MANIFOLDS WITH CONSTANT p-SECTIONAL CURVATURE

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Recently Thorpe has made clear the relationship between the p-sectional curvature and the Pontrjagin classes ([1]). He proved that if the p-sectional curvature is constant, then some of the Pontrjagin classes vanish. Making use of this fact we shall investigate in this note some topological properties of compact orientable Riemannian manifolds.

1. In this note we denote by X_n a compact orientable n-dimensional Riemannian manifold and by p_k the Pontrjagin class of dimension $4k$.

The following theorem is basic for our purpose:

THEOREM 1 (*Thorpe* [1]). *If the p-sectional curvature of* X_n *is constant, then* $p_k = 0,\ k \geqq p/2$, *where p denotes an even integer.*

First we consider the case where $n = 4m$ and $p = 4$. In this case we have from the above theorem

$$p_k = 0, \quad k \geqq 2.$$

Hence we have

$$(1.\ 1) \qquad \tau = \left[\frac{\sqrt{p_1}}{\operatorname{tgh}\sqrt{p_1}}\right][X_{4m}] = \left[1 + \sum_{k\geqq 1}(-1)^{k-1}\frac{2^{2k}}{(2k)!}B_k p_1^k\right][X_{4m}] \qquad ([2])$$

$$= (-1)^{m-1}\frac{2^m}{(2m)!}B_m p_1^m\,[X_{4m}]$$

and

$$(1.\ 2) \quad A = \left[\frac{2\sqrt{p_1}}{\sinh 2\sqrt{p_1}}\right][X_{4m}] = \left[-1 + 2\sum_{k\geqq 1}(-1)^k\frac{2^{2k}(2^{2k-1}-1)}{(2k)!}B_k p_1^k\right][X_{4m}]$$

$$= (-1)^{m+1}\frac{2^{2m+1}(2^{2m-1}-1)}{(2m)!}B_m p_1^m\,[X_{4m}]$$

where τ or A denotes the index or the A-genus of X_{4m} respectively and B_k's denote the Bernoulli numbers. We have from (1.1) and (1.2)

$$A = -2(2^{2m-1} - 1)\tau . \tag{1. 3}$$

We recall the

THEOREM 2 (*Atiyah and Hirzebruch* [5]) *If a compact orientable differentiable* X_{4m} *is differentiably imbedded in the* $(8m-2q)$*-dimensional sphere, then* $A(X_{4m}) \equiv 0 \bmod 2^{q+1}$. *If moreover* $q \equiv 2 \bmod 4$, *then* $A(X_{4m}) \equiv 0 \bmod 2^{q+2}$.

We have from (1.3) and the above theorem the

THEOREM 3. *If* X_{4m} *is of constant* 4*-sectional curvature and is differentiably imbedded in the* E_{8m-2q}, *where* E_n *denotes the euclidean space of dimension* n, *then* $\tau \equiv 0 \bmod 2^q$. *If moreover* $q \equiv 2 \bmod 4$, *then* $\tau \equiv 0 \bmod 2^{q+1}$.

Since the index is a cobordism invariant, the above theorem holds for any manifold which is cobordant to such a manifold. We have from (1.3) and the Theorem 2 of [6] the

THEOREM 4. *If* X_{4m} *is of constant* 4*-sectional curvature and its index is odd, then such an* X_{4m} *is indecomposable.*

Furthermore we have the

THEOREM 5. *If* X_{4m} *is of constant* 4*-sectional curvature and differentiably imbedded in the* E_{6m}, *then* $2X_{4m}$ *is "bord".*

PROOF. In this case we have

$$\bar{p}_m = 0 \tag{1. 4}$$

where $\bar{p}_k$ denotes the dual Pontrjagin class of dimension $4k$, i.e.

$$\Big(1 + \sum_{k \geqq 1} p_k\Big)\Big(1 + \sum_{j \geqq 1} \bar{p}_j\Big) = 1 . \tag{1. 5}$$

Meanwhile we have from the first assumption and (1.5)

(1. 6) $$\bar{p}_m = (-1)^m p_1{}^m .$$

Hence we have

(1. 7) $$p_1^m = 0 ,$$

i.e. all the Pontrjagin numbers vanish, which leads to the conclusion. ([3])
Q.E.D.

A similar fact is the following

THEOREM 6. *If X_{4m} is of constant $2m$-sectional curvature and is differentiably imbedded in the E_{4m+4}, then $2X_{4m}$ is "bord".*

PROOF. In this case we have

(1. 8) $$p_m = 0$$

and

(1. 9) $$\bar{p}_k = 0 \quad k \geqq 2 .$$

Hence we have from (1.5)

(1.10) $$p_m = (-1)^m \bar{p}_1^m$$

which leads to $\bar{p}_1^m = 0$. Therefore we see from (1.5) that all the Pontrjagin numbers vanish. Q.E.D.

2. It is clear from (1.1) and (1.2) that

THEOREM. 7. *If X_{4m} is of constant 4-sectional curvature and either its index or its A-genus is zero, then all Pontrjagin numbers of X_{4m} become zero and hence $2X_{4m}$ is "bord".*

Next we consider the manifolds which do not admit the constant p-sectional curvature. For example we consider the complex projective space $P_n(c)$ whose Pontrjagin classes are given by

(2. 1) $$p = (1 + g^2)^{n+1} , \quad g \in H^2(P_n(c), Z) .$$

Hence any q-sectional curvature of $P_n(c)$ $(2 \leqq q \leqq n)$ cannot be constant. Precisely speaking we have the

THEOREM 8. *The complex projective space $P_n(c)$ doesn't admit any Riemann metric on which some q-sectional curvature* $(2 \leqq q \leqq n)$ *becomes constant.*

The similar facts are found in the cases of the quaternion projective spaces etc..

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