A THEOREM ON REGULAR VECTOR FIELDS AND ITS APPLICATIONS TO ALMOST CONTACT STRUCTURES

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Introduction. In the paper [1], Boothby-Wang dealt with the period function λ of the associated vector field of the regular contact form on a compact contact manifold and proved that λ is differentiable and constant ([6]).

In this note we prove a theorem on the proper and regular vector field, by this we can give a simple proof to one of their result. Moreover, as a natural consequence, this procedure enables us to generalize Morimoto's theorem (Theorem 5, [4]), concerning the period function on a normal almost contact manifold.

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1. **Regular vector fields**. Let M be a connected differentiable manifold and X be a differentiable vector field on M such that X does not vanish everywhere. We assume that the distribution defined by X is regular and X is proper, i.e., X generates the global 1-parameter group $\exp tX(-\infty<t<\infty)$ of transformations of M. For the terminologies we refer to [5]. We can find always a 1-form w satisfying $w(X)=1$. Now the next assumption is that there exists a 1-form w such that $w(X) = 1$ and $L(X)w = i(X)\,dw = 0$, where $L(X)$ or $i(X)$ is the operator of the Lie derivative or interior product operator by X.

First we see that the quotient space M/X is a Hausdorff space, because X is proper and regular. Hence by Palais' theorem ([5], Chap. I, § 5), M/X is a differentiable manifold and the projection $\pi: M \rightarrow M/X$ is a differentiable map.

Let h be an arbitrary Riemannian metric in M/X. The tensor g in M defined by $g = \pi^*h + w\otimes w$ is easily seen to be a Riemannian metric in M, π^* and $\otimes$ denoting the dual of π and tensor product respectively. Clearly we have $g(X, X) = 1$, and we see that the relation $L(X)g = 0$ holds good. Namely X is a unit and Killing vector field with respect to g. Thus each trajectory of X is a geodesic and the parameter t in $\exp tX$ is nothing but the arc length of it.

Suppose that there is a point p and a positive number λ such that

$\exp \lambda X \cdot p = p$, and $\exp tX \cdot p \neq p$ for $0 < t < \lambda$. Denote by $l(q)$ the leaf passing through a point q of M and let U be a sufficiently small coordinate neighborhood of p which is regular with respect to X. If we take an arbitrary point x in U such that x does not belong to $l(p)$, then we can draw the shortest geodesic $c(x)$ from x to $l(p)$. Clearly at the intersecting point $\bar{x}$ of $c(x)$ and $l(p)$, both geodesics are orthogonal.

Now as $\exp \lambda X$ is an isometry, the image of the geodesic $c(x)$ is the geodesic passing through $\bar{x}$. On the other hand, by the regularity, for any point q of $c(x)$ $\exp \lambda X \cdot q$ belongs to $l(q)$. Hence we have $\exp \lambda X \cdot x = x$, and so the period function λ is constant on U, and on M as M is connected.

If there exists a point p such that $\exp tX \cdot p \neq p$ for any t, then this is the same for all point in M.

THEOREM. *For a proper and regular vector field X on M, the following three conditions are equivalent.*

(i) *The period function λ of X is constant (finite or infinite).*
(ii) *There exists a 1-form w such that $w(X)=1$ and $L(X)w = 0$.*
(iii) *There exists a Riemannian metric g such that $g(X,X) = 1$ and $L(X)g = 0$.*

Since we have proved (ii) $\to$ (iii) $\to$ (i), the next process is (i) $\to$ (ii). By virtue of (i), M can be considered as a principal fibre bundle whose structural group is a toroidal group S^1 or a real additive group R. If we take an infinitesimal connection w on M such that X is a fundamental vector field, then (ii) holds goods.

2. Applications. We denote by ϕ, ξ and η the structure tensors of an almost contact structure. As the first application we have the following (see [4])

THEOREM A. *Let M be an almost contact manifold such that $L(\xi)\eta = 0$ and ξ is a proper and regular vector field.*

(i) *If $\exp t\xi \cdot p \neq p$ for some point p in M and for any t, then M is a principal fibre bundle with the group R.*
(ii) *If we have $\exp \lambda\xi \cdot p = p$ for some point p in M and a real number λ, then M is a principal fibre bundle with the group S^1.*

In both cases, η defines an infinitesimal connection on M.

If ξ is an associated vector field of the contact form η, we have $L(\xi)\eta = 0$, consequently we have

COROLLARY 1. *In a contact manifold with the contact from η, if an associated vector field is proper and regular, then M is a principal fibre bundle with the group R or S^1 according as* (i) *or* (ii) *in Theorem A is satisfied. On M the contact form defines an infinitesimal connection and M/ξ is a symplectic manifold.*

In this Corollary, as $L(\xi)d\eta=0$, the symplectic structure W on M/ξ is defined by the relation $\pi^*W = d\eta$.

If the manifold is compact, every vector field is proper. Thus we get the following (see [1])

COROLLARY 2. *Let M be a compact contact manifold with a regular contact form η. Then M is a principal S^1-bundle over the symplectic manifold M/ξ with η as a connection form of an infinitessimal connection.*

THEOREM B. *In the contact manifold, if the assoiated vector field ξ is proper and regular. Then we can find an almost contact metric structure (ϕ, ξ, η, g) associated to the contact form η such that $L(\xi)\phi = 0$, equivalently ξ is a Killing vector field.*

PROOF. Let $\pi: M \to M/\xi$ be a fibering given in Corollary 1 and let W be the symplectic structure on M/ξ such that $d\eta=\pi^*W$. Further we can define an almost kählerian structure F and the metric h on M/ξ satisfying $W(u, v)=h(u, Fv)$ and $h(Fu, Fv)=h(u, v)$ for any vector fields u, v on M/ξ ([2]). Therefore by the result in [3] M has (ϕ, ξ, η, g)-structure associated to η such that $L(\xi)\phi=0=L(\xi)g$ and $g=\pi^*h+\eta\otimes\eta$. Therefore M is a K-contact manifold.

For brevity, we say that an almost contact structure (ϕ, ξ, η) is of type P if it is constructed in a principal fibre bundle M with the structural group S^1 or R and with an almost complex manifold B as its base, using an almost complex structure of B and an infinitesimal connection η as in [3].

THEOREM C. *Suppose that in an almost contact manifold ξ is proper and regular, then $L(\xi)\phi=0$ if and only if the almost contact structure is of type P.*

PROOF. From $L(\xi)\phi = 0$ it follows that $L(\xi)\eta = 0$. So, M is a principal fibre bundle with the group S^1 or R and on M η defines an infinitesimal connection. We denote by u^* the horizontal lift of a vector field u on M/ξ with respect to η. It is natural to define an almcst complex structure F on M/ξ by

$$Fu = \pi\phi u^* ,$$

where ϕu^* is considered at the point q containd in the leaf over the origin of u, and $\pi\phi u^*$ does not depend upon the choice of the point q in the leaf as $L(\xi)\phi=0$. Then Theorem C follows immediately.

REMARK. The tensor $N_2=(N^i_j)$ is defined ([7]) by $N_2(X)=L(\xi)\phi\cdot X$ for a vector field X. Theorem C gives a geometrical meaning of $N_2=0$ in the case when ξ is proper and regular. In contact manifold $N_2=0$ is equivalent to the fact that ξ is a Killing vector field.

COROLLARY 3. *Let M be an almost contact manifold such that ξ is proper and regular. If $L(\xi)\phi=0$, then we can find an associated Riemannian metric to the almost contact structure so that ξ is a Killing vector field.*

This follows from Theorem C and the similar argument in the proof of Theorem B.

BIBLIOGRAPHY

[1] W. M. BOOTHBY AND H. C. WANG, On contact manifolds, Ann. of Math., 68(1958), 721-734.

[2] Y. HATAKEYAMA, On the existence of Riemann metrics associated with a 2-form of rank 2r, Tôhoku Math. Journ., 14(1962), 162-166.

[3] ——————, Some notes on differentiable manifolds with almost contact structures, Tôhoku Math. Journ., 15(1963), 176-181.

[4] A. MORIMOTO, On normal almost contact structures with a regularity, Tôhoku Math. Journ., 16(1964), 90-104.

[5] R. S. PALAIS, A global formulation of the Lie theory of transformation groups, Mem. of Amer. Math. Soc., 22(1957).

[6] MATHEMATICAL REVIEWS, 1961, # 3015.

[7] S. SASAKI AND Y. HATAKEYAMA, On differentiable manifolds with certain structures which are closely related to almost contact structure, II, Tôhoku Math. Journ., 13 (1961), 281-294.

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