LEBESGUE CONSTANTS FOR A FAMILY OF SUMMABILITY METHODS

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1. Introduction. The unboundedness of the sequence of Lebesgue constants implies the existence of a continuous function whose Fourier series diverges at a point, and this is also the case with many summability methods. The estimation of such constants for various summability methods has been calculated by K. Ishiguro [2], [3], A. E. Livingston [4], and L. Lorch [5], [6], [7], [8].

In this paper we shall study the behavior of the Lebesgue constants for a family of summability methods. A. Meir [9] has introduced a family of summability methods which is defined by two parameters a and q, and has shown that this family contains Borel, Valiron, Euler, Taylor and S_α-transformation.

If we define $L_F(a, q(p))$ by the Lebesgue constants for this family of summability methods, then we obtain the following formula:

$$(1.1)\qquad L_F(a, q(p)) = \frac{2}{\pi^2}\log 4aq(p) + A + O(\log q/\sqrt{q}\,) \text{ as } p \to \infty,$$

where A is the constant such as

$$(1.2)\qquad A = -\frac{c}{\pi^2} + \frac{2}{\pi}\int_0^1 \frac{\sin u}{u}\,du - \frac{2}{\pi}\int_1^\infty \left(\frac{2}{\pi} - |\sin u|\right)\frac{du}{u}$$

and

$$c = \int_0^1 \frac{1-e^{-u}}{u} - \int_1^\infty \frac{1}{ue^u}\,du,$$

which is Euler-Mascheroni's constant.

The proof of formula (1.1) consists of two parts; 1°) the case where $q=q(p)$ is integer and 2°) the case where $q=q(p)$ is not integer. In the last section we shall show that from (1.1) we can obtain Lebesgue constants for Borel, Valiron, Euler, Taylor and S_α-transformation which are contained in this family of summability methods.

Finally I wish to express my gratitude to Professor G. Sunouchi for his kind suggestions.

2. The Family F(a, q(p)) of Summability Methods. After A. Meir [9], let us say the summability matrix $[c_{pk}]$ belongs to $F(a, q(p))$ if it satisfies the following conditions: p is a discrete or continuous parameter; a is a positive constant; $q=q(p)$ is a positive increasing function which tends to infinity as $p\to\infty$; for every fixed δ: $1/2<\delta<2/3$

$$c_{pk}=\sqrt{\frac{a}{\pi q}}\,e^{-\frac{a}{q}(k-q)^2}\left\{1+O\left(\frac{|k-q|+1}{q}\right)+O\left(\frac{|k-q|^3}{q^2}\right)\right\} \tag{2.1}$$

as $p\to\infty$ uniformly in k for $|k-q|\leqq q^{\delta}$,

$$c_{p0}+\sum_{|k-q|>q^{\delta}} kc_{pk}=O\left(\exp(-q^{\eta})\right) \tag{2.2}$$

where η is some positive number independent of p, and

$$c_{pk}\geqq 0\,. \tag{2.3}$$

It is known that the family $F(a, q(p))$ with appropriate a and $q(p)$ contains such summability methods as Borel, Valiron, Euler, Taylor and S_α-transformation, see G. H. Hardy [1] and A. Meir [9].

Let a function $f(x)$ be integrable in Lebesgue's sense over the interval $-\pi\leqq x\leqq\pi$ and periodic with period 2π.

If we define $S_n(f;x)$ by the n-th partial sum of the Fourier series of $f(x)$ and $t_p(f;x)$ by the transformation of $S_n(f;x)$ by means of summability matrix $[c_{pk}]$, then we have

$$t_p(f;x)=\sum_{k=0}^{\infty}c_{pk}S_k(f;x)=\sum_{k=0}^{\infty}\frac{1}{\pi}\int_{-\pi}^{\pi}f(x+u)\frac{c_{pk}\sin\left(k+\frac{1}{2}\right)u}{2\sin u/2}\,du\,.$$

When we suppose that for all p

$$\sum_{k=0}^{\infty}k|c_{pk}|<\infty\,, \tag{2.4}$$

$$t_p(f;x)=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x+u)\frac{1}{2\sin u/2}\sum_{k=0}^{\infty}c_{pk}\sin\left(k+\frac{1}{2}\right)u\,du\,.$$

If the summability matrix $[c_{pk}]$ belongs to $F(a, q(p))$, then the condition

(2.4) is satisfied and Lebesgue constants for this methods $L_F(a, q(p))$ are defined as follows:

$$L_F(a, q(p)) = \frac{2}{\pi}\int_0^{\pi/2}\left|\frac{1}{\sin u}\sum_{k=0}^{\infty} c_{pk}\sin(2k+1)u\right|du. \tag{2.5}$$

3. Three Lemmas. To prove formula (1.1), we require the following three lemmas.

LEMMA 3.1. *When p tends to infinity, we get:*

$$\frac{2}{\pi}\int_0^{\pi/2}\frac{1}{\sin u}\sum_{|k-q|\leqq q^\delta}\sqrt{\frac{a}{\pi q}}\,e^{-\frac{a}{q}(k-q)^2}\frac{|k-q|+1}{q}|\sin(2k+1)u|\,du \tag{3.1.1}$$
$$= O(\log q/\sqrt{q})$$

and

$$\frac{2}{\pi}\int_0^{\pi/2}\frac{1}{\sin u}\sum_{|k-q|\leqq q^\delta}\sqrt{\frac{a}{\pi q}}\,e^{-\frac{a}{q}(k-q)^2}\frac{|k-q|^3}{q^2}|\sin(2k+1)u|\,du \tag{3.1.2}$$
$$= O(\log q/\sqrt{q}).$$

PROOF. i) We can suppose $0 < 1/q < \pi/2$ for sufficiently large p, and set I_{11}, I_{12} as follows:

$$\frac{2}{\pi}\int_0^{\pi/2}\frac{1}{\sin u}\sum_{|k-q|\leqq q^\delta}\sqrt{\frac{a}{\pi q}}\,e^{-\frac{a}{q}(k-q)^2}\frac{|k-q|+1}{q}|\sin(2k+1)u|\,du$$

$$= \frac{2}{\pi}\left(\int_0^{1/q}+\int_{1/q}^{\pi/2}\right)\frac{1}{\sin u}\sum_{|k-q|\leqq q^\delta}\sqrt{\frac{a}{\pi q}}\,e^{-\frac{a}{q}(k-q)^2}\frac{|k-q|+1}{q}|\sin(2k+1)u|\,du$$

$$= I_{11} + I_{12}.$$

Since we have

$$\sum_{|k-q|\leqq q^\delta}\sqrt{\frac{a}{\pi q}}\,e^{-\frac{a}{q}(k-q)^2}\frac{|k-q|+1}{q}|\sin(2k+1)u|$$

$$\leqq\sqrt{\frac{a}{\pi q}}\sum_{|k-q|\leqq q^\delta}e^{-\frac{a}{q}(k-q)^2}\frac{|k-q|+1}{q}\{2|k-q|+(2q+1)\}|u|$$

$$= O\left(\sqrt{\frac{a}{\pi q}} \sum_{|k-q|\leqq q^{\delta}} e^{-\frac{a}{q}(k-q)^2}\left(\frac{(k-q)^2}{q} + |k-q| + 1\right)|u|\right)$$

$$= O(\sqrt{q}\,|u|),$$

then we obtain

$$I_{11} = O(\sqrt{q} \int_0^{1/q} \frac{u}{\sin u}\, du) = O(1/\sqrt{q}),$$

and

$$I_{12} = O\left(\int_{1/q}^{\pi/2} \frac{1}{\sin u} \sum_{|k-q|\leqq q^{\delta}} \sqrt{\frac{a}{\pi q}}\, e^{-\frac{a}{q}(k-q)^2} \frac{|k-q|+1}{q}\, du\right)$$

$$= O\left(\frac{1}{\sqrt{q}} \int_{1/q}^{\pi/2} \frac{du}{\sin u}\right) = O(\log q/\sqrt{q}), \quad \text{as} \quad p \to \infty .$$

Therefore (3.1.1) has been proved.

ii) We can prove (3.1.2) by the same method as in (i).
We suppose that $0 < 1/q < \pi/2$ similar as in (i) and set I_{21}, I_{22} as follows:

$$\frac{2}{\pi} \int_0^{\pi/2} \frac{1}{\sin u} \sum_{|k-q|\leqq q^{\delta}} \sqrt{\frac{a}{\pi q}}\, e^{-\frac{a}{q}(k-q)^2} \frac{|k-q|^3}{q^2}\, |\sin(2k+1)\,u|\, du$$

$$= \frac{2}{\pi}\left(\int_0^{1/q} + \int_{1/q}^{\pi/2}\right) \frac{1}{\sin u} \sum_{|k-q|\leqq q^{\delta}} \sqrt{\frac{a}{\pi q}}\, e^{-\frac{a}{q}(k-q)^2} \frac{|k-q|^3}{q^2}\, |\sin(2k+1)u|\, du$$

$$= I_{21} + I_{22} .$$

Since we have

$$\sum_{|k-q|\leqq q^{\delta}} \sqrt{\frac{a}{\pi q}}\, e^{-\frac{a}{q}(k-q)^2} \frac{|k-q|^3}{q^2}\, |\sin(2k+1)u|$$

$$\leqq \sqrt{\frac{a}{\pi q}} \sum_{|k-q|\leqq q^{\delta}} e^{-\frac{a}{q}(k-q)^2} \frac{|k-q|^3}{q^2} \{2|k-q| + (2q+1)\}\, |u|$$

$$= O\left(\sqrt{\frac{a}{\pi q}} \sum_{|k-q|\leqq q^{\delta}} e^{-\frac{a}{q}(k-q)^2}\left(\frac{(k-q)^4}{q^2} + \frac{|k-q|^3}{q}\right)|u|\right)$$

$$= O(\sqrt{q}\,|u|),$$

then we obtain

$$I_{21} = O\left(\sqrt{q}\int_0^{1/q}\frac{u}{\sin u}du\right) = O(1/\sqrt{q})$$

and

$$I_{22} = O\left(\frac{2}{\pi}\int_{1/q}^{\pi/2}\frac{1}{\sin u}\sum_{|k-q|\leqq q^\delta}\sqrt{\frac{a}{\pi q}}\,e^{-\frac{a}{q}(k-q)^2}\frac{|k-q|^3}{q^2}du\right)$$

$$= O\left(\frac{1}{\sqrt{q}}\int_{1/q}^{\pi/2}\frac{du}{\sin u}\right) = O(\log q/\sqrt{q}) \quad \text{as} \quad p\to\infty\,.$$

Therefore (3.1.2) has been proved.

LEMMA 3.2. *If $q=q(p)$ is an integer valued function of p, then we have*

$$\frac{2}{\pi}\int_0^{\pi/2}\left|\frac{1}{\sin u}\sum_{|k-q|\leqq q^\delta}\sqrt{\frac{a}{\pi q}}\,e^{-\frac{a}{q}(k-q)^2}\sin(2k+1)u\right|du \tag{3.2}$$

$$= \frac{2}{\pi}\int_0^{\pi/2}\left|\frac{e^{-\frac{qu^2}{a}}\sin(2q+1)u}{\sin u}\right|du + o(1/q)\,, \quad \textit{as} \quad p\to\infty\,.$$

PROOF. When we set $n=k-q(p)$, we have

$$\sum_{|k-q|\leqq q^\delta}\sqrt{\frac{a}{\pi q}}\,e^{-\frac{a}{q}(k-q)^2}\sin(2k+1)u$$

$$= \Im\left\{e^{i(2q+1)u}\sqrt{\frac{a}{\pi q}}\sum_{|n|\leqq q^\delta}e^{-\frac{a}{q}n^2+2uni}\right\}$$

$$= \Im\left\{e^{i(2q+1)u}\sqrt{\frac{a}{\pi q}}\left(\sum_{n=-\infty}^{+\infty} - \sum_{|n|>q^\delta}\right)e^{-\frac{q}{a}n^2+2uni}\right\}.$$

Using the property of Theta function [10], we get

$$\sqrt{\frac{a}{\pi q}}\sum_{n=-\infty}^{+\infty}e^{-\frac{a}{q}n^2+2uni} = \sum_{n=-\infty}^{+\infty}e^{-\frac{a}{q}(u-n\pi)^2}$$

and for $0\leqq u\leqq \pi/2$

$$\sum_{|k-q|\leqq q^{\delta}} \sqrt{\frac{a}{\pi q}}\, e^{-\frac{a}{q}(k-q)^2} \sin(2k+1)u$$

$$= \Im\left\{e^{i(2q+1)u} \sum_{n=-\infty}^{+\infty} e^{-\frac{q}{a}(u-n\pi)^2}\right\} + O\left(\sum_{|n|>q^{\delta}} \sqrt{\frac{a}{\pi q}}\, e^{-\frac{a}{q}n^2} |\sin(2n+2q+1)u|\right)$$

$$= e^{-\frac{qu^2}{a}} \sin(2q+1)u + O(qe^{-\frac{qu^2}{4a}}|u|) + O(qe^{-aq^{2\delta-1}}|u|)$$

$$= e^{-\frac{qu^2}{a}} \sin(2q+1)u + O(qe^{-aq^{2\delta-1}}|u|).$$

Therefore we get

$$\frac{2}{\pi}\int_0^{\pi/2}\left|\frac{1}{\sin u}\sum_{|n|\leqq q^{\delta}} \sqrt{\frac{a}{\pi q}}\, e^{-\frac{an^2}{q}} \sin(2n+2q+1)u\right| du$$

$$= \frac{2}{\pi}\int_0^{\pi/2}\left|\frac{e^{-\frac{qu^2}{a}} \sin(2q+1)u}{\sin u}\right| du + o(1/q) \quad \text{as} \quad p\to\infty.$$

and consequently we have proved Lemma 3.2.

LEMMA 3.3 *Let $f(u,q)$ be defined over $q\geqq 0$ and $0\leqq u\leqq \frac{\pi}{2}$, and let $\frac{\partial f}{\partial u}=f_u(u,q)$ exist for $q>0$, and $f_u(u,q)$ be integrable over $[0,\pi/2]$.*

If $\int_0^{\pi/2}|f_u(u,q)|du = O(\sqrt{q})$ and $f\left(\frac{\pi}{2},q\right)=O(\sqrt{q})$ as $q\to\infty$, then we have

$$\int_0^{\pi/2} f(u,q)\left\{\frac{2}{\pi} - |\sin(2q+1)u|\right\} du = O(1/\sqrt{q}), \quad \text{as} \quad q\to\infty.$$

In order to prove this lemma, see L. Lorch [5].

4. Lebesgue Constants. In this section we calculate Lebesgue constants for a family of summability methods whose matrix $[c_{pk}]$ belongs to $F(a,q(p))$.

THEOREM. *Let $L_F(a,q(p))$ denote the Lebesgue constants for a family of summability methods whose matrix $[c_{pk}]$ belongs to $F(a,q(p))$.*

Then we get the following formula:

$$(1.1)\quad L_F(a,q(p)) = \frac{2}{\pi^2}\log 4aq(p) + A + O(\log q/\sqrt{q}) \quad \text{as} \quad p\to\infty,$$

where constant A is defined by (1.2).

PROOF. 1°) The case where $q = q(p)$ is integer.
From (2.5) we have

$$L_F(a, q(p)) = \frac{2}{\pi}\int_0^{\pi/2}\left|\frac{1}{\sin u}\left(\sum_{|k-q|\leqq q^\delta} + \sum_{|k-q|>q^\delta}\right) c_{pk}\sin(2k+1)u\right| du.$$

We set n, $L(a, q)$ and E as follows:

$$n = k - q(p)$$

$$L(a, q) = \frac{2}{\pi}\int_0^{\pi/2}\left|\frac{1}{\sin u}\sum_{|k-q|\leqq q^\delta} c_{pk}\sin(2k+1)u\right| du \tag{4.1}$$

$$E = \frac{2}{\pi}\int_0^{\pi/2}\left|\frac{1}{\sin u}\sum_{|k-q|>q^\delta} c_{pk}\sin(2k+1)u\right| du. \tag{4.2}$$

Using (2.2), we have

$$\begin{aligned} |E| &= O\left(\int_0^{\pi/2}\frac{1}{\sin u}\left(c_{po} + \sum_{|k-q|>q^\delta} kc_{pk}\right)|u|\,du\right) \\ &= O(e^{-q^\eta}) = o(1/q) \end{aligned} \tag{4.3}$$

We get from Lemma 3.2,

$$\begin{aligned} &\frac{2}{\pi}\int_0^{\pi/2}\left|\frac{1}{\sin u}\sum_{|n|\leqq q^\delta} e^{-\frac{an^2}{q}}\sin(2k+1)u\right| du \\ &= \frac{2}{\pi}\int_0^{\pi/2}\left|\frac{1}{\sin u} e^{-\frac{qu^2}{a}}\sin(2q+1)u\right| du + o(1/q). \end{aligned} \tag{4.4}$$

Applying (4.3) and Lemma 3.1 to (4.1), we obtain

$$\begin{aligned} L(a, q) = \frac{2}{\pi}\int_0^{\pi/2}\Bigg|\frac{1}{\sin u}\sum_{|n|\leqq q^\delta}\sqrt{\frac{a}{\pi q}}\,e^{-\frac{an^2}{q}} \\ \left\{1 + O\left(\frac{|n|+1}{q}\right) + O\left(\frac{|n|^3}{q^2}\right)\right\}\sin(2k+1)u\Bigg| du \end{aligned} \tag{4.5}$$

$$= \frac{2}{\pi}\int_0^{\pi/2}\left|\frac{1}{\sin u}\, e^{-\frac{qu^2}{a}} \sin(2q+1)u\right| du + O(\log q/\sqrt{q})$$

$$= \frac{2}{\pi}\int_0^{\pi/2}\left|\frac{e^{-\frac{qu^2}{a}}}{u} \sin(2q+1)u\right| du + O(1/q) + O(\log q/\sqrt{q}).$$

If we define $L(q)$ as follows;

$$L(q) = \frac{2}{\pi}\int_0^{\pi/2} \frac{1}{u}\,|\sin(2q+1)u|\, du,$$

then from L. Lorch [5], [6] we obtain

$$L(q) = \frac{4}{\pi^2}\log q - \frac{2}{\pi^2}\int_0^{\pi} \frac{\Gamma'(u/\pi)}{\Gamma(u/\pi)} \sin u\, du + O(1/q) \tag{4.6}$$

$$= \frac{4}{\pi^2}\log q + \frac{4}{\pi^2}\log \pi + \frac{2c}{\pi^2} + A + O(1/q),$$

where A and c are defined by (1.2).

If we set $d(q) = L(q) - L(a, q)$, then from (4.5) we have

$$d(q) = L(q) - L(a, q)$$

$$= \frac{2}{\pi}\int_0^{\pi/2} \frac{1}{u}\,(1-e^{-\frac{qu^2}{a}})\,|\sin(2q+1)u|\, du$$

$$= \frac{2}{\pi}\int_0^{\pi/2} f(u, q)\,|\sin(2q+1)u|\, du + O(\log q/\sqrt{q})$$

where $f(u, q) = \frac{1}{u}\,(1-e^{-\frac{qu^2}{a}})$.

Since the function $f(u, q)$ satisfies the conditions of Lemma 3.3, (see L. Lorch [5]), we have

$$d(q) = \frac{4}{\pi^2}\int_0^{\pi/2} \frac{1}{u}(1-e^{-\frac{qu^2}{a}})\, du + O(\log q/\sqrt{q})$$

$$= \frac{2}{\pi^2}\int_0^1 \frac{1}{u}(1-e^{-u})\, du + \frac{2}{\pi^2}\int_1^{qu^2/4a} \frac{1}{u}(1-e^{-u})\, du + O(\log q/\sqrt{q})$$

$$= \frac{2}{\pi^2}\log\frac{q\pi^2}{4a} + \frac{2}{\pi^2}\left\{\int_0^1 \frac{1}{u}(1-e^{-u})\,du - \int_1^\infty \frac{1}{ue^u}\,du\right\}$$

$$+ \frac{2}{\pi^2}\int_{q\pi^2/4a}^\infty \frac{du}{ue^u} + O(\log q/\sqrt{q})$$

$$= \frac{2}{\pi^2}\log\frac{q\pi^2}{4a} + \frac{2c}{\pi^2} + O(\log q/\sqrt{q})\,.$$

Consequently we obtain

$$(4.7)\qquad L(a,q) = L(q) - d(q)$$

$$= \frac{4}{\pi^2}\log q + \frac{4}{\pi^2}\log\pi + \frac{2c}{\pi^2} + A - \frac{2}{\pi^2}\log\frac{q\pi^2}{4a} - \frac{2c}{\pi^2} + O(\log q/\sqrt{q})$$

$$= \frac{2}{\pi^2}\log 4aq + A + O(\log q/\sqrt{q})\,.$$

From (4.1), (4.2), and (4.7), we get

$$L_F(a, q(p)) = L(a,q) + o(1/q)$$

$$= \frac{2}{\pi^2}\log 4aq(p) + A + O(\log q/\sqrt{q}), \quad \text{as} \quad p\to\infty\,.$$

Therefore we have proved (1.1) when $q=q(p)$ is integer.

2°) The case when $q=q(p)$ is not integer.
Let $[q]$ denote the integral part of $q=q(p)$ and $q_0=[q]+1$.
We set D_1, D_2, D_3, D_4 as follows:

$$(4.8)\qquad \left|\frac{2}{\pi}\int_0^{\pi/2}\left|\frac{1}{\sin u}\sum_{|k-q|\leq q^\delta}\sqrt{\frac{a}{\pi q}}\,e^{-\frac{a}{q}(k-q)^2}\sin(2k+1)u\right|du\right.$$

$$\left. - \frac{2}{\pi}\int_0^{\pi/2}\left|\frac{1}{\sin u}\sum_{|k-q_0|\leq q_0^\delta}\sqrt{\frac{a}{\pi q_0}}\,e^{-\frac{a}{q_0}(k-q_0)^2}\sin(2k+1)u\right|du\right|$$

$$\leqq \frac{2}{\pi}\int_0^{\pi/2}\frac{1}{\sin u}\left(\sum_{q<k\leqq q+q^\delta} + \sum_{q-q^\delta\leqq k<q}\right)\left|\left(\sqrt{\frac{a}{\pi q}}\,e^{-\frac{a}{q}(k-q)^2}\right.\right.$$

$$\left.\left. - \sqrt{\frac{a}{\pi q_0}}\,e^{-\frac{a}{q_0}(k-q_0)^2}\right)\sin(2k+1)u\right|du$$

$$+\frac{2}{\pi}\int_0^{\pi/2}\frac{1}{\sin u}\sum_{q+q^\delta<k\leqq q_0+q_0^\delta}\sqrt{\frac{a}{\pi q_0}}\,e^{-\frac{a}{q_0}(k-q_0)^2}\,|\sin(2k+1)u|\,du$$

$$+\frac{2}{\pi}\int_0^{\pi/2}\frac{1}{\sin u}\sum_{q-q^\delta\leqq k<q_0-q_0^\delta}\sqrt{\frac{a}{\pi q}}\,e^{-\frac{a}{q}(k-q)^2}\,|\sin(2k+1)u|\,du$$

$$=D_1+D_2+D_3+D_4\,,$$

where we take p large enough.

i) In case where $q<q_0\leqq k\leqq q+q^\delta$, we have

$$0\leqq(k-q_0)/\sqrt{q_0}<(k-q)/\sqrt{q}<(k-[q])/\sqrt{[q]}\,.$$

Hence the following inequality results:

$$\left|\sqrt{\frac{a}{\pi q}}\,e^{-\frac{a}{q}(k-q)^2}-\sqrt{\frac{a}{\pi q_0}}\,e^{-\frac{a}{q_0}(k-q_0)^2}\right| \tag{4.9}$$

$$\leqq\sqrt{\frac{a}{\pi q_0}}\left(e^{-\frac{a}{q_0}(k-q_0)^2}-e^{-\frac{a}{q}(k-q)^2}\right)$$

$$=O\left(\frac{1}{\sqrt{q_0}}\int_{(k-q_0)/\sqrt{q_0}}^{(k-[q])/\sqrt{[q]}}|xe^{-ax^2}|\,dx\right)$$

$$=O\left(\frac{1}{\sqrt{q_0}}\,e^{-\frac{a}{q_0}(k-q_0)^2}\left(\frac{(k-[q])^2}{[q]}-\frac{(k-q_0)^2}{q_0}\right)\right)$$

$$=O\left(\frac{1}{\sqrt{q_0}}\,e^{-\frac{a}{q_0}(k-q_0)^2}\left(\frac{(k-q_0)^2}{q_0^{\,2}}+\frac{|k-q_0|}{q_0}+\frac{1}{q_0}\right)\right).$$

We shall estimate D_1, by dividing the range of integration of D_1 in (4.8) into two parts. We set D_{11}, D_{12} as follows:

$$D_1=\frac{2}{\pi}\left(\int_0^{1/q}+\int_{1/q}^{\pi/2}\right)\frac{1}{\sin u}\sum_{q<k\leqq q+q^\delta}\left|\left(\sqrt{\frac{a}{\pi q}}\,e^{-\frac{a}{q}(k-q)^2}-\sqrt{\frac{a}{\pi q_0}}\,e^{-\frac{a}{q_0}(k-q_0)^2}\right)\sin(2k+1)u\right|du$$

$$=D_{11}+D_{12}\,,$$

where we can suppose $0<1/q<\pi/2$ for sufficiently large p.

From (4.9), we get

$$D_{11}=O\Big(\int_0^{1/q}\frac{1}{\sin u}\sum_{q<k\leqq q+q^\delta}\frac{1}{\sqrt{q_0}}e^{-\frac{a}{q_0}(k-q_0)^2}$$

$$\times\Big(\frac{(k-q_0)^2}{q_0^2}+\frac{|k-q_0|}{q_0}+\frac{1}{q_0}\Big)(2|k-q|+2q+1)\,udu\Big)$$

$$=O\Big(\sqrt{q}\int_0^{1/q}\frac{u}{\sin u}du\Big)=O(1/\sqrt{q})$$

and

$$D_{12}=O\Big(\int_{1/q}^{\pi/2}\frac{1}{\sin u}\sum_{q<k\leqq q+q^\delta}\frac{1}{\sqrt{q_0}}e^{-\frac{a}{q_0}(k-q_0)^2}\Big(\frac{(k-q_0)^2}{q_0^2}+\frac{|k-q_0|}{q_0}+\frac{1}{q_0}\Big)du\Big)$$

$$=O\Big(\frac{1}{\sqrt{q}}\int_{1/q}^{\pi/2}\frac{1}{\sin u}du\Big)=O(\log q/\sqrt{q}).$$

Therefore

$$D_1=D_{11}+D_{12}=O(\log q/\sqrt{q}). \tag{4.10}$$

ii) In case where $k\leqq[q]<q<q_0$, we have

$$(k-q_0)/\sqrt{q_0}<(k-q)/\sqrt{q}<(k-[q])/\sqrt{[q]}\leqq 0.$$

Hence the following inequality results similarly to (4.9):

$$\Big|\sqrt{\frac{a}{\pi q}}e^{-\frac{a}{q}(k-q)^2}-\sqrt{\frac{a}{\pi q_0}}e^{-\frac{a}{q_0}(k-q_0)^2}\Big| \tag{4.11}$$

$$\leqq\sqrt{\frac{a}{\pi q}}\Big(e^{-\frac{a}{q}(k-q)^2}-e^{-\frac{a}{q_0}(k-q_0)^2}\Big)$$

$$=O\Big(\frac{1}{\sqrt{q}}\int_{(k-q_0)/\sqrt{q_0}}^{(k-[q])/\sqrt{[q]}}|xe^{-ax^2}|\,dx\Big)$$

$$=O\Big(\frac{1}{\sqrt{[q]}}e^{-\frac{a}{[q]}(k-[q])^2}\Big(\frac{(k-q_0)^2}{q_0}-\frac{(k-[q])^2}{[q]}\Big)\Big)$$

$$=O\Big(\frac{1}{\sqrt{[q]}}e^{-\frac{a}{[q]}(k-[q])^2}\Big(\frac{(k-[q])^2}{[q]^2}+\frac{|k-[q]|}{[q]}+\frac{1}{[q]}\Big)\Big).$$

We divide the range of integration of D_2 in (4.8) into two parts for sufficiently large p and set D_{21}, D_{22} as follows:

$$D_2 = \frac{2}{\pi}\left(\int_0^{1/q} + \int_{1/q}^{\pi/2}\right)\frac{1}{\sin u}\left|\sum_{q-q^\delta \leqq k<q}\left(\sqrt{\frac{a}{\pi q}}\, e^{-\frac{a}{q}(k-q)^2} - \sqrt{\frac{a}{\pi q_0}}\, e^{-\frac{a}{q_0}(k-q_0)^2}\right)\sin(2k+1)u\right| du$$

$$= D_{21} + D_{22}\,.$$

From (4.11), we get

$$D_{21} = O\left(\int^{1/q}\frac{1}{\sin u}\sum_{q-q^\delta\leqq k<q}\frac{1}{\sqrt{[q]}}e^{-\frac{a}{[q]}(k-[q])^2} \times\left(\frac{(k-[q])^2}{[q]^2} + \frac{|k-[q]|}{[q]} + \frac{1}{[q]}\right)(2|k-q|+2q+1)u\, du\right)$$

$$= O\left(\sqrt{q}\int_0^{1/q}\frac{u}{\sin u}du\right) = O(1/\sqrt{q})$$

and

$$D_{22} = O\left(\int_{1/q}^{\pi/2}\frac{1}{\sin u}\sum_{q-q^\delta\leqq k<q}\frac{1}{\sqrt{[q]}}\, e^{-\frac{a}{[q]}(k-[q])^2}\left(\frac{(k-[q])^2}{[q]^2} + \frac{|k-[q]|}{[q]} + \frac{1}{[q]}\right)du\right)$$

$$= O\left(\frac{1}{\sqrt{q}}\int_{1/q}^{\pi/2}\frac{1}{\sin u}du\right) = O(\log q/\sqrt{q})\,.$$

Therefore

$$D_2 = D_{21} + D_{22} = O(\log q/\sqrt{q})\,. \tag{4.12}$$

Next we shall estimate D_3, D_4 and we get for sufficiently large p,

$$D_3 = \frac{2}{\pi}\int_0^{\pi/2}\frac{1}{\sin u}\sum_{q+q^\delta<k\leqq q_0+q_0^\delta}\sqrt{\frac{a}{\pi q_0}}\, e^{-\frac{a}{q_0}(k-q_0)^2}\,|\sin(2k+1)u|\, du \tag{4.13}$$

$$= O\left(\sqrt{q}\, e^{-aq^{2\delta-1}}\int_0^{\pi/2}\frac{u}{\sin u}du\right) = O(\sqrt{q}\, e^{-aq^{2\delta-1}})$$

$$= o(1/q)\,,$$

and

$$(4.14)\qquad D_4=\frac{2}{\pi}\int_0^{\pi/2}\frac{1}{\sin u}\sum_{q-q^\delta\leqq k<q_0-q_0^\delta}\sqrt{\frac{a}{\pi q}}\,e^{-\frac{a}{q}(k-q)^2}\,|\sin(2k+1)u|\,du$$

$$=O\left(\sqrt{q}\,e^{-aq^{2\delta-1}}\int_0^{\pi/2}\frac{u}{\sin u}\,du\right)=O(\sqrt{q}\,e^{-aq^{2\delta-1}})$$

$$=o(1/q)\,.$$

Using (4.8), (4.10), (4.12), (4.13), (4.14) and setting $n_0=k-q_0$, we obtain

$$\frac{2}{\pi}\int_0^{\pi/2}\left|\frac{1}{\sin u}\sum_{|k-q|\leqq q^\delta}\sqrt{\frac{a}{\pi q}}\,e^{-\frac{a}{q}(k-q)^2}\sin(2k+1)u\right|du$$

$$=\frac{2}{\pi}\int_0^{\pi/2}\left|\frac{1}{\sin u}\sum_{|n_0|\leqq q_0^\delta}\sqrt{\frac{a}{\pi q_0}}\,e^{-\frac{an_0^2}{q_0}}\sin(2k+1)u\right|du+O(\log q/\sqrt{q})\,.$$

Since we have from (4.7)

$$\frac{2}{\pi}\int_0^{\pi/2}\left|\frac{1}{\sin u}\sum_{|n_0|\leqq q_0^\delta}\sqrt{\frac{a}{\pi q_0}}\,e^{-\frac{an_0^2}{q_0}}\sin(2k+1)u\right|du$$

$$=\frac{2}{\pi^2}\log 4aq_0+A+O(\log q_0/\sqrt{q_0})\,,$$

we obtain

$$(4.15)\qquad \frac{2}{\pi}\int_0^{\pi/2}\left|\frac{1}{\sin u}\sum_{|k-q|\leqq q^\delta}\sqrt{\frac{a}{\pi q}}\,e^{-\frac{a}{q}(k-q)^2}\sin(2k+1)u\right|du$$

$$=\frac{2}{\pi^2}\log 4aq(p)+A+O(\log q/\sqrt{q})\,.$$

From Lemma 3.1, (4.3) and (4.15), we have

$$L_F(a,q(p))=\frac{2}{\pi}\int_0^{\pi/2}\left|\frac{1}{\sin u}\sum_{|k-q|\leqq q^\delta}\sqrt{\frac{a}{\pi q}}\,e^{-\frac{a}{q}(k-q)^2}\sin(2k+1)u\right|du+O(\log q/\sqrt{q})$$

$$=\frac{2}{\pi^2}\log 4a\,q(p)+A+O(\log q/\sqrt{q})\,,\quad\text{as}\quad p\to\infty\,.$$

Thus we have obtained the Lebesgue constants for a family of summability methods whose matrix $[c_{pk}]$ belongs to $F(a, q(p))$.

5. Lebesgue constants for Borel, Valiron, Euler, Taylor and S_α-transformation.

i) Borel-transformation. See L. Lorch [5].

The summability matrix of this transformation is defined by

$$c_{pk} = e^{-p}\frac{p^k}{k!} \qquad (k = 0, 1, 2, \cdots),$$

where $p > 0$, $a = \frac{1}{2}$, and $q = p$, see A. Meir [9].

Therefore from (1.1) we get Lebesgue constants for Borel-transformation L_B as follows:

$$L_B = L_F\left(\frac{1}{2}, p\right) = \frac{2}{\pi^2}\log 2p + A + O(\log p/\sqrt{p})$$

This Lebesgue constants have been obtained already by L. Lorch [5] whose remainder term is $O(1/\sqrt{p})$.

ii) Valiron-transformation.

The summability matrix is defined by

$$c_{pk} = \sqrt{\frac{\alpha}{\pi p}}\, e^{-\frac{\alpha}{p}(k-p)^2} \qquad (p = 1, 2, \cdots, \quad k = 0, 1, 2, \cdots)$$

where $\alpha > 0$, $a = \alpha$ and $q = p$, see A. Meir [9].

Therefore from Theorem 1°) in section 4 we get Lebesgue constants for Valiron-transformation $L_{(V,\alpha)}$ as follows:

$$L_{(V,\alpha)} = L_F(\alpha, p) = \frac{2}{\pi^2}\log 4\alpha p + A + O(1/\sqrt{p}).$$

iii) Euler-transformation. See L. Lorch [7] and A. E. Livingston [4].

The summability matrix of this transformation is defined by

$$c_{pk} = \binom{p}{k}\alpha^k(1-\alpha)^{p-k}, \qquad (p = 1, 2, \cdots, k = 0, 1, 2, \cdots)$$

where $0 < \alpha < 1$, $a = 1/2(1-\alpha)$ and $q = \alpha p$, see A .Meir [9].

Therefore we get from (1.1) Lebesgue constants for Euler-transformation $L_{(E,\alpha)}$ as follows:

$$L_{(E,\alpha)} = L_F(1/2(1-\alpha), \alpha p)$$

$$= \frac{2}{\pi^2} \log \frac{2\alpha p}{1-\alpha} + A + O(\log p/\sqrt{p}).$$

L. Lorch has obtained Lebesgue constants for $(E, \frac{1}{2})$ in [7] and has shown that the remainder term is $O(1/\sqrt{p})$.

iv) Taylor-transformation. See K. Ishiguro [2].

The summability matrix is defined by

$$c_{pk} = \begin{cases} 0 & (0 \leqq k \leqq p-1) \\ r^{p+1}\binom{k}{p}(1-r)^{k-p} & (p \leqq k) \end{cases}$$

where $0 < r < 1$, $a = r/2(1-r)$ and $q = p/r$, see A. Meir [9].

Therefore we get from (1.1) Lebesgue constants for Taylor-transformation $L_{(T,r)}$ as follows:

$$L_{(T,r)} = L_F(r/2(1-r), p/r)$$

$$= \frac{2}{\pi^2} \log \frac{2p}{1-r} + A + O(\log p/\sqrt{p}).$$

v) S_α-transformation. See K. Ishiguro [3].

The summability matrix of this transformation is defined by

$$c_{pk} = (1-\alpha)^{p+1}\binom{p+k}{k}\alpha^k \qquad (k = 0, 1, 2, \cdots, p = 1, 2, \cdots)$$

where $0 < \alpha < 1$, $a = (1-\alpha)/2$ and $q = \alpha p/(1-\alpha)$, see A. Meir [9].

Therefore we get from (1.1) Lebesgue constants for S_α-transformation $L_{(s,\alpha)}$ as follows:

$$L_{(s,\alpha)} = L_F((1-\alpha)/2, \alpha p/(1-\alpha))$$

$$= \frac{2}{\pi^2} \log 2\alpha p + A + O(\log p/\sqrt{p}).$$

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