

# ON GENERALIZED CESÀRO MEANS OF INTEGRAL ORDER

DENNIS C. RUSSELL.<sup>1)</sup>

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**1. Introduction.** It is the object of this paper to consider some of the properties of a  $\lambda$ -type generalization  $(C, \lambda, \kappa)$  of Cesàro summability, which reduces to  $(C, \kappa)$  when  $\lambda_n = n$ . We shall be concerned mainly with the relations between  $(C, \lambda, \kappa)$  and other summability methods, notably the Riesz method  $(R, \lambda, \kappa)$  and a more general method  $(G, \lambda)$  defined by means of a function  $g$ . Except in this introductory section, we shall deal almost entirely with methods of integral order (we draw attention to this by writing  $p$  in place of  $\kappa$ ), and we suppose throughout that  $\lambda = \{\lambda_n\}$  is a sequence satisfying

$$0 \leq \lambda_0 < \lambda_1 < \dots < \lambda_n \rightarrow \infty.$$

Given any series<sup>2)</sup>  $\sum a_v$ , and any  $\kappa \geq 0$ , denote

$$A^\kappa(\omega) = \sum_{\lambda_v < \omega} (\omega - \lambda_v)^\kappa a_v;$$

if  $\omega^{-\kappa} A^\kappa(\omega) \rightarrow s$  as  $\omega \rightarrow +\infty$

then we say that  $\sum a_v$  is Riesz summable  $(R, \lambda, \kappa)$  to  $s$ . When  $\omega \rightarrow \infty$  through the sequence  $\{\lambda_n\}$ , we obtain the definition of 'discontinuous' Riesz summability  $(R^*, \lambda, \kappa)$ , and we may then relax the restriction on  $\kappa$  to  $\kappa > -1$ ; thus  $\sum a_v$  is summable  $(R^*, \lambda, \kappa)$  to  $s$  if  $\lambda_n^{-\kappa} A^\kappa(\lambda_n) \rightarrow s$ .

It is of course trivial that<sup>3)</sup>, for any  $\{\lambda_n\}$  and any  $\kappa \geq 0$ ,

$$(R, \lambda, \kappa) \subseteq (R^*, \lambda, \kappa).$$

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- 1) This paper was written while the author was a Fellow at the Summer Research Institute of the Canadian Mathematical Congress, Vancouver, 1965.
  - 2) Unless otherwise specified, limits of summation or integration are assumed throughout to be 0,  $\infty$ .
  - 3) Given two summability methods  $A, B$ , we say that  $A$  is included in  $B$  (written  $A \subseteq B$ ) if every series summable- $A$  is also summable- $B$  (to the same value);  $A$  and  $B$  are equivalent (written  $A \sim B$ ) if each includes the other.

The converse inclusion, that

$$(R^*, \lambda, \kappa) \subseteq (R, \lambda, \kappa),$$

is trivial for  $\kappa=0$  and for  $\kappa=1$  (in the case  $\kappa=1$  this follows since  $A^1(\omega)$  is linear in  $\lambda_n \leq \omega \leq \lambda_{n+1}$ ), and has been proved by Jurkat [7] to hold for  $0 < \kappa < 1$ , without restriction on  $\{\lambda_n\}$ . Results for  $\kappa > 1$  have been obtained by Kuttner [11]–[15] and Peyerimhoff [27], and although the problem has not yet been completely disposed of, certain restrictions on  $\{\lambda_n\}$  have been shown to be either necessary or sufficient for the inclusion to hold when  $\kappa > 1$ .

In the special case  $\lambda_n = n$ , it is well-known (see, for example, the references given by Kuttner [11]), that  $(R, \lambda, \kappa)$  is equivalent to Cesàro summability  $(C, \kappa)$ , for any  $\kappa \geq 0$ ; that is,

$$(R, n, \kappa) \sim (C, \kappa).$$

Riesz [22] has shown that the equivalence

$$(R^*, n, \kappa) \sim (C, \kappa)$$

holds when  $-1 < \kappa < 1$ ; Kuttner [11] has extended this to  $-1 < \kappa < 2$ , and has shown also that equivalence fails for  $\kappa \geq 2$ .

In problems (particularly on inclusion relations or summability factors) involving the ‘continuous’ Riesz method  $(R, n, \kappa)$ , the equivalence with the Cesàro method  $(C, \kappa)$ , which has a discrete matrix with an easily calculated inverse, often enables a treatment to be simplified by using  $(C, \kappa)$ -means. A corresponding simplification would occur in problems on the general Riesz method  $(R, \lambda, \kappa)$  if we could obtain a generalized Cesàro method  $(C, \lambda, \kappa)$ , which would reduce to the  $(C, \kappa)$  method for  $\lambda_n = n$ , and for which

$$(R, \lambda, \kappa) \sim (C, \lambda, \kappa).$$

Such a method  $(C, \lambda, \kappa)$  has been defined by Jurkat [8]; in his definition,  $(C, \lambda, \kappa)$  coincides with  $(C, \kappa)$  when  $\lambda_n = n$  and  $\kappa$  is a non-negative integer, and is equivalent to (but does not coincide with)  $(C, \kappa)$  when  $\lambda_n = n$  and  $\kappa$  is non-integral. An almost identical definition<sup>4)</sup> of  $(C, \lambda, \kappa)$  has been given, for integral  $\kappa$  only, by Burkill [3]; this is equivalent to Jurkat’s method, for any  $\{\lambda_n\}$ , and also coincides with  $(C, \kappa)$  when  $\lambda_n = n$ . Both Jurkat and Burkill

4) See §3, where this definition is given. The two definitions coincide when  $\lambda_0 = 0$ ; but a difference in  $\lambda_0$  (or in any finite number of the  $\lambda_n$ ) cannot affect the summability properties of the method.

obtained different sufficient conditions (in the form of restrictions on  $\{\lambda_n\}$ ) in order that  $(R, \lambda, \kappa)$  should be equivalent to  $(C, \lambda, \kappa)$ , but Jurkat imposed additional restrictions on  $\{\lambda_n\}$  in the case of non-integral  $\kappa$  and I propose to deal in this paper with an attempt to lighten the restrictions only in the integral case; it may be that an alternative definition of  $(C, \lambda, \kappa)$  would be desirable for non-integral  $\kappa$ . I shall deal separately (§§4, 5) with the inclusions

$$(C, \lambda, \kappa) \subseteq (R, \lambda, \kappa)$$

and

$$(R, \lambda, \kappa) \subseteq (C, \lambda, \kappa),$$

where  $\kappa$  is a non-negative integer, showing that the first of these is true without restriction on  $\{\lambda_n\}$ , and that the second is true (i) when  $\kappa=0, 1, 2$ , without restriction on  $\{\lambda_n\}$ , and (ii) when  $\kappa \geq 3$ , under a restriction on  $\{\lambda_n\}$  which is weaker than either Jurkat's or Burkill's. It will be useful for our purpose (and also of independent interest) to examine (§3) the relation between  $(C, \lambda, \kappa)$ -means of different integral orders  $\kappa$ , mainly in the form of limitation theorems, though it follows almost at once that

$$(C, \lambda, \kappa_1) \subseteq (C, \lambda, \kappa_2), \quad 0 \leq \kappa_1 \leq \kappa_2.$$

Though we shall not be concerned here with  $(C, \lambda, \kappa)$ -means of negative order, it should be noted that Maddox [19] has given a definition of  $(C, \lambda, -1)$  summability (which coincides with the definition of  $(C, -1)$  summability when  $\lambda_n = n$ ) and has established inclusion and summability factor properties of the method. Some related methods are discussed in [26].

The problem of finding necessary and/or sufficient conditions in order that a general summability method  $A$  should satisfy

$$(R, \lambda, \kappa) \subseteq A$$

has been considered by Maddox [18]. With  $(R, \lambda, \kappa)$  replaced by  $(C, \lambda, \kappa)$ , Jurkat [8] had previously given a result in this direction in the case where  $\kappa$  is an integer and  $A$  is a normal method (i.e. its matrix is triangular with non-zero diagonal elements). Kuttner [10] has considered the problem when  $A$  is a generalized Abel method  $(A, \lambda, \kappa')$ . In [23] I have given necessary and sufficient conditions in order that  $B \subseteq A$ , where  $B$  is a normal method satisfying a certain 'mean-value theorem' introduced by Jurkat and Peyerimhoff [9]; and in particular,  $B$  can be taken to be  $(R^*, \lambda, \kappa)$ ,  $0 < \kappa \leq 1$  (which is equivalent to  $(R, \lambda, \kappa)$  in this range of value of  $\kappa$ ). I have dealt [24, 25] with the case where  $A$  is a generalized Riemann method  $(\mathfrak{R}, \lambda, \mu)$  (Burkill and Petersen [4]

and Burkill [3] had considered this with  $\kappa=1$ ,  $\kappa$  an integer, respectively), and also [25] with sufficient conditions when  $A$  is a method  $(G, \lambda)$  defined as follows:

$\sum a_v$  is summable- $(G, \lambda)$  to  $s$  if

$$\sum_v g(\lambda_v, h) a_v \rightarrow s \quad \text{as } h \rightarrow 0+,$$

where  $g$  is a function having certain properties which will be specified later (the Riesz, Riemann, Abel methods are special cases). In this last case, the question of finding easily applicable necessary conditions for inclusion appears to be more difficult, especially for non-integral  $\kappa$ ; however, when  $\kappa$  is an integer we can use the relation between  $(R, \lambda, \kappa)$  and  $(C, \lambda, \kappa)$  given in this paper, and hence examine necessary conditions in order that

$$(C, \lambda, \kappa) \subseteq (G, \lambda), \quad \kappa \text{ an integer};$$

a result of this form is given in §6.

It will be apparent from the above discussion (and also from comments of Kuttner [12, 13] and Maddox [16]) that the significance of many of these results depends upon the extent to which the restrictions on  $\{\lambda_n\}$  can be lightened. Since we refer later to a number of different such restrictions, it will be useful to state quite clearly which of them have relations of implication between them, and which of them are mutually independent; this is done in the following section.

**2. Relations between Different Conditions on  $\lambda$ .** The following conditions on  $\lambda = \{\lambda_n\}$ , which is assumed always to be a sequence of non-negative numbers strictly increasing to  $\infty$ , are among those which have occurred from time to time in work on Riesz means; most of them are referred to later in this paper. (1) and (5) appear to have been first used by Jurkat [6], (2) by Kuttner [12], (4) by Burkill and Petersen [4], (9) (with  $\lambda_n$  on the right in place of  $\lambda_{n+1}$ ) by Russell [25] and Rangachari [21]; condition (8), which appears for the first time in this paper, was suggested to me by Professor D. Borwein in place of a more restrictive condition, similar to (7a), which I had assumed at first.

In the usual notation, we write

$$\Delta b_n = b_n - b_{n+1}, \quad \Lambda_n = \lambda_{n+1} / (\lambda_{n+1} - \lambda_n),$$

and  $\nearrow$  or  $\searrow$  for monotonicity (in the wide sense). Note that we always have

$$\Delta\lambda_n < 0, \quad \lambda_{n+1}/\lambda_n > 1, \quad \Lambda_n > 1.$$

$$(1) \quad (a) \quad 0 < a \leq \frac{\Delta\lambda_n}{\Delta\lambda_{n-1}} \quad (b) \quad \frac{\Delta\lambda_n}{\Delta\lambda_{n-1}} \leq b < \infty.$$

$$(2) \quad \Lambda_{n-1} = O(\Lambda_n).$$

$$(3) \quad \frac{\lambda_{n+1}}{\lambda_n} = O(1).$$

$$(4) \quad (a) \quad 0 < a' \leq |\Delta\lambda_n| \quad (b) \quad |\Delta\lambda_n| \leq b' < \infty.$$

$$(5) \quad (a) \quad \Lambda_n \nearrow, \text{ equivalent to } (b) \quad \frac{\lambda_{n+1}}{\lambda_n} \searrow.$$

$$(6) \quad (a) \quad \Lambda_n = O(1), \text{ equivalent to } (b) \quad \liminf \frac{\lambda_{n+1}}{\lambda_n} > 1.$$

$$(7) \quad (a) \quad \lim \frac{1}{\lambda_n} \max_{1 \leq \nu \leq n} |\Delta\lambda_{\nu-1}| = 0, \text{ equivalent to } (b) \quad \lim \frac{\lambda_{n+1}}{\lambda_n} = 1.$$

$$(8) \quad \liminf \frac{1}{\lambda_n} \max_{1 \leq \nu \leq n} |\Delta\lambda_{\nu-1}| = 0.$$

$$(9) \quad \Lambda_n^p = O(\lambda_{n+1}^\mu)$$

for some pair of numbers  $\mu, p$  with  $\mu \geq p > 0$ .

For the sake of clarity, we omit the parentheses in referring to conditions 1 to 9, and use the logical symbols:

$\rightarrow$  (implies),  $\wedge$  (and),  $\vee$  (inclusive or),  $\sim$  (negation).

There are 110 possible relations of implication between the 11 conditions on  $\lambda$  which are listed (counting  $1a, 1b, 4a, 4b$  separately). 15 of these are true — that is, of the form  $p \rightarrow q$  where any sequence  $\lambda$  satisfying  $p$  must necessarily also satisfy  $q$ ; however, most of these can be verified immediately, and it seems enough to mention in more detail only the following:

*Relations between 1, 2, 3.* Note that

$$\frac{\Delta\lambda_n}{\Delta\lambda_{n-1}} = \frac{\Lambda_{n-1}}{\Lambda_n} \cdot \frac{\lambda_{n+1}}{\lambda_n} > \frac{\Lambda_{n-1}}{\Lambda_n} > \frac{1}{\Lambda_n};$$

then the equation shows that  $2 \wedge 3 \rightarrow 1b$ , the central inequality that  $1b \rightarrow 2$ , and the inequality between the extremes that  $6 \rightarrow 1a$ . Also, if  $1b$  then

$$\lambda_{n+1} - \lambda_n \leq b(\lambda_n - \lambda_{n-1}) < b\lambda_n, \quad \text{whence } \lambda_{n+1}/\lambda_n < b+1,$$

so that  $1b \rightarrow 3$  (and since  $1b \rightarrow 2$  it follows that  $1b \rightarrow 2 \wedge 3$ ).

*Relation between 6 and 8.* If 6 then  $\lambda_{n+1}/\lambda_n \geq c > 1$  for every  $n$ , and then

$$\frac{1}{\lambda_n} \max_{1 \leq \nu \leq n} |\Delta \lambda_{\nu-1}| \geq \frac{|\Delta \lambda_{n-1}|}{\lambda_n} = 1 - \frac{\lambda_{n-1}}{\lambda_n} \geq 1 - \frac{1}{c} > 0;$$

hence  $6 \rightarrow \tilde{8}$  (and  $8 \rightarrow \tilde{6}$ ).

*Equivalence of 7a and 7b.* Suppose 7b. Then, by the first of the inequalities just employed above,

$$0 < 1 - \frac{\lambda_{n-1}}{\lambda_n} \leq \frac{1}{\lambda_n} \max_{1 \leq \nu \leq n} |\Delta \lambda_{\nu-1}| \rightarrow 0,$$

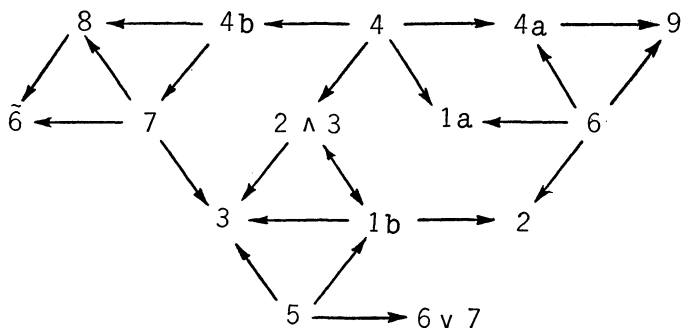
so that  $7b \rightarrow 7a$ . Suppose 7a. Now  $\max_{1 \leq \nu \leq n} |\Delta \lambda_{\nu-1}| = |\Delta \lambda_{\nu_n-1}|$ , where  $\{\nu_n\}$ , which is clearly non-decreasing or can be chosen to be so, satisfies  $1 \leq \nu_n \leq n$ . Either  $\{\nu_n\}$  is bounded, in which case  $|\Delta \lambda_{\nu_n-1}|$  is bounded and hence  $\lambda_n^{-1} |\Delta \lambda_{\nu_n-1}| \rightarrow 0$ ; or else  $\{\nu_n\}$  is unbounded, in which case (since it is non-decreasing) it tends to  $+\infty$ , and then

$$\lambda_n^{-1} |\Delta \lambda_{\nu_n-1}| \leq \lambda_{\nu_n}^{-1} |\Delta \lambda_{\nu_n-1}| = 1 - (\lambda_{\nu_n-1}/\lambda_{\nu_n}) \rightarrow 1 - 1 = 0;$$

hence  $7a \rightarrow 7b$ .

*Condition 9.* If  $p=0$  then 9 always holds, trivially, for any  $\mu \geq 0$ . Also if 9 holds for some pair  $\mu, p$  then, since  $\lambda_n \nearrow$ , it clearly holds for any pair  $\mu', p$  with  $\mu' > \mu$ . It is obvious that, since  $\Lambda_n = \lambda_{n+1}/|\Delta \lambda_n|$ ,  $4a \rightarrow 9$  for  $\mu \geq p$ ; but note that we do not require 9 to hold for every pair  $\mu, p$  with  $\mu \geq p > 0$  — for, if we did, then in particular it would have to hold for  $\mu = p$ , and 9 would then be equivalent to the simple condition 4a.

Each of the 15 valid implications (together with those arising from the additional results  $2 \wedge 3 \rightarrow 1b$ ,  $4 \rightarrow 1$ ,  $5 \rightarrow 6 \vee 7$ ,  $8 \rightarrow \tilde{6}$ , where 4 means  $4a \wedge 4b$ , and 1 means  $1a \wedge 1b$ ) can be traced out on the following diagram:



The other 95 of the 110 possible implications are false, that is they are of the form  $\sim(p \rightarrow q)$ ; in other words, there exists a sequence  $\lambda$  satisfying  $p \wedge \tilde{q}$ . 95 counter-examples of such sequences  $\lambda$  can be selected from the illustrations which follow ( $\lambda_n = n$  is not needed for this purpose, but is included since it is an important special case); for example, to show that 2 and 8 are independent of each other, we note that  $2 \wedge \tilde{8}$  is satisfied by (12), or by (13), and  $\tilde{2} \wedge 8$  by (15), (16), (18). In the first six of the illustrations,  $\{\lambda_n\}$  is defined and its behaviour relative to the conditions 1 to 9 stated concisely; the verifications are left to the reader. In the other three illustrations some of the salient features are briefly indicated.

(10)  $\lambda_n = n$ :  $1 \wedge 2 \wedge 3 \wedge 4 \wedge 5 \wedge \tilde{6} \wedge 7 \wedge 8 \wedge 9$ .

(11)  $\lambda_n = \log(n+1)$ :  $1 \wedge 2 \wedge 3 \wedge \tilde{4a} \wedge 4b \wedge 5 \wedge \tilde{6} \wedge 7 \wedge 8 \wedge \tilde{9}$ .

(12)  $\lambda_n = 2^n$ :  $1 \wedge 2 \wedge 3 \wedge 4a \wedge \tilde{4b} \wedge 5 \wedge 6 \wedge \tilde{7} \wedge \tilde{8} \wedge 9$ .

(13)  $\lambda_n = 2^{n^2}$ :  $1a \wedge \tilde{1b} \wedge 2 \wedge \tilde{3} \wedge 4a \wedge \tilde{4b} \wedge \tilde{5} \wedge 6 \wedge \tilde{7} \wedge \tilde{8} \wedge 9$ .

(14)  $\lambda_{2n} = n, \lambda_{2n+1} = n + \theta_n$ , where  $0 < c_1 \leq \theta_n \leq c_2 < \frac{1}{2}$ :

$$1 \wedge 2 \wedge 3 \wedge 4 \wedge \tilde{5} \wedge \tilde{6} \wedge 7 \wedge 8 \wedge 9.$$

(15)  $\lambda_{2n} = n, \lambda_{2n+1} = n + \theta_n$ , where  $0 < \theta_n \nearrow 1$  (or  $\theta_n \rightarrow 0$ ):

$$\tilde{1a} \wedge \tilde{1b} \wedge \tilde{2} \wedge 3 \wedge \tilde{4a} \wedge 4b \wedge \tilde{5} \wedge \tilde{6} \wedge 7 \wedge 8.$$

The truth of 9 in this case depends on  $\theta_n$  and on the relative values of  $\mu$  and  $p$ ; thus in the respective cases

$$(\theta_n)^{-1} = \text{(i) } \log(n+3), \quad \text{(ii) } n+2, \quad \text{(iii) } 2^{n+1},$$

condition 9 is:   
 (i) true if  $\mu > p > 0$ ,   
 (ii) true if  $\mu \geq 2p > 0$ ,   
 (iii) false for every positive  $\mu$  and  $p$ .

$$(16) \quad \lambda_\nu = k_i + \nu \quad \text{for } n_i \leq \nu < n_{i+1} \quad (i = 0, 1, 2, \dots),$$

where  $\{n_i\}$  is an increasing sequence of positive integers,  $k_{i+1} - k_i$  is positive and increasing, and  $n_i/k_i \rightarrow 0, n_{i+1}/k_i \rightarrow +\infty$ . It follows from these conditions that  $k_i/k_{i+1} \rightarrow 0, k_{i+1} - k_i \rightarrow +\infty, n_i/n_{i+1} \rightarrow 0$  (a suitable choice for  $\{n_i\}, \{k_i\}$  would be  $n_i = 2^{i^2}, k_i = i \cdot 2^{i^2}$ ). Then  $\lambda_\nu - \lambda_{\nu-1} = 1 (\nu \neq n_i), \lambda_{n_i} - \lambda_{n_i-1} = k_i - k_{i-1} + 1$ , so that 4a (hence 9),  $\tilde{4}b, \tilde{1}a$  (hence  $\tilde{6}$ ),  $\tilde{1}b$  (hence  $\tilde{5}$ ). Now if we let  $n \rightarrow \infty$  through the values  $n = n_{i+1} - 1$ , we have:

$$(i) \quad \frac{\lambda_n}{\lambda_{n+1}} = \frac{k_i + n_{i+1} - 1}{k_{i+1} + n_{i+1}} < \frac{k_i + n_{i+1}}{k_{i+1}} \rightarrow 0;$$

thus  $\tilde{3}$  (hence  $\tilde{1}b, \tilde{4}b, \tilde{5}, \tilde{7}$ ).

$$(ii) \quad \Lambda_n = \frac{k_{i+1} + n_{i+1}}{k_{i+1} - k_i + 1} \rightarrow 1, \quad \Lambda_{n-1} = k_i + n_{i+1} - 1 \rightarrow +\infty;$$

thus  $\tilde{2}$  (hence  $\tilde{1}b, \tilde{5}, \tilde{6}$ ).

$$(iii) \quad \frac{1}{\lambda_n} \max_{1 \leq \nu \leq n} (\lambda_\nu - \lambda_{\nu-1}) = \frac{k_i - k_{i-1} + 1}{k_i + n_{i+1} - 1} \leq \frac{k_i}{n_{i+1}} \rightarrow 0;$$

thus 8 (hence  $\tilde{6}$ ).

Combining the results, we see that  $\{\lambda_n\}$  satisfies

$$\tilde{1}a \wedge \tilde{1}b \wedge \tilde{2} \wedge \tilde{3} \wedge 4a \wedge \tilde{4}b \wedge \tilde{5} \wedge \tilde{6} \wedge \tilde{7} \wedge 8 \wedge 9.$$

$$(17) \quad \lambda_\nu = i + (\nu - n_i)/(n_{i+1} - n_i) \quad \text{for } n_i \leq \nu \leq n_{i+1} \quad (i = 0, 1, 2, \dots),$$

where  $n_{i+1} - n_i$  is positive and increasing, with  $n_{i+1}/n_i \rightarrow +\infty$  (for example,  $n_i = 2^{i^2}$ ). Here

$$\lambda_{\nu+1} - \lambda_\nu = 1/(n_{i+1} - n_i) \quad (n_i \leq \nu < n_{i+1}),$$

so that  $\tilde{4}a$  (hence  $\tilde{6}$ ),  $4b$  (hence 3,  $\tilde{6}, \tilde{7}, 8$ ); and, since  $|\Delta\lambda_\nu| \searrow$ , we have  $\Delta_\nu \nearrow$ ,



so that 5 (hence 1*b*, 2, 3). Also, for  $\nu = n_i$ ,  $\frac{\Delta\lambda_\nu}{\Delta\lambda_{\nu-1}} = \frac{n_i - n_{i-1}}{n_{i+1} - n_i} \rightarrow 0$ , so that  $\tilde{1}a$  (hence  $\tilde{6}$ ). Finally, the condition on  $\{n_i\}$  ensures that  $n_{i+1} - n_i > c \cdot 2^i$  ( $c > 0$ ), and hence

$$\Lambda_{n_i}^p \lambda_{n_{i+1}}^{-\mu} \sim i^{p-\mu} (n_{i+1} - n_i)^p > c i^{p-\mu} 2^{pi} \rightarrow \infty$$

for any positive  $\mu$  and  $p$ , so that  $\tilde{9}$  (hence  $\tilde{4}a$ ,  $\tilde{6}$ ). Combining the results,  $\{\lambda_n\}$  satisfies

$$\tilde{1}a \wedge 1b \wedge 2 \wedge 3 \wedge \tilde{4}a \wedge 4b \wedge 5 \wedge \tilde{6} \wedge 7 \wedge 8 \wedge \tilde{9}.$$

$$(18) \quad \lambda_\nu = n_i + \{(v - n_i)(n_{i+1} - n_i)\}^{\frac{1}{2}} \quad \text{for } n_i \leq \nu \leq n_{i+1} \quad (i = 0, 1, 2, \dots),$$

where  $n_{i+1} - n_i \rightarrow +\infty$ , but such that  $n_{i+1} - n_i = o(n_i^2)$  (a suitable choice is  $n_i = 2^i$ ). An easy calculation shows that, for  $n_i \leq \nu < n_{i+1}$ ,

$$\lambda_{\nu+1} - \lambda_\nu = \frac{\sqrt{(n_{i+1} - n_i)}}{\sqrt{(v+1 - n_i)} + \sqrt{(v - n_i)}} > \frac{1}{2},$$

so that 4*a* (hence 9); but  $|\Delta\lambda_{n_i}| = \sqrt{(n_{i+1} - n_i)} \rightarrow \infty$ , so that  $\tilde{4}b$ . So long as the suffixes all remain within  $[n_i, n_{i+1}]$ , we also see that

$$|\Delta\lambda_\nu| \searrow, \text{ hence } \Lambda_\nu \nearrow, \text{ hence } \lambda_{\nu+1}/\lambda_\nu \searrow;$$

thus

$$\begin{aligned} 1 < \frac{\lambda_{\nu+1}}{\lambda_\nu} &\leq \frac{\lambda_{n_{i+1}}}{\lambda_{n_i}} \quad (n_i \leq \nu < n_{i+1}) \\ &= \frac{n_i + \sqrt{(n_{i+1} - n_i)}}{n_i} = 1 + o(1), \end{aligned}$$

so that 7 (hence 3,  $\tilde{6}$ , 8). It then follows that

$$\frac{\Lambda_{\nu-1}}{\Lambda_\nu} = \frac{\lambda_\nu}{\lambda_{\nu+1}} \frac{\Delta\lambda_\nu}{\Delta\lambda_{\nu-1}} \sim \frac{\Delta\lambda_\nu}{\Delta\lambda_{\nu-1}}$$

and by taking  $\nu = n_i$  and noting that  $|\Delta\lambda_{n_{i-1}}| \leq 1$ ,  $|\Delta\lambda_{n_i}| \rightarrow \infty$ , we obtain  $\tilde{2}$  (hence  $\tilde{1}b$ ,  $\tilde{5}$ ,  $\tilde{6}$ ). Finally, it is easily verified that (for any increasing  $\{n_i\}$ )  $(\Delta\lambda_\nu)/(\Delta\lambda_{\nu-1}) \geq 1/(1 + \sqrt{2})$  for every  $\nu$ , so that 1*a* holds, and  $\{\lambda_n\}$  therefore satisfies

$$1a \wedge \tilde{1}b \wedge \tilde{2} \wedge 3 \wedge 4a \wedge \tilde{4}b \wedge \tilde{5} \wedge \tilde{6} \wedge 7 \wedge 8 \wedge 9.$$

**3. Relations between  $(C, \lambda, p)$ -Means of Different Orders.** Given a non-negative integer  $p$ , a sequence  $\{\lambda_n\}$  of non-negative numbers strictly increasing to  $\infty$ , and any series  $\sum a_n$ , we define, for  $n=0, 1, 2, \dots$ ,

$$(19) \quad C_n^o = \sum_{\nu=0}^n a_\nu, \quad C_n^p = \sum_{\nu=0}^n (\lambda_{n+1} - \lambda_\nu) \cdots (\lambda_{n+p} - \lambda_\nu) a_\nu \quad (p=1, 2, \dots);$$

the  $(C, \lambda, p)$ -means  $t_n^p$  of  $\sum a_n$  are defined by

$$t_n^o = C_n^o, \quad t_n^p = (\lambda_{n+1} \cdots \lambda_{n+p})^{-1} C_n^p \quad (p = 1, 2, \dots),$$

and we say that  $\sum a_n$  is summable  $(C, \lambda, p)$  to  $s$  if

$$t_n^p \rightarrow s.$$

Denoting convergence by  $I$  (the identity transformation) we have trivially, for any  $\{\lambda_n\}$ ,

$$(20) \quad \begin{aligned} (C, \lambda, 0) &\equiv (R^*, \lambda, 0) \equiv (R, \lambda, 0) \equiv I, \\ (C, \lambda, 1) &\equiv (R^*, \lambda, 1) \sim (R, \lambda, 1). \end{aligned}$$

Note that, for  $n=0, 1, 2, \dots$  (and defining  $C_{-1}^r = 0$ ),

$$(21) \quad \begin{aligned} C_n^{p+1} - C_{n-1}^{p+1} &= \sum_{\nu=0}^n (\lambda_{n+1} - \lambda_\nu) \cdots (\lambda_{n+p+1} - \lambda_\nu) a_\nu - \sum_{\nu=0}^{n-1} (\lambda_n - \lambda_\nu) \cdots (\lambda_{n+p} - \lambda_\nu) a_\nu \\ &= (\lambda_{n+p+1} - \lambda_n) C_n^p; \end{aligned}$$

and it follows directly that<sup>5)</sup>

$$(22) \quad C_n^{p+1} = \sum_{\nu=0}^n (\lambda_{\nu+p+1} - \lambda_\nu) C_\nu^p.$$

**THEOREM 1.** *If  $C_n^p = o(\eta_n)$ , where  $0 < \eta_n \nearrow$ , then  $C_n^{p+1} = o(\lambda_{n+p+1} \eta_n)$ . We may replace  $o$  by  $O$  throughout.*

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5) Although the definition of  $C_\nu^k$  is slightly different, this is the formula following 1(13) of Jurkat [8]. Most of the results of this section are elementary consequences of (21) and (22), but are given for the sake of completeness.

PROOF. Under the hypothesis of the theorem, it follows at once from (22) that

$$C_n^{p+1} = o(\eta_n) \sum_{\nu=0}^n (\lambda_{\nu+p+1} - \lambda_\nu) = o(\lambda_{n+p+1} \eta_n).$$

COROLLARY 1A.  $(C, \lambda, p) \subseteq (C, \lambda, p+1)$  ( $p=0, 1, 2, \dots$ ).

PROOF. We may suppose, without loss of generality, that  $\sum a_\nu$  is summable  $(C, \lambda, p)$  to zero, and the corollary follows from Theorem 1 on taking  $\eta_n = 1$  ( $p=0$ ),  $\eta_n = \lambda_{n+1} \cdots \lambda_{n+p}$  ( $p \geq 1$ ).

COROLLARY 1B.  $(C, \lambda, p)$  is regular for every non-negative integer  $p$ .

If  $C^p$  denotes the  $(C, \lambda, p)$ -matrix then (22) (when expressed in terms of the means  $t_\nu^p$  and  $t_n^{p+1}$ ) defines a matrix  $L_p$  such that  $C^{p+1} = L_p C^p$ , and Corollary 1A is equivalent to the statement that, for  $p=0, 1, 2, \dots$ ,  $L_p$  is regular (a  $T$ -matrix) —  $L_0$  is the sequence-to-sequence matrix of  $(C, \lambda, 1)$ . Since  $L_p$  is normal, it has an inverse  $L_p^{-1}$  (which, in fact, is easily calculated — see (28)) and Maddox [19] defines the  $(C, \lambda, -1)$ -matrix as  $L_0^{-1}$ . Some further properties of  $L_p$  and  $L_p^{-1}$  also appear in a forthcoming note [26].

To proceed from a  $(C, \lambda, p)$ -mean to a  $(C, \lambda, r)$ -mean of lower order, we have the following limitation theorems.

THEOREM 2. Let  $C_n^p = o(\eta_n)$  ( $\eta_n > 0$ ), and denote  $\eta_{n,r} = \max_{n-r \leq i \leq n} \eta_i$ ; then, for  $r=0, 1, \dots, p$ ,

$$(23) \quad C_n^r = o\{\eta_{n,p-r}/(\lambda_{n+r+1} - \lambda_n)^{p-r}\}.$$

We may replace  $o$  by  $O$  throughout.

PROOF. (23) is certainly true for  $r=p$ , since it then reduces so the hypothesis. Suppose that (23) holds for some  $r$  in  $0 < r \leq p$ ; then, by (21),

$$(24) \quad \begin{aligned} C_n^{r-1} &= \frac{C_n^r - C_{n-1}^r}{\lambda_{n+r} - \lambda_n} = \frac{1}{\lambda_{n+r} - \lambda_n} \left\{ \frac{o(\eta_{n,p-r})}{(\lambda_{n+r+1} - \lambda_n)^{p-r}} + \frac{o(\eta_{n-1,p-r})}{(\lambda_{n+r} - \lambda_{n-1})^{p-r}} \right\} \\ &= \frac{1}{\lambda_{n+r} - \lambda_n} \frac{o(\eta_{n,p-r+1})}{(\lambda_{n+r} - \lambda_n)^{p-r}}, \end{aligned}$$

since  $\eta_{n,p-r} \leq \eta_{n,p-r+1}$ ,  $\eta_{n-1,p-r} \leq \eta_{n,p-r+1}$ , and  $\lambda_n \nearrow$ . It now follows by induction that (23) holds for  $r=0, 1, \dots, p$ .

COROLLARY 2. If  $C_n^p = o(\eta_n)$ , where  $0 < \eta_n \nearrow$ , then

$$C_n^r = o\{\eta_n/|\Delta\lambda_n|^{p-r}\} \quad (r = 0, 1, \dots, p).$$

Jurkat [8, Satz 7] gives this result, where  $p$  and  $r$  need not be integers, subject to the restrictions

$$[(1b)] \quad \frac{\Delta\lambda_n}{\Delta\lambda_{n-1}} = O(1),$$

$$\frac{\eta_n}{|\Delta\lambda_n|^p} \nearrow;$$

neither of these restrictions is needed when  $p$  and  $r$  are integers.

There is an alternative form of Theorem 2 in which we replace  $\eta_n$  by  $\lambda_{n+1} \cdots \lambda_{n+p} \eta'_n$  and express the result in terms of the  $(C, \lambda, r)$ -means  $t_n^r = (\lambda_{n+1} \cdots \lambda_{n+p})^{-1} C_n^r$ . Since the two forms are not completely equivalent we give a short separate proof.

THEOREM 3. Let  $p$  be a non-negative integer,  $\eta'_n > 0$ , and denote

$$(25) \quad \Lambda_{n,r} = \lambda_{n+r+1}/(\lambda_{n+r+1} - \lambda_n) \quad (r = 0, 1, 2, \dots),$$

and 
$$\eta'_{n,r} = \max_{n-r \leq i \leq n} \eta'_i.$$

If 
$$t_n^p = o(\eta'_n)$$

then

$$(26) \quad t_n^r = o(\eta'_{n,p-r} \Lambda_{n,r}^{p-r}) \quad (r=0, 1, \dots, p).$$

We may replace  $o$  by  $O$  throughout.

PROOF. It is easily verified that, since  $\lambda_n \nearrow$ ,

$$(27) \quad \Lambda_{n,r} < \Lambda_{n,r-1} \quad \text{and} \quad \Lambda_{n-1,r} < \Lambda_{n,r-1};$$

and also

$$(27)' \quad \eta'_{n,r} \leq \eta'_{n,r+1} \quad \text{and} \quad \eta'_{n-1,r} \leq \eta'_{n,r+1}.$$

The proof is now similar to that of Theorem 2, except that it is convenient to express (24) in terms of the  $t_n^r$ , namely

$$(28) \quad t_n^{r-1} = \frac{\lambda_{n+r} t_n^r - \lambda_n t_{n-1}^r}{\lambda_{n+r} - \lambda_n}.$$

Now the formula in (26) is true for  $r = p$ , by hypothesis; and if it holds for some integer  $r$  in  $0 < r \leq p$  then, substituting in (28) and using  $\lambda_n < \lambda_{n+r}$ ,

$$\begin{aligned} t_n^{r-1} &= o(\Lambda_{n,r-1} \eta'_{n,p-r} \Lambda_{n,r}^{p-r}) + o(\Lambda_{n,r-1} \eta'_{n-1,p-r} \Lambda_{n-1,r}^{p-r}) \\ &= o(\eta'_{n,p-r+1} \Lambda_{n,r-1}^{p-r+1}), \end{aligned}$$

by (27) and (27)'. The required result now follows by induction.

**COROLLARY 3A.** Denote  $\Lambda_n = \lambda_{n+1}/(\lambda_{n+1} - \lambda_n)$ . If  $t_n^p = o(1)$  then  $t_n^r = o(\Lambda_n^{p-r})$  ( $r=0, 1, \dots, p$ ),

**PROOF.** By (27),  $\Lambda_{n,r} \leq \Lambda_{n,0} = \Lambda_n$  ( $r=0, 1, \dots$ ), and using this inequality in Theorem 3, with  $\eta'_n=1$ , we get the result.

Note, incidentally, that Corollary 3A is directly analogous to one form of the limitation theorem for Riesz means (see Borwein [1], Lemma 2; or for  $\kappa=p$ , a non-negative integer, see Bosanquet [2], Lemma 3; we write  $o$  in place of  $O$ ); thus denoting the Riesz mean of order  $\kappa$  by  $R^\kappa(\omega) = \omega^{-\kappa} A^\kappa(\omega)$ , the result is:

$$R^\kappa(\omega) = o(1) \text{ implies } R^r(\omega) = o(\Lambda_n^{\kappa-r}) \quad (\lambda_n < \omega \leq \lambda_{n+1}, r=0, 1, \dots, [\kappa]).$$

**COROLLARY 3B.** If  $\Lambda_n = O(1)$  or, what is the same thing, if

$$[(6)] \quad \liminf (\lambda_{n+1}/\lambda_n) > 1$$

then  $(C, \lambda, p)$  is equivalent to convergence for any integer  $p$ .

**PROOF.** If (6) holds then, by Corollary 3A,  $t_n^p = o(1)$  implies  $t_n^0 = o(1)$ .

**4. The Inclusion  $(C, \lambda, p) \subseteq (R, \lambda, p)$ .** In considering an inclusion relation of the form  $C \subseteq A$ , it is desirable to be able to express the  $A$ -means of a series  $\sum a_n$  in terms of its  $C$ -means, and then to consider conditions under which the resulting transformation is regular. This problem is simplified when the matrix of  $C$  has a readily calculated inverse, as is the case with the  $(C, \lambda, p)$  method, where the inverse can be expressed in terms of divided differences; both Jurkat [8] and Burkill [3] make use of this, though in the former case the notation is somewhat different from that adopted here. Thus,

given a function  $f$  defined in an interval  $[a, b]$ , and distinct points  $x_i$  in this interval, we denote  $f[x] = f(x)$  and

$$(29) \quad f[x_0, x_1, \dots, x_n] = \frac{f[x_0, \dots, x_{n-1}] - f[x_1, \dots, x_n]}{x_0 - x_n} \quad (n=1, 2, \dots);$$

for an exposition of the properties of divided differences see, for example, Milne-Thomson [20, Chapter I]. Since  $f[x_0, \dots, x_n]$  is independent of the order of the arguments, we may suppose that  $a = x_0 < x_1 < \dots < x_n = b$ . If the derivative  $f^{(n)}(x)$  exists in  $(x_0, x_n)$ , and  $f^{(n-1)}(x)$  is continuous also at the endpoints  $x_0, x_n$ , then [see 20, p. 6]

$$(30) \quad f[x_0, \dots, x_n] = \frac{1}{n!} f^{(n)}(\xi), \text{ for some } \xi \text{ in } x_0 < \xi < x_n.$$

Thus in the special case where, in the interval  $[x_0, x_n]$ ,  $f$  is a polynomial of degree less than  $n$ ,  $f[x_0, \dots, x_n] = 0$ . Any divided difference is expressible in terms of the functional values at the points  $x_i$  as follows [see 20, p. 7]:

$$(31) \quad f[x_0, \dots, x_n] = \sum_{i=0}^n \frac{f(x_i)}{\rho_i}, \text{ where } \rho_i = \prod_{j=0, j \neq i}^n (x_i - x_j),$$

and  $\Pi'$  indicates omission of the zero factor given by  $j=i$ .

LEMMA 1. *Let  $g(x)$  be defined for  $x \geq 0$ . Then*

$$(32) \quad \sum_{v=0}^n g(\lambda_v) a_v = \sum_{r=0}^p (-1)^r g[\lambda_{n+1}, \dots, \lambda_{n+r+1}] C_n^r + (-1)^{p+1} \sum_{v=0}^n g[\lambda_v, \dots, \lambda_{v+p+1}] (\lambda_{v+p+1} - \lambda_v) C_v^p.$$

PROOF. The proof is by induction on  $p$ . First, using

$$C_v^0 - C_{v-1}^0 = a_v \quad \text{and} \quad g[\lambda_v, \lambda_{v+1}] (\lambda_{v+1} - \lambda_v) = g(\lambda_{v+1}) - g(\lambda_v),$$

we easily verify that (32) is true for  $p=0$  (and any  $n \geq 0$ ).

Now for any non-negative integer  $p$  we have, using in succession (21), partial summation, and (29),

$$\begin{aligned}
 & \sum_{\nu=0}^n g[\lambda_\nu, \dots, \lambda_{\nu+p+1}] (\lambda_{\nu+p+1} - \lambda_\nu) C_\nu^p \\
 &= \sum_{\nu=0}^n g[\lambda_\nu, \dots, \lambda_{\nu+p+1}] (C_\nu^{p+1} - C_{\nu-1}^{p+1}) \\
 &= \sum_{\nu=0}^n \{g[\lambda_\nu, \dots, \lambda_{\nu+p+1}] - g[\lambda_{\nu+1}, \dots, \lambda_{\nu+p+2}]\} C_\nu^{p+1} + g[\lambda_{n+1}, \dots, \lambda_{n+p+2}] C_n^{p+1} \\
 &= - \sum_{\nu=0}^n g[\lambda_\nu, \dots, \lambda_{\nu+p+2}] (\lambda_{\nu+p+2} - \lambda_\nu) C_\nu^{p+1} + g[\lambda_{n+1}, \dots, \lambda_{n+p+2}] C_n^{p+1}
 \end{aligned}$$

and by substituting this in (32) we see that the right hand side of (32) has the same value for any non-negative integer  $p$ ; since for  $p = 0$  it is equal to the left hand side (which is independent of  $p$ ), the result follows.

Now define

$$c_\omega(x) = \begin{cases} (\omega - x)^p & (0 \leq x < \omega), \\ 0 & (x \geq \omega). \end{cases}$$

Given  $\omega > 0$ , let  $n$  be the integer such that  $\lambda_n < \omega \leq \lambda_{n+1}$ ; then the Riesz sum is

$$A^p(\omega) = \sum_{\lambda_\nu < \omega} (\omega - \lambda_\nu)^p a_\nu = \sum_{\nu=0}^n c_\omega(\lambda_\nu) a_\nu$$

and the Riesz mean is

$$R^p(\omega) = \omega^{-p} A^p(\omega).$$

We now employ Lemma 1 with  $g(x) = c_\omega(x)$ . First, since  $c_\omega(x) = 0$  for  $x \geq \omega$ , and each of the points  $\lambda_i$  ( $i = n+1, \dots, n+r+1$ ) satisfies  $\lambda_i \geq \omega$ , we have

$$c_\omega[\lambda_{n+1}, \dots, \lambda_{n+r+1}] \equiv 0 \quad (r=0, 1, 2, \dots).$$

Further, if  $0 \leq \nu \leq n-p-1$  then  $\lambda_{\nu+p+1} \leq \lambda_n < \omega$  and  $c_\omega(x)$  is then a polynomial of degree  $p$  throughout the range  $\lambda_\nu \leq x \leq \lambda_{\nu+p+1}$  — any divided difference of order greater than  $p$ , taken at points  $x_i$  in this range, will then vanish; in particular,

$$c_\omega[\lambda_\nu, \dots, \lambda_{\nu+p+1}] = 0 \quad \text{for } 0 \leq \nu \leq n-p-1.$$

In case  $n < p$ , define  $C_\nu^p = 0$  for  $\nu < 0$ . Then, by Lemma 1,

$$(33) \quad A^p(\omega) = (-1)^{p+1} \sum_{\nu=n-p}^n c_\omega[\lambda_\nu, \dots, \lambda_{\nu+p+1}] (\lambda_{\nu+p+1} - \lambda_\nu) C_\nu^p \quad (\lambda_n < \omega \leq \lambda_{n+1}).$$

This is the same as the expression for  $T(\omega)$  given half-way down p.58 of Burkill [3]; it is stated there as being obtained from [3, Lemma 1], which assumes the restriction on  $\lambda$  stated in this paper as (4). In terms of the Riesz mean  $R^p(\omega)$  and the  $(C, \lambda, p)$ -mean  $t_\nu^p$ , (33) may be written

$$(34) \quad R^p(\omega) = \sum_{\nu=n-p}^n \alpha_\nu^p(\omega) t_\nu^p \quad (\lambda_n < \omega \leq \lambda_{n+1})$$

where

$$(35) \quad \alpha_\nu^p(\omega) = (-1)^{p+1} \omega^{-p} c_\omega[\lambda_\nu, \dots, \lambda_{\nu+p+1}] (\lambda_{\nu+p+1} - \lambda_\nu) \lambda_{\nu+1} \dots \lambda_{\nu+p}.$$

THEOREM 4.  $(C, \lambda, p) \subseteq (R, \lambda, p) \quad (p = 0, 1, 2, \dots)$ .

PROOF. Denote  $h_\nu(x) = \frac{(\omega - x)^p}{(x - \lambda_{n+1}) \dots (x - \lambda_{\nu+p+1})}$  where  $0 \leq n - p \leq \nu \leq n$  and  $\lambda_\nu \leq x \leq \lambda_n < \omega \leq \lambda_{n+1} < \dots < \lambda_{\nu+p+1}$ . By the expansion formula (31) for divided differences, we then have

$$\begin{aligned} c_\omega[\lambda_\nu, \dots, \lambda_{\nu+p+1}] &= \sum_{i=\nu}^{\nu+p+1} \frac{c_\omega(\lambda_i)}{\beta_{\nu i}}, \quad \text{where } \beta_{\nu i} = \prod_{j=\nu}^{\nu+p+1} (\lambda_i - \lambda_j) \\ &= \sum_{i=\nu}^n \frac{(\omega - \lambda_i)^p}{\beta_{\nu i}}, \quad \text{since } c_\omega(\lambda_i) = 0 \text{ for } \lambda_i \geq \lambda_{n+1} \geq \omega; \end{aligned}$$

and also

$$h_\nu[\lambda_\nu, \dots, \lambda_n] = \sum_{i=\nu}^n \frac{h_\nu(\lambda_i)}{\beta'_{\nu i}}, \quad \text{where } \beta'_{\nu i} = \prod_{j=\nu}^n (\lambda_i - \lambda_j).$$

But, by definition of  $h_\nu(x)$ ,

$$\frac{h_\nu(\lambda_i)}{\beta'_{\nu i}} = \frac{(\omega - \lambda_i)^p}{(\lambda_i - \lambda_{n+1}) \dots (\lambda_i - \lambda_{\nu+p+1}) \beta'_{\nu i}} = \frac{(\omega - \lambda_i)^p}{\beta_{\nu i}}$$

and hence

$$(36) \quad c_\omega[\lambda_\nu, \dots, \lambda_{\nu+p+1}] = h_\nu[\lambda_\nu, \dots, \lambda_n] \quad (0 \leq n - p \leq \nu \leq n).$$



Also, by property (30) for divided differences,

$$(37) \quad h_\nu[\lambda_\nu, \dots, \lambda_n] = \frac{h_\nu^{(n-\nu)}(\xi)}{(n-\nu)!} \quad \text{for some } \xi \text{ in } \lambda_\nu \leq \xi \leq \lambda_n.$$

Now the denominator of  $h_\nu(x)$  contains  $\nu + p - n + 1$  factors, all negative, so that

$$(-1)^{\nu+p-n+1} h_\nu(x) > 0.$$

Further, logarithmic differentiation of  $h_\nu(x)$  gives

$$\frac{h'_\nu(x)}{h_\nu(x)} = -\frac{n-\nu-1}{\omega-x} - \sum_{i=n+1}^{\nu+p+1} \left( \frac{1}{\omega-x} - \frac{1}{\lambda_i-x} \right) \equiv y(x)$$

and since  $\lambda_i \geq \omega$  for  $i \geq n+1$  we have

$$y(x) \leq 0 \quad \text{for } \lambda_\nu \leq x \leq \lambda_n, \quad \nu \leq n-1.$$

Then, since  $y(x)$  alternates in sign on successive differentiations, we can differentiate  $h'_\nu(x)$  any number of times using Leibniz' theorem and prove easily by induction that

$$(38) \quad (-1)^{\nu+p-n+r+1} h_\nu^{(r)}(x) \geq 0 \quad (\lambda_\nu \leq x \leq \lambda_n)$$

for  $n-p \leq \nu \leq n, r=0$ ; or for  $n-p \leq \nu \leq n-1, r=0, 1, 2, \dots$ .

A combination of (36), (37), (38) (with  $r = n-\nu$ ) now yields

$$(-1)^{p+1} c_\omega[\lambda_\nu, \dots, \lambda_{\nu+p+1}] \geq 0 \quad (n-p \leq \nu \leq n)$$

and since the other factors on the right of (35) are also positive, it now follows that

$$\alpha_\nu^p(\omega) \geq 0 \quad (n-p \leq \nu \leq n, \lambda_n < \omega \leq \lambda_{n+1}),$$

so that (34) is a positive transformation. Now we clearly have

$$\lim_{\omega \rightarrow \infty} \alpha_\nu^p(\omega) = 0 \quad (\nu = 0, 1, 2, \dots)$$

since  $\alpha_\nu^p(\omega) \equiv 0$  for  $\nu < n-p$ . Hence (by the Toeplitz theorem) in order that  $t_\nu^p \rightarrow s$  always implies  $R^p(\omega) \rightarrow s$  it is necessary and sufficient that

$$(39) \quad \lim_{\omega \rightarrow \infty} \sum_{\nu=n-p}^n \alpha_{\nu}^p(\omega) = 1.$$

Since, for  $n > p$ , the transformation (34) is independent of  $\lambda_0$ , let

$$\lambda_0 = 0, \quad a_0 = 1, \quad a_i = 0 \quad (i > 0);$$

then  $t_{\nu}^p=1$  for every  $\nu$ , and  $R^p(\omega)=1$  for every  $\omega > 0$ , and substitution in (34) gives at once

$$\sum_{\nu=n-p}^n \alpha_{\nu}^p(\omega) = 1 \quad (\lambda_n < \omega \leq \lambda_{n+1}, \quad n > p).$$

Thus (39) holds and we conclude that, for any sequence  $\{\lambda_n\}$  of non-negative numbers increasing to infinity, and for any non-negative integer  $p$ ,  $(C, \lambda, p) \subseteq (R, \lambda, p)$ .

COROLLARY 4. *Let  $t_{\nu}^p = o(\eta_{\nu}')$  ( $\eta_{\nu}' > 0$ ) and denote  $\eta'_{n,p} = \max_{n-p \leq \nu \leq n} \eta_{\nu}'$ . Then  $R^p(\omega) = o(\eta'_{n,p})$  ( $\lambda_n < \omega \leq \lambda_{n+1}$ ).*

PROOF. This follows at once from the theorem on putting  $\bar{t}_{\nu}^p = (\eta_{\nu}')^{-1} t_{\nu}^p = o(1)$  and noting that

$$|R^p(\omega)| \leq \eta'_{n,p} \sum_{\nu=n-p}^n \alpha_{\nu}^p(\omega) |t_{\nu}^p|.$$

REMARK 1. If we put  $\eta_{\nu} = \lambda_{\nu+1} \cdots \lambda_{\nu+p} \eta_{\nu}'$  in Corollary 4, the result can be put in the form

$$C_{\nu}^p = o(\eta_{\nu}) \text{ implies } A^p(\omega) = o \left\{ \omega^p \max_{n-p \leq \nu \leq n} \frac{\eta_{\nu}}{\lambda_{\nu+1} \cdots \lambda_{\nu+p}} \right\},$$

but, writing  $\eta_{n,p} = \max_{n-p \leq \nu \leq n} \eta_{\nu}$ , it is possible to obtain an alternative result (which in some cases may be better than Corollary 4 — compare with the forms of Theorems 2 and 3), namely:

$$(40) \quad C_{\nu}^p = o(\eta_{\nu}) \text{ implies } A^p(\omega) = o(\eta_{n,p}) \quad (\lambda_n < \omega \leq \lambda_{n+1}).$$

To prove this, we note that the transformation (33) contains only  $p+1$  terms, so that (40) will follow if we can show that, for  $\lambda_n < \omega \leq \lambda_{n+1}$  and  $n-p \leq \nu \leq n$ ,

$$c_\omega[\lambda_\nu, \dots, \lambda_{\nu+p+1}](\lambda_{\nu+p+1} - \lambda_\nu) = O(1) \text{ independently of } n.$$

But

$$c_\omega[\lambda_\nu, \dots, \lambda_{\nu+p+1}](\lambda_{\nu+p+1} - \lambda_\nu) = c_\omega[\lambda_{\nu+1}, \dots, \lambda_{\nu+p+1}] - c_\omega[\lambda_\nu, \dots, \lambda_{\nu+p}]$$

and it is therefore enough to show that

$$(41) \quad c_\omega[\lambda_\nu, \dots, \lambda_{\nu+p}] = O(1) \text{ for } n-p \leq \nu \leq n+1, \lambda_n < \omega \leq \lambda_{n+1}.$$

Now although  $c_\omega^{(p)}(x)$  may not exist at  $x = \omega$ ,  $c_\omega^{(p-1)}(x)$  exists everywhere (and is continuous); thus, by (29) and (30),

$$\begin{aligned} c_\omega[\lambda_\nu, \dots, \lambda_{\nu+p}] &= \frac{c_\omega[\lambda_{\nu+1}, \dots, \lambda_{\nu+p}] - c_\omega[\lambda_\nu, \dots, \lambda_{\nu+p-1}]}{\lambda_{\nu+p} - \lambda_\nu} \\ &= \frac{c_\omega^{(p-1)}(\xi_2) - c_\omega^{(p-1)}(\xi_1)}{(p-1)!(\lambda_{\nu+p} - \lambda_\nu)} \end{aligned}$$

where  $\xi_1 \geq \lambda_\nu$ ,  $\xi_2 \geq \lambda_{\nu+1} > \lambda_\nu$ . Calculation of the derivative then shows that, for  $\xi \geq \lambda_\nu$ ,

$$|c_\omega^{(p-1)}(\xi)| \leq p! \max(0, \omega - \xi) \leq p! (\lambda_{n+1} - \lambda_\nu) \leq p! (\lambda_{\nu+p} - \lambda_\nu),$$

provided that  $\nu \geq n-p+1$ ; and (41) therefore holds for these values of  $\nu$ . In the excluded case  $\nu = n-p$ , we have  $c_\omega(x) = (\omega - x)^p$  for  $x \leq \lambda_n < \omega$ , and hence

$$c_\omega[\lambda_{n-p}, \dots, \lambda_n] = (-1)^p;$$

thus (41) holds in any case, and (40) follows.

This proof of (40) provides a simpler proof of Theorem 4 if we know that

$$[(3)] \quad \lambda_{n+1} = O(\lambda_n).$$

For, if  $\eta_n = \lambda_{n+1} \cdots \lambda_{n+p}$  and (3) holds, then, for  $\lambda_n < \omega$ ,

$$\eta_{n,p} = \eta_n = O(\lambda_n^p) = O(\omega^p);$$

(40) then shows that:

$$\text{If (3) holds then } (C, \lambda, p) \subseteq (R, \lambda, p).$$

However, the more delicate analysis in the proof of Theorem 4 shows that (3)

can be dispensed with entirely.

REMARK 2. It is well-known (see, for example, [5], Corollary 1.62) that if

$$[(6)] \quad \liminf (\lambda_{n+1}/\lambda_n) > 1$$

then  $(R, \lambda, \kappa)$  is equivalent to convergence for any  $\kappa \geq 0$ . This fact, together with Theorem 4 and Corollary 1B, shows that if (6) holds then

$$I \subseteq (C, \lambda, p) \subseteq (R, \lambda, p) \subseteq I$$

and we get an alternative proof of Corollary 3.

**5. The Inclusion  $(R, \lambda, p) \subseteq (C, \lambda, p)$ .** In this section we deduce information about  $(C, \lambda, p)$ -means from knowledge of the  $(R, \lambda, p)$ -means (i.e. in the opposite direction to the results of §4). In order to obtain an inclusion theorem we shall impose a restriction on  $\lambda$  when  $p \geq 3$ , namely that given in (2). Some lemmas are required.

LEMMA 2. Let  $Q(t)$  be a polynomial in  $t$ , of degree  $p$ , and define coefficients  $\theta_r^p(\omega)$  by

$$(42) \quad Q(\omega - x) = \sum_{r=0}^p \theta_r^p(\omega) x^r.$$

Then

$$(43) \quad \sum_{\lambda_\nu < \omega} Q(\lambda_\nu) a_\nu = \sum_{r=0}^p \theta_r^p(\omega) A^r(\omega).$$

PROOF. Put  $x = \omega - \lambda_\nu$  in (42), multiply by  $a_\nu$ , and sum over all values of  $\nu$  such that  $\lambda_\nu < \omega$ ; then

$$\sum_{\lambda_\nu < \omega} Q(\lambda_\nu) a_\nu = \sum_{\lambda_\nu < \omega} a_\nu \sum_{r=0}^p \theta_r^p(\omega) (\omega - \lambda_\nu)^r = \sum_{r=0}^p \theta_r^p(\omega) A^r(\omega),$$

on interchanging the order of summation and using the basic definition of  $A^r(\omega)$ .

LEMMA 2'. If  $\mu_{n,r} = \lambda_{n+r+1} - \lambda_{n+1}$  and coefficients  $\theta_r^{n,p}$  are defined by

$$(42)' \quad x(x + \mu_{n,1})(x + \mu_{n,2}) \cdots (x + \mu_{n,p-1}) = \sum_{r=1}^p \theta_r^{n,p} x^r$$

then

$$(43)' \quad C_n^p = \sum_{r=1}^p \theta_r^{n,p} A^r(\lambda_{n+1}) \quad (p=1, 2, 3, \dots).$$

PROOF. Let  $Q(t) = (\lambda_{n+1}-t) \cdots (\lambda_{n+p}-t)$ ; then, by (19),

$$C_n^p = \sum_{v=0}^n Q(\lambda_v) a_v ;$$

also

$$\begin{aligned} Q(\lambda_{n+1}-x) &= [\lambda_{n+1}-(\lambda_{n+1}-x)][\lambda_{n+2}-(\lambda_{n+1}-x)] \cdots [\lambda_{n+p}-(\lambda_{n+1}-x)] \\ &= x(x+\mu_{n,1})(x+\mu_{n,2}) \cdots (x+\mu_{n,p-1}). \end{aligned}$$

The result follows Lemma 2 on taking  $\omega = \lambda_{n+1}$ .

LEMMA 3. Let  $p$  be a positive integer and  $0 < \xi_n \nearrow$ . If

$$(44) \quad A^p(\omega) = o(\xi_n) \quad (\lambda_n < \omega \leq \lambda_{n+1})$$

then

$$(45) \quad C_n^p = o(\xi_n) \lambda_{n+2} \cdots \lambda_{n+p} \left[ \frac{\lambda_{n+p} - \lambda_n}{\lambda_{n+p}(\lambda_{n+1} - \lambda_n)} \right]^{p-1}.$$

We may replace  $o$  by  $O$  throughout.

PROOF. We first recall the limitation theorem for Riesz means in the form corresponding to Corollary 2 (see, for example, [5], Theorem 1.61) that (44) implies

$$(46) \quad A^r(\omega) = o(\xi_n / |\Delta \lambda_n|^{p-r}) \quad (\lambda_n < \omega \leq \lambda_{n+1}; r = 0, 1, \dots, p).$$

From the definition of  $\theta_r^{n,p}$  given in (42)' we see that, for fixed  $n$ ,  $\theta_r^{n,p}$  is the sum of all the products of  $p-r$  different  $\mu_{n,i}$ ; since  $\mu_{n,i} = \lambda_{n+i+1} - \lambda_{n+1}$  increases with  $i$  (for each fixed  $n$ ), it follows that

$$(47) \quad 0 < \theta_r^{n,p} \leq K_p \mu_{n,r} \mu_{n,r+1} \cdots \mu_{n,p-1}.$$

Substitution of (46) and (47) in the result (43)' of Lemma 2' now yields

$$(48) \quad C_n^p = o(\xi_n) \sum_{r=1}^p \mu_{n,r} \mu_{n,r+1} \cdots \mu_{n,p-1} / (\lambda_{n+1} - \lambda_n)^{p-r}.$$

But  $\mu_{n,i}/\lambda_{n+i+1}$  increases with  $i$ , and hence

$$\frac{\mu_{n,i}}{\lambda_{n+i+1}} \leq \frac{\mu_{n,p-1}}{\lambda_{n+p}} < \frac{\lambda_{n+p} - \lambda_n}{\lambda_{n+p}} \quad (i=1, 2, \dots, p-1);$$

substitution of this inequality into (48) now gives

$$\begin{aligned} C_n^p &= o(\zeta_n) \sum_{r=1}^p \lambda_{n+r+1} \lambda_{n+r+2} \cdots \lambda_{n+p} \left[ \frac{\lambda_{n+p} - \lambda_n}{\lambda_{n+p}(\lambda_{n+1} - \lambda_n)} \right]^{p-r} \\ &\equiv o(\zeta_n) \sum_{r=1}^p b_{n,r}, \text{ say.} \end{aligned}$$

Now  $b_{n,r} = \frac{\lambda_{n+1}^r \lambda_{n+r+1} \cdots \lambda_{n+p}}{\lambda_{n+1}^p} (l_{n,p})^{p-r}$ , where

$$(49) \quad l_{n,p} = \frac{\lambda_{n+1}(\lambda_{n+p} - \lambda_n)}{\lambda_{n+p}(\lambda_{n+1} - \lambda_n)};$$

and, since  $l_{n,p} \geq 1$ ,  $b_{n,r} \searrow$  as  $r \nearrow$ . Hence  $C_n^p = o(\zeta_n b_{n,1})$ , and this is the required result (45).

REMARK 3. Writing  $\zeta_n = \lambda_{n+1}^p \zeta'_n$  and with  $l_{n,p}$  as defined in (49), we can deduce from Lemma 3 a result in terms of the means; thus: *Let  $\lambda_{n+1}^p \zeta'_n \nearrow$  and  $\lambda_n < \omega \leq \lambda_{n+1}$ ; then*

$$(50) \quad R^p(\omega) = o(\zeta'_n) \text{ implies } t_n^p = o\{\zeta'_n (l_{n,p})^{p-1}\}.$$

A slight improvement might apparently be effected by avoiding the use of the inequality  $\mu_{n,p-1} < \lambda_{n+p} - \lambda_n$  which appears just after (48), and concluding the proof of Lemma 3 very nearly as before. Thus write

$$l'_{n,p} = \frac{\lambda_{n+1}(\lambda_{n+p} - \lambda_{n+1})}{\lambda_{n+p}(\lambda_{n+1} - \lambda_n)};$$

then (50) can be modified to

$$(50)' \quad R^p(\omega) = o(\zeta'_n) \text{ implies } t_n^p = o(\zeta'_n) \max\{1, (l'_{n,p})^{p-1}\};$$

however, we see that  $l'_{n,p} < l_{n,p} < l'_{n,p} + 1$ , so that (50) and (50)' are in fact equivalent statements.

Before deducing an inclusion theorem, it is worth noting that if we sacrifice some of the generality of hypothesis (44) we can use, in place of (46), the improved limitation theorem of Bosanquet (already mentioned after our

Corollary 3A) to yield the following result :

LEMMA 3'. *Let  $p$  be a positive integer,  $p + \alpha \geq 0$ , and  $l_{n+1,p-1}$  be defined as in (49), with  $(l_{n+1,0})^0$  interpreted as 1. If*

$$(44)' \quad R^p(\omega) = o(\omega^\alpha)$$

then

$$(45)' \quad t_n^p = o\{\lambda_{n+1}^\alpha (l_{n+1,p-1})^{p-1}\}.$$

We may replace  $o$  by  $O$  throughout.

PROOF. Bosanquet's result [2, Lemma 3, with  $o$  in place of  $O$ ] can be stated as follows: if (44)' holds then

$$A^r(\omega) = o(\omega^r \lambda_n^\alpha \Lambda_n^{p-r}) \quad \text{for } \lambda_n < \omega \leq \lambda_{n+1}, \quad r=0, 1, \dots, p.$$

Now  $A^r(\omega)$  is a continuous function of  $\omega$ , if  $r > 0$ ; and hence the order condition on  $A^r(\omega)$  is valid also for  $\omega = \lambda_n$  and  $r=1, 2, \dots, p$ . Thus, putting  $\omega = \lambda_n$  and then replacing  $n$  by  $n+1$ , we get

$$(46)' \quad A^r(\lambda_{n+1}) = o(\lambda_{n+1}^{r+\alpha} \Lambda_{n+1}^{p-r}), \quad r=1, 2, \dots, p.$$

Since the summation in (43)' starts at  $r=1$ , we can now follow through the proof of Lemma 3, using (46)' in place of (46) and omitting the inequality  $\mu_{n,p-1} < \lambda_{n+p} - \lambda_n$  which follows (48). We thus arrive at

$$C_n^p = o(\lambda_{n+1}^\alpha) \sum_{r=1}^p b'_{n,r}$$

where  $b'_{n,r} = \lambda_{n+1}^r \lambda_{n+r+1} \dots \lambda_{n+p} (l_{n+1,p-1})^{p-r}$ , and conclude, by the same method as in Lemma 3, that  $C_n^p = o(\lambda_{n+1}^\alpha b'_{n,1})$ . When expressed in terms of the means  $t_n^p$ , this is the required result (45)'.

When the hypothesis is  $(C, \lambda, p)$ -summability, the case in which  $l_{n,p}$  or  $l_{n+1,p-1}$  is bounded is clearly of interest in deducing an inclusion theorem; and then, since  $l_{n,1} = 1$  and  $l_{n,r}$  increases with  $r$ , Lemma 3' will be more effective than Lemma 3 for this purpose. We first draw attention to the condition

$$[(2)] \quad \Lambda_{n-1} = O(\Lambda_n),$$

and then prove:

Let  $q$  be a fixed positive integer greater than 1; then

$$(51) \quad l_{n,q} = O(1) \text{ if and only if (2) holds.}$$

For:  $l_{n,r} \nearrow$  as  $r \nearrow$ , so that  $l_{n,q} \geq l_{n,2} > \Lambda_n/\Lambda_{n+1}$  for  $q \geq 2$ , and (2) is therefore necessary for boundedness of  $l_{n,q}$ . Conversely, if (2) holds, then

$$\frac{\lambda_{n+q} - \lambda_n}{\lambda_{n+q}} \leq \frac{\lambda_{n+q} - \lambda_{n+q-1}}{\lambda_{n+q}} + \dots + \frac{\lambda_{n+1} - \lambda_n}{\lambda_{n+1}} = \frac{1}{\Lambda_{n+q-1}} + \dots + \frac{1}{\Lambda_n} = O\left(\frac{1}{\Lambda_n}\right)$$

and hence  $l_{n,q} = O(1)$ .

THEOREM 5. Let  $p$  be a non-negative integer; if  $p \geq 3$  assume that

$$[(2)] \quad \Lambda_{n-1} = O(\Lambda_n).$$

Then 
$$(R, \lambda, p) \subseteq (C, \lambda, p).$$

PROOF. We may suppose, without loss of generality, that  $\sum a_n = 0$  ( $R, \lambda, p$ ); the result now follows from (51) and Lemma 3' (with  $\alpha = 0$ ).

It would, incidentally, be of interest to know whether (2) is a necessary condition for the inclusion  $(R, \lambda, p) \subseteq (C, \lambda, p)$  when  $p \geq 3$ , but I have been unable to prove this, or to find a counter-example. (Added in proof: see [28])

REMARK 4. Since, by Theorem 4, the reverse inclusion to that of Theorem 5 is true unrestrictedly, it follows that if (when  $p \geq 3$ ) (2) holds, then

$$(52) \quad (C, \lambda, p) \sim (R, \lambda, p) \quad (p = 0, 1, 2, \dots).$$

We note that (see §2) (2) is implied by either of the conditions

$$[(4)] \quad 0 < a' \leq \lambda_{n+1} - \lambda_n \leq b' < \infty;$$

$$[(5)] \quad \Lambda_n \nearrow.$$

Hence we have:

COROLLARY 5A. [Burkill 3, Theorem 1] (4) implies (52).

COROLLARY 5B. [Jurkat 8, Satz 1] (5) implies (52).



As remarked in §1, Jurkat has given a definition of  $(C, \lambda, \kappa)$  for non-integral  $\kappa$ . He has shown in [8], Satz 1, that  $(C, \lambda, \kappa)$  and  $(R, \lambda, \kappa)$  are equivalent for all positive  $\kappa$  if

$$\Lambda_n \nearrow, |\Delta \lambda_n| \text{ is monotonic, } n |\Delta \lambda_n|^{\kappa - [\kappa]} \nearrow.$$

This result has been used by Maddox [17, Theorem A] to show that (subject to these restrictions on  $\lambda$ ) a necessary and sufficient condition that

$$\sum |a_n \varepsilon_n| \text{ converges whenever } \sum a_n \text{ is summable } (R, \lambda, \kappa)$$

is  $\sum \Lambda_n^\kappa |\varepsilon_n| < \infty$ . For any  $\kappa > 0$ , it is enough to assume our condition (2) for the sufficiency part of the proof (as is clear from Maddox' proof). Conversely, when  $\kappa = p$ , a positive integer, use of our Theorem 4 and Maddox' method (using his key Lemma 4) shows that a necessary condition for the result (without restriction on  $\lambda$ ) is  $\sum \Lambda_{n,p-1}^p |\varepsilon_n| < \infty$ , where  $\Lambda_{n,p-1}$  is given by (25); and if (2) holds then this implies  $\sum \Lambda_n^p |\varepsilon_n| < \infty$ .

We drew attention in §1 to the problem of finding necessary or sufficient conditions for the inclusion relation  $(R^*, \lambda, \kappa) \subseteq (R, \lambda, \kappa)$ . Kuttner [15] has shown that if  $\kappa = 2$ , then a necessary condition for this relation is that  $\Lambda_n = O(1)$  (both methods then being equivalent to convergence), and conjectures that the same result holds for  $\kappa > 2$ . Now it is not difficult to show that, for any  $\kappa > 1$ , (2) is a necessary condition for this inclusion relation; consequently, if  $p$  is a positive integer we have, by Theorems 4 and 5,

$$(R^*, \lambda, p) \subseteq (R, \lambda, p) \text{ if and only if } (R^*, \lambda, p) \subseteq (C, \lambda, p).$$

Although the second inclusion appears to be as difficult to deal with as the first, at least this may provide, for integral  $\kappa$ , an alternative line of attack on the problem.

**6. Relation of  $(G, \lambda)$  to  $(R, \lambda, p)$  and  $(C, \lambda, p)$ .** We consider here the summability method  $(G, \lambda)$  referred to in the Introduction. Suppose that  $g(x)$  is defined for  $x \geq 0$ ; given a series  $\sum a_n$ , denote

$$G_\lambda(h) = \sum_{v=0}^{\infty} g(\lambda_v h) a_v \quad (h > 0)$$

whenever this last series converges. We say that  $\sum a_n$  is summable  $(G, \lambda)$  to  $s$  if  $G_\lambda(h)$  exists in some interval  $(0, h_0)$  and  $G_\lambda(h) \rightarrow s$  as  $h \rightarrow 0+$ . The conditions to be imposed on  $g$  will be

$$(53) \quad g(0+) = g(0) = 1,$$

(54)  $g^{(p)}$  exists and is of bounded variation in  $[0, X]$  for any  $X > 0$ ,

$$(55) \quad g^{(r)}(x) = O(x^{-\mu}) \text{ as } x \rightarrow +\infty \quad (r = 0, 1, \dots, p),$$

where  $p$  is a non-negative integer and  $\mu \geq 0$ . The first condition is necessary for regularity of  $(G, \lambda)$ ; we impose the second since we shall be making use of Stieltjes integrals and this will ensure their existence; the third condition anticipates possible application to generalized Riemann summability (see [24] and [25]), since it is satisfied by  $g(x) = (\sin x/x)^\mu$  (with the restriction  $\mu > p$  if  $\mu$  is not an integer).

A theorem dealing with sufficient conditions for  $(R, \lambda, \kappa) \subseteq (G, \lambda)$  (where  $\kappa$  need not be an integer) is given in a forthcoming paper [Russell, 25]; the result for  $\kappa = p$ , a non-negative integer, is as follows:

**THEOREM 6.** *Let  $p$  be a non-negative integer,  $\mu \geq p$ , and  $\Lambda_n^p = o(\lambda_n^\mu)$ ; and let  $g$  denote a function with the properties (53), (54), (55). If*

$$(56) \quad \int_0^\infty x^p |dg^{(p)}(x)| < \infty$$

then

$$(57) \quad (R, \lambda, p) \subseteq (G, \lambda).$$

Since (by Theorem 4) (57) implies

$$(58) \quad (C, \lambda, p) \subseteq (G, \lambda),$$

Theorem 6 also provides sufficient conditions for (58). Conversely, necessary conditions for (58) are also necessary conditions for (57) (of course, if we postulate condition (2) on  $\lambda$  then, by Remark 4, (57) and (58) are equivalent statements). The main result of this section will show that, subject to some reasonable restrictions on  $\lambda$  (in fact, we assume (3), (8), (9)), (56) is a necessary condition for (58). Since we shall make use of Lemma 1, we first need a lemma giving conditions under which the first sum on the right of (32) tends to zero as  $n \rightarrow \infty$ .

**LEMMA 4.** *Suppose that  $p$  is a non-negative integer,  $\mu \geq 0$ ,  $g^{(p)}(x)$  exists for  $x \geq 0$ , and (55) holds. Let  $C_n^p = o(\eta_n)$ , where  $0 < \eta_n \nearrow$ , and*

$$(59) \quad \eta_n = O(\lambda_{n+1}^\mu), \quad \eta_n = O(\lambda_{n+1}^\mu |\Delta \lambda_n|^p).$$

Then

$$(60) \quad \sum_{r=0}^p (-1)^r g[\lambda_{n+1}, \dots, \lambda_{n+r+1}] C_n^r = o(1) \quad \text{as } n \rightarrow \infty.$$

PROOF. By (30) and (55) we have, for  $r = 0, 1, \dots, p$  and for some  $\xi_r$  in  $\lambda_{n+1} \leq \xi_r \leq \lambda_{n+r+1}$ ,

$$|g[\lambda_{n+1}, \dots, \lambda_{n+r+1}]| = |g^{(r)}(\xi_r)/r!| \leq K \xi_r^{-\mu} \leq K \lambda_{n+1}^{-\mu},$$

where  $K$  is independent of  $n$  and  $r$ ; also, by Corollary 2,  $C_n^p = o(\eta_n)$  implies

$$C_n^r = o(\eta_n / |\Delta \lambda_n|^{p-r}) \quad (r = 0, 1, \dots, p).$$

Using these estimates, together with (59), we obtain

$$\begin{aligned} \sum_{r=0}^p (-1)^r g[\lambda_{n+1}, \dots, \lambda_{n+r+1}] C_n^r &= o(\eta_n \lambda_{n+1}^{-\mu}) \sum_{r=0}^p |\Delta \lambda_n|^{r-p} \\ &= o(\eta_n \lambda_{n+1}^{-\mu})(1 + |\Delta \lambda_n|^{-p}) \\ &= o(1) + o(1). \end{aligned}$$

LEMMA 4'. Let  $g$  satisfy (55),  $\mu \geq p$ , and  $\sum a_\nu = 0(C, \lambda, p)$ . If  $p \geq 1$  assume that (9) holds, and if  $p \geq 2$  assume in addition that (3) holds. Then (60) follows.

PROOF. Take  $\eta_n = 1$  ( $p = 0$ ),  $\eta_n = \lambda_{n+1} \cdots \lambda_{n+p}$  ( $p \geq 1$ ). Then (3) implies

$$\eta_n = O(\lambda_{n+1}^p) = O(\lambda_{n+1}^\mu) \quad \text{for } \mu \geq p;$$

while (3) and (9) together imply

$$\eta_n = O(\lambda_{n+1}^p) = O(\lambda_n^p |\Delta \lambda_n|^p) = O(\lambda_{n+1}^\mu |\Delta \lambda_n|^p).$$

Thus the hypotheses of Lemma 4 hold, and the conclusion follows.

THEOREM 7. Let  $p$  be a non-negative integer,  $\mu \geq p$ , and  $g$  denote a function with the properties (53), (54), (55). Assume that

$$[(8)] \quad \liminf_{n \rightarrow \infty} \frac{1}{\lambda_n} \max_{1 \leq \nu \leq n} (\lambda_\nu - \lambda_{\nu-1}) = 0;$$

if  $p \geq 1$  assume also that

$$[(9)] \quad \Lambda_n^p = O(\lambda_{n+1}^p);$$

if  $p \geq 2$  assume in addition that

$$[(3)] \quad \lambda_{n+1} = O(\lambda_n).$$

Then in order that  $(C, \lambda, p) \subseteq (G, \lambda)$  it is necessary that

$$[(56)] \quad \int_0^\infty x^p |dy^{(p)}(x)| < \infty.$$

PROOF. Using Lemma 1 and replacing  $\lambda_\nu$  by  $\lambda_\nu h$  ( $h > 0$ ) in (32), we obtain

$$(61) \quad \sum_{\nu=0}^n g(\lambda_\nu h) a_\nu = \sum_{r=0}^p (-h)^r g[\lambda_{n+1} h, \dots, \lambda_{n+r+1} h] C_n^r + \sum_{\nu=0}^n \gamma_{h,\nu}^p t_\nu^p,$$

where

$$(62) \quad \gamma_{h,\nu}^p = (-1)^{p+1} h^{p+1} g[\lambda_\nu h, \dots, \lambda_{\nu+p+1} h] (\lambda_{\nu+p+1} - \lambda_\nu) \lambda_{\nu+1} \dots \lambda_{\nu+p}.$$

Now the hypotheses of Lemma 4' are included in the hypotheses of this theorem and hence, for every series summable  $(C, \lambda, p)$  (to zero, without loss of generality), the first sum on the right of (61) tend to zero as  $n \rightarrow \infty$ , for any fixed  $h$ . Further, if  $(C, \lambda, p) \subseteq (G, \lambda)$  then, whenever  $t_\nu^p = o(1)$ ,  $G_\lambda(h)$  must exist for each  $h$  in some interval  $(0, H)$  (i.e. the series defining it must converge) and  $G_\lambda(h) = o(1)$  as  $h \rightarrow 0+$ . It then follows, on letting  $n \rightarrow \infty$  in (61), that

$$G_\lambda(h) = \sum_{\nu=0}^\infty \gamma_{h,\nu}^p t_\nu^p \quad (0 < h < H)$$

and that this transformation must be regular. But then, by the Toeplitz theorem, it is necessary that, for some  $h_0 \leq H$ ,

$$(63) \quad \sup_{0 < h < h_0} \sum_{\nu=0}^\infty |\gamma_{h,\nu}^p| = M < \infty.$$

Let us, for the moment, replace  $\lambda_\nu h$  by  $x_\nu$ ; let  $X > 0$  be arbitrarily given, let  $\{x_\nu\}$  denote a partition

$$0 = x_{-1} \leq x_0 < x_1 < \dots < x_{n+p} = X,$$

with norm

$$\delta = \max_{0 \leq v \leq n+p} (x_v - x_{v-1}),$$

and consider the sum

$$\begin{aligned} (64) \quad S(\{x_v\}, g, p) &= \sum_{v=0}^{n-1} |g[x_v, \dots, x_{v+p+1}](x_{v+p+1} - x_v) x_{v+1} \cdots x_{v+p}| \\ &= \sum_{v=0}^{n-1} |g[x_{v+1}, \dots, x_{v+p+1}] - g[x_v, \dots, x_{v+p}]| x_{v+1} \cdots x_{v+p} \\ &= \frac{1}{p!} \sum_{v=0}^{n-1} |g^{(p)}(\xi_{v+1}) - g^{(p)}(\xi_v)| x_{v+1} \cdots x_{v+p}, \end{aligned}$$

where  $x_v < \xi_v < x_{v+p}$  ( $v = 0, 1, \dots, n$ ) for  $p \geq 1$ . Ideally, we should like this last sum to be a Riemann-Stieltjes sum for the integral  $\int_0^X x^p |dg^{(p)}(x)|$ , but we cannot deduce this at once, since  $\{\xi_v\}$  is not necessarily monotonic, nor do  $x_{v+1}, \dots, x_{v+p}$  necessarily lie between  $\xi_v$  and  $\xi_{v+1}$ . Accordingly, we let  $N = N(n)$  be the integer such that  $Np \leq n < (N+1)p$ ; then we note that, when  $p \geq 1$ ,

$$0 = x_{-1} \leq x_0 < \xi_0 < x_p < \xi_p < x_{2p} < \xi_{2p} < \dots < \xi_{Np} < x_{(N+1)p} \leq x_{n+p} = X.$$

Now, for  $(r-1)p \leq v < rp$ , we have  $x_{v+1} \cdots x_{v+p} \geq x_{(r-1)p}^p > x_{(r-1)p}^p$ , and hence

$$\begin{aligned} (65) \quad \sum_{r=1}^N |g^{(p)}(\xi_{rp}) - g^{(p)}(\xi_{(r-1)p})| x_{(r-1)p}^p \\ = \sum_{r=1}^N x_{(r-1)p}^p \left| \sum_{v=(r-1)p}^{rp-1} \{g^{(p)}(\xi_{v+1}) - g^{(p)}(\xi_v)\} \right| \\ \leq \sum_{v=0}^{Np-1} |g^{(p)}(\xi_{v+1}) - g^{(p)}(\xi_v)| x_{v+1} \cdots x_{v+p} \\ \leq p! S(\{x_v\}, g, p). \end{aligned}$$

Since  $\max_{0 \leq v \leq n+p} (x_v - x_{v-1}) = \delta$ , we have  $x_{rp} - x_{(r-1)p} \leq p\delta$  and

$$x_{rp}^p - x_{(r-1)p}^p < p x_{rp}^{p-1} (x_{rp} - x_{(r-1)p}) < p^2 X^{p-1} \delta = K\delta \quad (r = 1, 2, \dots, N);$$

thus since, by hypothesis,  $g^{(p)}$  is of bounded variation in  $[0, X]$ , it follows that, given  $\epsilon > 0$ ,

$$(66) \quad \sum_{r=1}^N |g^{(p)}(\xi_{rp}) - g^{(p)}(\xi_{(r-1)p})| (x_{rp}^p - x_{(r-1)p}^p) \\ \leq K\delta \int_0^{\xi_{Np}} |dg^{(p)}(x)| < \varepsilon \text{ for } \delta < \eta_1(\varepsilon).$$

Further,  $X - \xi_{Np} < x_{n+p} - x_{np} < 2p\delta$  and  $\xi_0 < x_p \leq (p+1)\delta$ , whence

$$(67) \quad |g^{(p)}(\xi_0) - g^{(p)}(0)| x_0^p \leq X^p \int_0^{\xi_0} |dg^{(p)}(x)| \\ \leq X^p \int_0^{(p+1)\delta} |dg^{(p)}(x)| < \varepsilon \text{ for } \delta < \eta_2(\varepsilon),$$

$$(68) \quad |g^{(p)}(X) - g^{(p)}(\xi_{Np})| x_{(N+1)p}^p \leq X^p \int_{\xi_{Np}}^X |dg^{(p)}(x)| \\ \leq X^p \int_{X-2p\delta}^X |dg^{(p)}(x)| < \varepsilon \text{ for } \delta < \eta_3(\varepsilon).$$

Now define  $\xi_{-p} = 0$ ,  $\xi_{(N+1)p} = X$ , and note that

$$\xi_{(r-1)p} \leq x_{rp} \leq \xi_{rp}, \quad \xi_{rp} - \xi_{(r-1)p} < 2p\delta \quad (r=0, 1, \dots, N+1).$$

Then, since  $\int_0^X x^p |dg^{(p)}(x)|$  exists as a Riemann-Stieltjes integral ( $g^{(p)}$  is of bounded variation and  $x^p$  is continuous), it follows that

$$(69) \quad \left| \sum_{r=0}^{N+1} |g^{(p)}(\xi_{rp}) - g^{(p)}(\xi_{(r-1)p})| x_{rp}^p - \int_0^X x^p |dg^{(p)}(x)| \right| < \varepsilon \text{ for } \delta < \eta_4(\varepsilon).$$

Writing  $\eta(\varepsilon) = \min(\eta_1, \eta_2, \eta_3, \eta_4)$ , a combination of (65)-(69) now yields at once, for a positive integer  $p$ ,

$$(70) \quad p! S(\{x_\nu\}, g, p) \geq \int_0^X x^p |dg^{(p)}(x)| - 4\varepsilon \text{ for } \delta < \eta(\varepsilon).$$

In the case  $p=0$ ,  $S(\{x_\nu\}, g, 0) = \sum_{\nu=0}^{n-1} |g(x_{\nu+1}) - g(x_\nu)|$  and (70) follows trivially.

We shall reach the required conclusion of the theorem by combining (63) and (70), with a suitable choice of  $\{x_\nu\}$ .

Employing the hypothesis (8), there is an increasing sequence  $\{n_i\}$  such that

$$(71) \quad \delta^{(i)} \equiv \frac{1}{\lambda_{n_i+p}} \max_{1 \leq v \leq n_i+p} (\lambda_v - \lambda_{v-1}) \rightarrow 0 \quad \text{as } i \rightarrow \infty.$$

With  $X > 0$  arbitrarily fixed, let  $\{h_i\}$  be the sequence

$$h_i = X/\lambda_{n_i+p} \quad (i = 1, 2, \dots);$$

then  $0 < h_i < h_0$  for  $i > i_0$ , and  $h_i \rightarrow 0+$ . Write

$$x_v^{(i)} = \lambda_v h_i \quad (0 \leq v \leq n_i + p);$$

then, for each  $i$ ,  $\{x_v^{(i)}\}$  forms a partition of  $[0, X]$  with norm  $\delta = \delta^{(i)}X$ , where  $\delta^{(i)}$  is given by (71); and hence, given  $\eta(\varepsilon) > 0$ , we can find  $I(\varepsilon) > i_0$  such that  $\delta < \eta(\varepsilon)$  for  $i > I(\varepsilon)$ . Then, from (62), (63), (64),

$$\begin{aligned} M &\geq \sup_{i > i_0} \sum_{v=0}^{\infty} |\gamma_{h_i, v}^p| \geq \sup_{i > i_0} S(\{x_v^{(i)}\}, g, p) \\ &\geq S(\{x_v^{(i)}\}, g, p) \quad \text{for any } i > i_0 \\ &\geq \frac{1}{p!} \int_0^X x^p |d g^{(p)}(x)| - \frac{4}{p!} \varepsilon \quad \text{for } i > I(\varepsilon), \text{ by (70).} \end{aligned}$$

Since  $M$  and the integral are independent of  $i$ , it now follows that

$$\int_0^X x^p |d g^{(p)}(x)| \leq p! M$$

for arbitrary  $X > 0$ ; thus, letting  $X \rightarrow \infty$ , we obtain

$$\int_0^\infty x^p |d g^{(p)}(x)| \leq p! M$$

and this completes the proof of the theorem.

We may choose, from the diagram of implications given in §2, conditions on  $\lambda$  which imply those postulated in Theorem 7—for example,

$$[(7)] \quad \lambda_{n+1}/\lambda_n \rightarrow 1$$

implies (3) and (8). Or

$$[(4)] \quad 0 < a' \leq \lambda_{n+1} - \lambda_n \leq b' < \infty$$

implies (3), (8) and (9) — and, incidentally (Corollary 5A)  $(C, \lambda, p)$  and  $(R, \lambda, p)$  are then equivalent. Though (4) is a somewhat restrictive condition, nonetheless we obtain a non-trivial corollary by using it in Theorems 6 and 7; thus:

COROLLARY 7. *Let  $\mu \geq p \geq 0$ ,  $g$  satisfy (53), (54), (55), and  $\lambda$  satisfy (4). Then  $(R, \lambda, p) \subseteq (G, \lambda)$  if and only if  $\int x^p |d g^{(p)}(x)| < \infty$ .*

A special case of Corollary 7 would be given by  $\lambda_n = n$ , and we should then obtain a necessary and sufficient condition in order that  $(C, p) \subseteq (G, n)$ . The special case of Theorems 6 and 7 in which  $(G, \lambda)$  is the Riemann method  $(\mathfrak{R}, \lambda, \mu)$  has been examined in more detail in [24] and [25].

It would be of interest if some of the results of the present paper could be extended, at the cost of minimal additional restrictions on  $\lambda$ , to non-integral values of  $p$ ; or if some of the results of Jurkat [8] in this direction could be obtained with lighter restrictions on  $\lambda$ .

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YORK UNIVERSITY,  
TORONTO 12, CANADA.