde{\mu}$ is true in B. Hence we have

$$\begin{aligned}\tilde{\mu}(B) &= \int_B <\xi, \eta> dt \wedge dV_N \wedge d\omega_{n-1} \\ &= \int_{(O,T)\times N} \left\{ \int_{E\cap\tau^{-1}(x)} <\xi, \eta> d\omega_{n-1} \right\} dt \wedge dV_N\end{aligned}$$

On the other hand, we have

$$\int_{E\cap\tau^{-1}(x)} <\xi, \eta> d\omega_{n-1} = c_{n-2} \int_0^{\frac{\pi}{2}} \cos\theta\cdot(\sin\theta)^{n-2}\, d\theta = \frac{c_{n-2}}{n-1} ,$$

hence we obtain

$$\tilde{\mu}(B) = \frac{c_{n-2}}{n-1} V_N(N)\cdot T. \qquad \text{q.e.d.}$$

REFERENCES

[1] W. AMBROSE, The Cartan structural equations in classical Riemannian geometry, Journ. Indian Math. Soc., 24(1960), 23-76.

[2] W. AMBROSE, The index theorem in Riemannian geometry, Annals of Math., 73 (1961), 49-86.

[3] L. N. PATTERSON, On the index theorem, Amer. Journ. of Math., 85(1963), 271-297.

[4] L. W. GREEN, Surfaces without conjugate points, Trans. Amer. Math. Soc., 76(1954), 529-546.

[5] L. W. GREEN, Auf Wiedersehensflächen, Annals of Math., 78(1963), 289-299.

[6] M. MORSE, The calculus of variations in the large, Amer. Math. Soc. Colloqium Publications, 1934.

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