PARTIALLY CONFORMAL TRANSFORMATIONS WITH RESPECT TO $(m-1)$-DIMENSIONAL DISTRIBUTIONS OF m-DIMENSIONAL RIEMANNIAN MANIFOLDS

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This paper is devoted to the geometry of transformations which have deep relation with an $(m-1)$-dimensional distribution D of m-dimensional Riemannian manifold M. A diffeomorphism φ of M to another m-dimensional Riemannian manifold N induces a mapping of the tangent space at any point x of M to that at φx of N. Also φ induces a mapping φ_D of the $(m-1)$-dimensional tangent subspace defined by D to that defined by φD. If φ_D is conformal for any point x of M, then we call φ an $(m-1)$-conformal transformation of M to N with respect to the distribution D. A conformal transformation in usual sense is of course an $(m-1)$-conformal transformation. An $(m-1)$-homothetic, or $(m-1)$-isometric transformation is naturally defined by its restriction to D. We denote by $D^\perp$ the orthocomplementary distribution to D. If an $(m-1)$-conformal transformation φ maps $D^\perp$ to $(\varphi D)^\perp$, then it is called special and denoted by an $(m-1)^s$-conformal transformation. As $D^\perp$ does not always admit a globally defined unit vector field ζ such that $\zeta_x \in D_x^\perp$ at every point x of M, we introduce a symbol ${}^{\varepsilon}\zeta$ (cf. §1). By this ${}^{\varepsilon}\zeta$ we can obtain the equation which characterizes an $(m-1)$-conformal transformation. An $(m-1)$-conformal transformation of M onto itself which preserves D is denoted by an $[m-1]$-conformal transformation.

Examples of such transformation appeared already in the theory of almost contact metric structures. As an almost contact Riemannian manifold admits a globally defined unit vector field ξ (see [14], [15] etc.), we can consider the orthogonal distribution to ξ. And a ϕ-preserving transformation of a contact Riemannian manifold is, in fact, an $[m-1]^s$-homothety ([8, 9], [17~20], etc.). Further, the existence of such transformations on certain contact Riemannian manifold characterizes the structure of the manifold itself ([19, 20]).

A trivial example is as follows: Let M, N be two Riemannian manifolds with metrics g, h respectively and denote by R a real line, then we can define Riemannian metrics on $M\times R$, $N\times R$ by $g+k$, $h+k$ respectively, where k is the usual metric on R. If $\varphi_0 : M \to N$ is a conformal transformation and $f: R\to R$ is an arbitrary diffeomorphism, then $\varphi : M\times R \to N\times R$ defined by $\varphi(x, t) = (\varphi_0 x, f(t))$, $t\in R$, is an $[m-1]^s$-conformal transformation.

In §1, we give the equation of an $(m-1)$-conformal transformation φ and the condition of speciality of $(m-1)$-conformal transformation. In §§4~6, we calculate φ-image of the Christoffel's symbol, the Riemannian curvature, the Ricci curvature and scalar curvature. Using these we investigate the properties of φ under the additional conditions on φ or on manifolds. In §7, we assume that ${}^{\varepsilon}\zeta$ is parallel along D and φ is an $(m-1)^s$-homothety, and have a relation of the sectional curvatures (Theorem 7.5). In §9, we consider the group Π of all $[m-1]$-conformal transformations and its subgroups. It is known that the set of all conformal transformations of a Riemannian manifold is a Lie group ([4]). But generally Π is not finite dimensional, so we want to find out the conditions on the manifold and a subgroup of Π so that the subgroup is a Lie group. And some answers are given in §15 (Theorem 15.9, 15.12).

Chapter II contains some studies of infinitesimal $(m-1)$-conformal transformations. The properties of conformal or infinitesimal conformal transformations (or homothetic, or isometric ones) of Riemannian manifolds are studied by many authors ([1], [3], [10], [16], [22], [23], etc.). In §16, we consider the case where M is compact and the scalar curvature is constant and obtain analogous results. Other extended investigation will be seen in other papers.

Contents are as follows:

Chapter I

Chapter II

Chapter I

1. Definition of an (m−1)-conformal transformation. Let M, N be two connected m-dimensional Riemannian manifolds ($m \geqq 3$) of class C^∞ and g, h two Riemannian metrics of M, N respectively. First we assume that M admits an $(m-1)$-dimensional distribution D, and we fix D throughout the paper. Then we also have an orthocomplementary 1-dimensional distribution. Now we consider a diffeomorphism $\varphi: M \to N$, and denote by φ^* the dual map of φ.

DEFINITION. If a diffeomorphism $\varphi: M \to N$ satisfies the following relation

$$(1.1) \qquad (\varphi^* h)_x(u, v) = \alpha(x)\, g_x(u, v)$$

for any point $x \in M$ and vector fields u, v on M such that $u, v \in D$, where α is a differentiable function on M, then we call φ an *$(m-1)$-conformal transformation* of M to N with respect to D. If α is constant, φ is an *$(m-1)$-homothety*. Furthermore if $\alpha=1$, φ is an *$(m-1)$-isometry*.

In order to express an $(m-1)$-conformal transformation by a tensor equation, let x be an arbitrary point of M. Then we can find an open neighborhood U of x and a vector field ζ_U on U, such that ζ_U is orthocomplementary to D and a unit vector field i.e., $g(\zeta_U, \zeta_U) = 1$. For some open covering $\{U\}$ of M we can define $\{\zeta_U\}$ corresponding ζ_U to each U in such a way that $\zeta_U = \zeta_V$ or $-\zeta_V$ holds on the intersection $U \cap V$, if it is not empty. $\{\zeta_U\}$ or its subfamily does not always define a vector field on M, so we use the symbol ${}^\varepsilon\zeta = \{\zeta_U\}$. We refer frequently this fixed covering $\{U\}$ in the sequel.

On each neighborhood U, we define a 1-form w_U by

$$(1.2) \qquad w_U(u) = g(\zeta_U, u)$$

for any vector field u on M. Then we have

$$(1.3) \qquad w_U(\zeta_U) = 1$$

and $w_U = 0$ is the equation of the distribution D. Similarly we use the notation ${}^\varepsilon w = \{w_U\}$.

We put $E = \varphi^* h - \alpha g$, clearly $E(u, v) = 0$ if ${}^\varepsilon w(u) = {}^\varepsilon w(v) = 0$. Next we put $E({}^{\varepsilon}\zeta, {}^{\varepsilon}\zeta) = \beta$, then β is a globally defined scalar field because it contains two ε's, and we define ${}^{\varepsilon}\theta$ by

$$(1.4) \qquad {}^{\varepsilon}\theta(u) = E(u, {}^{\varepsilon}\zeta) - \beta\, {}^{\varepsilon}w(u)$$

for any vector field u on M. ${}^{\varepsilon}\theta$ is determined as follows: Let x be any point of M then we have some open neighborhood $U \in \{U\}$ of x and ζ_U, w_U on U. These ζ_U, w_U determine θ_U of ${}^{\varepsilon}\theta$ on U. From the definition of E and ${}^{\varepsilon}\theta$, we see that

$$(1.5) \qquad {}^{\varepsilon}\theta({}^{\varepsilon}\zeta) = 0 .$$

Then we can verify

$$(1.6) \qquad \varphi^*h = \alpha g + {}^{\varepsilon}w \otimes {}^{\varepsilon}\theta + {}^{\varepsilon}\theta \otimes {}^{\varepsilon}w + \beta\, {}^{\varepsilon}w \otimes {}^{\varepsilon}w ,$$

where $\otimes$ means the tensor product. To see this, it is enough to compare both sides substituting the pairs $({}^{\varepsilon}\zeta, {}^{\varepsilon}\zeta)$, $({}^{\varepsilon}\zeta, u)$ and (u, v) where u and v are vector fields which belong to the distribution D. Though ${}^{\varepsilon}\zeta$, ${}^{\varepsilon}w$ and ${}^{\varepsilon}\theta$ are not tensor fields, restricting ourselves to some neighborhood we consider (1.6) as a tensor equation. Of course, for $U, V \in \{U\}$, the expressions $(1.6)_U$ in U and $(1.6)_V$ in V are equivalent, because ε's appear twice in the last three terms. It is evident that the decomposition of φ^*h given by (1.6) is unique in the sense that (1.5) holds. From the definition of β it follows that

$$(1.7) \qquad \alpha + \beta = h(\varphi\, {}^{\varepsilon}\zeta, \varphi\, {}^{\varepsilon}\zeta) \cdot \varphi ,$$

where we have used, and shall use φ to denote also the differential of φ.

As for the distribution φD induced by φ on N, one has ${}^{\delta}\xi$ and ${}^{\delta}\eta$ on N similar to ${}^{\varepsilon}\zeta$ and ${}^{\varepsilon}w$ on M, satisfying

$$(1.8) \qquad {}^{\delta}\eta({}^{\delta}\xi) = 1 , \qquad {}^{\delta}\eta(X) = h({}^{\delta}\xi, X)$$

for any vector field X on N. We also fix a covering $\{'V\}$ of N.

LEMMA 1.1. *For any $(m-1)$-conformal transformation φ, we have $\alpha > 0$ and $\alpha + \beta > 0$. And the following conditions are eqivalent*:

(i) ${}^{\varepsilon}\theta = 0$.

(ii) $\varphi\, {}^{\varepsilon}\zeta = {}^{\varepsilon\delta}\mu\, {}^{\delta}\xi$, *for some* ${}^{\varepsilon\delta}\mu$.

In the above lemma, ${}^{\varepsilon\delta}\mu$ is a symbol of $\{\mu_{U'V}\}$, and $\mu_{U'V}$ is a differentiable function on $\varphi U \cap 'V$. Namely for $x \in M$, if $y = \varphi x$, we take some neighborhoods

U of x and $'V$ of y, then $\varphi\zeta_U = \mu_{U'V}\xi_{'V}$ on $\varphi U \cap 'V$. In this case by (1.7), we see that $\mu^2_{U'V}(\varphi p) = \alpha(p)+\beta(p)$ for $p \in U \cap \varphi^{-1}('V)$.

PROOF. Following the above notation, we show (i) $\rightarrow$ (ii). By the definition of φD, we have $\varphi^* \eta_{'V} = \gamma_{'VU} w_U$ for some differentiable function $\gamma_{'VU}$ on $U \cap \varphi^{-1}('V)$. In the following we write this relation by

$$\varphi^{* \delta}\eta = {}^{\delta\varepsilon}\gamma\, {}^{\varepsilon}w\,. \tag{1.9}$$

Let u be any vector field on M such that ${}^{\varepsilon}w(u) = 0$. If ${}^{\varepsilon}\theta = 0$ holds in (1.6), then we have

$$h(\varphi\, {}^{\varepsilon}\zeta, \varphi u) = (\varphi^* h)({}^{\varepsilon}\zeta, u) = 0\,.$$

Thus $\varphi^{\varepsilon}\zeta$ is orthocomplementary to φD. This means that $\varphi^{\varepsilon}\zeta$ is proportional to ${}^{\delta}\xi$. And by (1.9), we have $\varphi^{\varepsilon}\zeta = ({}^{\delta\varepsilon}\gamma \cdot \varphi^{-1})^{\delta}\xi$. That is, we get

$$\varphi\, {}^{\varepsilon}\zeta = {}^{\varepsilon\delta}\mu\, {}^{\delta}\xi\,,$$

where

$${}^{\varepsilon\delta}\mu = {}^{\delta\varepsilon}\gamma \cdot \varphi^{-1}\,, \quad {}^{\delta\varepsilon}\gamma^2 = \alpha+\beta\,. \tag{1.10}$$

In the next place we prove (ii)$\rightarrow$(i). (1.4) means that

$$\begin{aligned} {}^{\varepsilon}\theta(u) &= h(\varphi u, \varphi^{\varepsilon}\zeta)\cdot\varphi - \alpha g(u, {}^{\varepsilon}\zeta) - \beta\, {}^{\varepsilon}w(u) \\ &= h(\varphi u, {}^{\varepsilon\delta}\mu\, {}^{\delta}\xi)\cdot\varphi \\ &= 0 \end{aligned}$$

for any vector field $u \in D$. This completes the proof.

PROPOSITION 1.2. *Let φ be an $(m-1)$-conformal transformations of M to N, and let ${}^{\delta\varepsilon}K_N$, ${}^{\delta\varepsilon}K_M$ be angles determined by $\varphi\, {}^{\varepsilon}\zeta$ and ${}^{\delta}\xi$, ${}^{\varepsilon}\zeta$ and $\varphi^{-1\,\delta}\xi$ respectively. Then we have*

$$\cos {}^{\delta\varepsilon}K_N = \frac{{}^{\delta\varepsilon}\gamma}{\sqrt{\alpha+\beta}}\, \varphi^{-1}\,, \tag{1.11}$$

$$\cos {}^{\delta\varepsilon}K_M = \mathrm{sgn}({}^{\delta\varepsilon}\gamma)\left(\frac{\alpha}{2\alpha+\beta-{}^{\delta\varepsilon}\gamma^2}\right)^{\frac{1}{2}}\,. \tag{1.12}$$

PROOF. In the formula

$$\cos {}^{\delta\varepsilon}K_N = \frac{h(\varphi^{\varepsilon}\zeta, {}^{\delta}\xi)}{\sqrt{h(\varphi^{\varepsilon}\zeta, \varphi^{\varepsilon}\zeta)}} \tag{1.13}$$

we substitute $(\varphi^*h)(\zeta, \zeta) = \alpha+\beta$ and

$$\begin{aligned} h(\varphi^{\varepsilon}\zeta, {}^{\delta}\xi) &= {}^{\delta}\eta(\varphi^{\varepsilon}\zeta) \\ &= (\varphi^{*\delta}\eta)({}^{\varepsilon}\zeta)\cdot\varphi^{-1} = {}^{\delta\varepsilon}\gamma\cdot\varphi^{-1}, \end{aligned}$$

then we get (1.11). Similarly e have the formula

$$\cos {}^{\delta\varepsilon}K = \frac{g({}^{\varepsilon}\zeta, \varphi^{-1\,\delta}\xi)}{\sqrt{g(\varphi^{-1\,\delta}\xi, \varphi^{-1\delta}\xi)}}. \tag{1.14}$$

First we have

$$\begin{aligned} g({}^{\varepsilon}\zeta, \varphi^{-1\,\delta}\xi) &= {}^{\varepsilon}w(\varphi^{-1\,\delta}\xi) \\ &= (\varphi^{-1*\,\varepsilon}w)({}^{\delta}\xi)\cdot\varphi = {}^{\delta\varepsilon}\gamma^{-1}. \end{aligned}$$

And in order to estimate $g(\varphi^{-1\,\delta}\xi, \varphi^{-1\,\delta}\xi)$, we decompose $\varphi^{-1\,\delta}\xi$ into orthogonal components as follows:

$$\varphi^{-1\,\delta}\xi = {}^{\varepsilon}w(\varphi^{-1\,\delta}\xi)\,{}^{\varepsilon}\zeta + (\varphi^{-1\,\delta}\xi - {}^{\varepsilon}w(\varphi^{-1\,\delta}\xi)\,{}^{\varepsilon}\zeta). \tag{1.15}$$

Then we have

$$g(\varphi^{-1\,\delta}\xi, \varphi^{-1\,\delta}\xi) = ({}^{\varepsilon}w(\varphi^{-1\,\delta}\xi))^2 + g(\varphi^{-1\,\delta}\xi - {}^{\varepsilon}w(\varphi^{-1\,\delta}\xi)\,{}^{\varepsilon}\zeta, \varphi^{-1\,\delta}\xi - {}^{\varepsilon}w(\varphi^{-1\,\delta}\xi)\,{}^{\varepsilon}\zeta),$$

where

$$({}^{\varepsilon}w(\varphi^{-1\,\delta}\xi))^2 = ((\varphi^{-1*\varepsilon}w)({}^{\delta}\xi))^2 = ({}^{\delta\varepsilon}\gamma)^{-2},$$

and

$$\begin{aligned} &\alpha g(\varphi^{-1\delta}\xi - {}^{\varepsilon}w(\varphi^{-1\delta}\xi)\,{}^{\varepsilon}\zeta, \varphi^{-1\delta}\xi - {}^{\varepsilon}w(\varphi^{-1\delta}\xi)\,{}^{\varepsilon}\zeta) \\ &\qquad = h({}^{\delta}\xi - \varphi({}^{\delta\varepsilon}\gamma^{-1}\cdot{}^{\varepsilon}\zeta), {}^{\delta}\xi - \varphi({}^{\delta\varepsilon}\gamma^{-1}\cdot{}^{\varepsilon}\zeta))\cdot\varphi \end{aligned}$$

since the 2nd term of the right hand side of (1.15) belongs to the distribution D. Then we have

$$h({}^{\delta}\xi, {}^{\delta}\xi) \cdot \varphi - 2({}^{\delta\varepsilon}\gamma)^{-1}\, h({}^{\delta}\xi, \varphi^{\varepsilon}\zeta) \cdot \varphi + ({}^{\delta\varepsilon}\gamma)^{-2}\, h(\varphi^{\varepsilon}\zeta, \varphi^{\varepsilon}\zeta) \cdot \varphi$$
$$= 1 - 2({}^{\delta\varepsilon}\gamma)^{-1}\, {}^{\delta}\eta(\varphi^{\varepsilon}\zeta) \cdot \varphi + ({}^{\delta\varepsilon}\gamma)^{-2}(\alpha+\beta)$$
$$= -1 + ({}^{\delta\varepsilon}\gamma)^{-2}(\alpha+\beta)\,.$$

Therefore, subsituting these into (1.14), we get (1.12). q.e.d.

It is geometrically obvious that φ^{-1} is also an $(m-1)$-conformal transformation of N to M with respect to the distribution φD, applying φ^{-1*} to (1.6) we have

$$h = (\alpha \cdot \varphi^{-1})\, \varphi^{-1*} g + \varphi^{-1*\varepsilon} w \otimes \varphi^{-1*\varepsilon}\theta + \varphi^{-1*\varepsilon}\theta \otimes \varphi^{-1*\varepsilon} w$$
$$+ (\beta \cdot \varphi^{-1})\, \varphi^{-1*\varepsilon} w \otimes \varphi^{-1*\varepsilon} w\,.$$

Hence

$$\varphi^{-1*} g = \left(\frac{1}{\alpha} \cdot \varphi^{-1}\right) h + {}^{\delta}\eta \otimes {}^{\delta}\lambda + {}^{\delta}\lambda \otimes {}^{\delta}\eta + \rho\, {}^{\delta}\eta \otimes {}^{\delta}\eta\,, \tag{1.16}$$

where we have put

$$ {}^{\delta}\lambda = -(\alpha^{-1}\, {}^{\delta\varepsilon}\gamma^{-1} \cdot \varphi^{-1})(\varphi^{-1*\varepsilon}\theta - {}^{\varepsilon}\theta(\varphi^{-1\,\delta}\xi)\, {}^{\delta}\eta)\,, \tag{1.17}$$

$$\rho = -(\alpha^{-1}\beta\, {}^{\delta\varepsilon}\gamma^{-2} \cdot \varphi^{-1}) - 2(\alpha^{-1}\, {}^{\delta\varepsilon}\gamma^{-1} \cdot \varphi^{-1})\, {}^{\varepsilon}\theta(\varphi^{-1\,\delta}\xi) \cdot \varphi^{-1}\,. \tag{1.18}$$

The right hand side of (1.17) contains ε, and so ${}^{\delta}\lambda$ may be written formally as ${}^{\varepsilon\delta}\lambda$. However ε appears twice in each term. Thus ${}^{\varepsilon\delta}\lambda$ does not depend on the choice of the neighborhood $U \in \{U\}$. Therefore we can omit ε from ${}^{\varepsilon\delta}\lambda$. Similarly the right hand side of (1.18) contains ε and δ twice in each term respectively. So ρ does not depend on the choice of neighborhoods.

DEFINITION. The most standard $(m-1)$-conformal transformation of M to N is one which satisfies ${}^{\varepsilon}\theta = 0$, we call such an $(m-1)$-conformal transformation a *special* $(m-1)$*-conformal transformation* and we denote it by an $(m-1)^{s}$*-conformal transformation.*

DEFINITION. If we consider an $(m-1)$-conformal transformation φ of M onto itself, we sometimes assume that φ preserves the distribution. And we denote such φ by an $[m-1]$*-conformal transformation.* Namely by an $[m-1]$-conformal transformation of M onto M we understand an $(m-1)$-conformal transformation such that $\varphi D = D$.

ABBREVIATION. In the subsequent sections, we abbreviate ε or δ in ${}^{\varepsilon}\zeta, {}^{\delta}\xi, {}^{\varepsilon\delta}\gamma, \cdots$, frequently in the case where there is no confusion.

2. Commutability of an $(m-1)^s$-homothety and the parallel translations. In this section, we study some properties of the $(m-1)^s$-homothety of M to N satisfying some additional conditions, concerning the parallel translations with respect to the Riemannian connections. We denote by τ and ∇, $'\tau$ and $'\nabla$ the parallel translations along certain curves and covariant differentiations with respect to the Riemannian connections for g, h respectively. We utilize the fundamental formula:

$$(2.1)\qquad 2g(\nabla_X Z, Y) = X\cdot g(Y,Z) + Z\cdot g(Y,X) - Y\cdot g(Z,X) + g(X,[Y,Z]) + g(Z,[Y,X]) - g(Y,[Z,X])$$

for any vector fields X, Y and Z.

THEOREM 2.1. *Let φ be an $(m-1)^s$-homothety of M to N and suppose that the distribution D in M is completely integrable. If a curve $l=\{l_t: 0\leqq t\leqq 1\}$ in M, joining two points l_0 and l_1, is a segment of an integral curve of the distribution D. And if u_{l_0} is a tangent vector at l_0 which belongs to D_{l_0} and $\tau_{l(t)}u_{l_0}\in D_{l_t}$ for any $t: 0\leqq t\leqq 1$, $l(t)=\{l_s: 0\leqq s\leqq t\}$, then we have*

$$\varphi_{l_1}\tau_l u_{l_0} = {}'\tau_{\varphi l}\varphi_{l_0}u_{l_0}\,.$$

PROOF. We can assume that l does not have any self-intersecting point. Let u be a vector field on M such that u coincides with $u_{l_t}=\tau_{l(t)}u_{l_0}$ on l_t: $0\leqq t\leqq 1$, and belongs to D. And let v be a vector field on M such that v is tangential to the curve l and belongs to D.

Of course, such u, v exist. In fact, let $\bar{u}$ be any vector field on M which coincides with u_{l_t} on l_t, then $u=\bar{u}-{}^{\varepsilon}w(\bar{u})^{\varepsilon}\zeta$ satisfies the required property. In this case ${}^{\varepsilon}w(\bar{u})^{\varepsilon}\zeta$ is a globally defined vector field, since it has two ε's.

Now, in (2.1) we set $X=\varphi v$, $Z=\varphi u$ and replace g by h, then we have

$$(2.2)\qquad 2h({}'\nabla_{\varphi v}\varphi u, Y) = \varphi v\cdot h(Y,\varphi u) + \varphi u\cdot h(Y,\varphi v) - Y\cdot h(\varphi u,\varphi v) + h(\varphi v,[Y,\varphi u]) + h(\varphi u,[Y,\varphi v]) - h(Y,[\varphi u,\varphi v])$$

for any vector field Y on N. By the assumption ${}^{\varepsilon}\theta=0$ and $w(u)=0$, and by (1.6), we have

$$\varphi v \cdot h(Y, \varphi u) = v \cdot (\varphi^* h)(\varphi^{-1} Y, u) \cdot \varphi^{-1}$$
$$= v \cdot (\alpha g(\varphi^{-1} Y, u)) \cdot \varphi^{-1} .$$

And we have

$$h(\varphi v, [Y, \varphi u]) = (\varphi^* h)(v, [\varphi^{-1} Y, u]) \cdot \varphi^{-1}$$
$$= \alpha g(v, [\varphi^{-1} Y, u]) \cdot \varphi^{-1} ,$$

$$(2.3) \qquad h(Y, [\varphi u, \varphi v]) = \alpha g(\varphi^{-1} Y, [u, v]) \cdot \varphi^{-1} + \beta w(\varphi^{-1} Y) w([u, v]) \cdot \varphi^{-1} .$$

As the distribution D is completely integrable, $w([u, v]) = 0$ holds good, and so

$$(2.4) \qquad h('\nabla_{\varphi v} \varphi u, Y) = \alpha g(\nabla_v u, \varphi^{-1} Y) \cdot \varphi^{-1} .$$

If u is parallel along l, $\nabla_v u = 0$ holds on l and we have $'\nabla_{\varphi v} \varphi u = 0$ on φl. q.e.d.

As a natural consequence, we see that, under the assumption in Theorem 2.1, if l is not only an integral curve of D, but also a geodesic, then φl is also geodesic. However this holds without the asumption of the complete integrability of D.

THEOREM 2.2. *Let φ be an $(m-1)^s$-homothety. If l is an integral curve of D and geodesic with respect to g, then φl is an integral curve of φD and geodesic with respect to h.*

PROOF. If l is a geodesic, in the above proof we may assume that $u = v$. In (2.2), we replace φu by φv, then (2.4) replaced u by v holds good, since the 2nd term of the right hand side of (2.3) is zero. And hence $\nabla_v v = 0$ on l means $'\nabla_{\varphi v} \varphi v = 0$ on φl.

THEOREM 2.3. *Suppose that the distribution D is completely integrable and each trajectory of ${}^\varepsilon\zeta$ is a geodesic. If an $(m-1)^s$-homothetic transformation φ satisfies β=constant. Then, denoting by $l = \{l_t : 0 \leqq t \leqq 1\}$ a segment of the trajectory of ${}^\varepsilon\zeta$, we have*

$$\varphi_{l_1} \tau_l u_{l_0} = {}' \tau_{\varphi l} \varphi_{l_0} u_{l_0}$$

for any tangent vector u_{l_0} at l_0 which belongs to D_{l_0}.

PROOF. As ζ is autoparallel and u_{l_0} is orthogonal to ζ_{l_0}, $\tau_{l(t)} u_{l_0}$ is also orthogonal to ζ_{l_t}. Let u be a vector field on M such that $u_{l_t} = \tau_{(t)} u_{l_0}$ and $w(u)=0$. Using (2.1) for $\varphi\zeta$ and φu, we get

$$2h('\nabla_{\varphi\zeta}\varphi u, Y)\cdot\varphi = 2\alpha g(\nabla_\zeta u, \varphi^{-1}Y) + \beta u\cdot w(\varphi^{-1}Y) + \beta w([\varphi^{-1}Y, u]) - \beta w(\varphi^{-1}Y)\, w([u,\zeta])$$

for any vector field Y on N. If we put $Y = \varphi v$, where v belongs to D, as D is integrable, we have

$$h('\nabla_{\varphi\zeta}\varphi u, \varphi v) = \alpha g(\nabla_\zeta u, v)\cdot\varphi^{-1}\,. \tag{2.5}$$

Next we put $Y=\varphi\zeta$, and notice that

$$w([\zeta,u]) = -L(\zeta)\, w\cdot u\,,$$

where $L(\zeta_U)$ means the operator of the Lie derivation with respect to ζ_U. It is known that $L(\zeta_U)w_U = 0$ if and only if each trajectory of ζ_U is a geodesic, since $w(\zeta)=1$. And so we have

$$h('\nabla_{\varphi\zeta}\varphi u, \varphi\zeta) = \alpha g(\nabla_\zeta u, \zeta)\cdot\varphi^{-1}\,. \tag{2.6}$$

By (2.5) and (2.6), $\nabla_\zeta u = 0$ on l means that $'\nabla_{\varphi\zeta}\varphi u = 0$ on φl.

THEOREM 2.4. *Suppose that the distribution D is completely integrable and $l=\{l_t : 0\leq t\leq 1\}$ is a segment of an integral curve of D. If an $(m-1)^s$-homothetic transformation φ satisfies β=constant and $^\varepsilon\zeta$ is parallel along l, then $^\delta\xi$ is parallel along φl.*

PROOF. Let v be a vector field stated in the proof of Theorem 2.1. By (2.1) we get

$$2h('\nabla_{\varphi v}\varphi\zeta, Y)\cdot\varphi = 2\alpha g(\nabla_v\zeta, \varphi^{-1}Y) + v\cdot\beta w(\varphi^{-1}Y) + \beta w([\varphi^{-1}Y, v]) - \beta w(\varphi^{-1}Y)\, w([\zeta, v])\,.$$

By the similar argument to the proof of Theorem 2.3 we have

$$h('\nabla_{\varphi v}\varphi\zeta, Y) = \alpha g(\nabla_v\zeta, \varphi^{-1}Y)\cdot\varphi^{-1}\,. \tag{2.7}$$

This completes the proof.

3. The case where each trajectory of ${}^{\varepsilon}\zeta$ is a geodesic. In this section we do not necessarily assume that α is constant.

THEOREM 3.1. *We assume that M and N admit an $(m-1)^s$-conformal transformation φ such that $\alpha+\beta$ is constant. If each trajectory of ${}^{\varepsilon}\zeta$ is a geodesic, each trajectory of ${}^{\delta}\xi$ is also geodesic.*

PROOF. In (2.1), putting $\varphi\zeta$ and Y, we get

$$2h('\nabla_{\varphi\zeta}\varphi\zeta, Y)\cdot\varphi = 2\zeta\cdot(\alpha+\beta)\, w(\varphi^{-1}Y) - \varphi^{-1}Y\cdot(\alpha+\beta) + 2(\alpha+\beta)\, w([\varphi^{-1}Y, \zeta])\,. \tag{3.1}$$

By the assumption that $\alpha+\beta$ is constant and that each trajectory of ζ is a geodesic, we see that the right hand side vanishes when we put $Y=\varphi\zeta$ and $Y=\varphi u$ respectively, u denoting a vector field which belongs to D. So we have $'\nabla_{\varphi\zeta}\varphi\zeta=0$. As $\varphi\zeta=\mu\xi$, $|\mu|^2=\alpha+\beta$, we see that $\nabla_{\xi}\xi=0$.

THEOREM 3.2. *Suppose that each trajectory of ${}^{\varepsilon}\zeta$ and ${}^{\delta}\xi$ is a geodesic. Let φ be an $(m-1)^s$-conformal transformation of M to N, and u be a vector field on M which belongs to D. Then we have $L(u)(\alpha+\beta)=0$.*

PROOF. We utilize (3.1) and putting $Y=\varphi u$, we have

$$2h('\nabla_{\varphi\zeta}\varphi\zeta, \varphi u)\cdot\varphi = -u\cdot(\alpha+\beta)\,. \tag{3.2}$$

On the other hand, as $(\varphi\zeta)_{\varphi x}=\mu_{\varphi x}\xi_{\varphi x}$, $\mu^2_{\varphi x}=(\alpha+\beta)_x$, $x\in M$, we get

$$'\nabla_{\varphi\zeta}\varphi\zeta = ('\nabla_{\varphi\xi}\mu)\,\xi\,, \tag{3.3}$$

where we have used $'\nabla_{\xi}\xi=0$. By (3.2) and (3.3), we have $u\cdot(\alpha+\beta)=0$.

PROPOSITION 3.3. *Suppose that each trajectory of ${}^{\varepsilon}\zeta$ and ${}^{\delta}\xi$ is a geodesic. Let φ be an $(m-1)^s$-conformal transformation of M to N such that $\zeta\cdot(\alpha+\beta)=0$. Then $\alpha+\beta$ is constant.*

PROOF. Any tangent vector v_x at $x\in M$ is written as

$$v_x = (v_x - w(v)\zeta_x) + w(v)\,\zeta_x\,,$$

where $v_x - w(v)\,\zeta_x\in D_x$. By Theorem 3.2 we have $(v_x - w(v)\,\zeta_x)(\alpha+\beta)=0\,.$

Thus we see that $\alpha+\beta$ is constant.

From Theorem 3.2 we see geometrically the following

PROPOSITION 3.4. *Suppose that each trajectory of* ${}^{\varepsilon}\zeta$ *and* ${}^{\delta}\xi$ *is a geodesic. Let* φ *be an* $(m-1)^s$*-conformal transformation of* M *to* N *and let* l *be a trajectory of* ${}^{\varepsilon}\zeta$. *If, for each* l *and for any points* $x, y \in l$, *we can join* x *and* y *by a piecewise differentiable integral curve of* D. *Then* $\alpha+\beta$ *is constant.*

4. **Transformation of the Christoffel's symbols.** Let φ be an $(m-1)$-conformal transformation of M to N and x be an arbitrary point of M and $y=\varphi x$. On some coordinate neighborhoods U of x and $'V$ of y, we have $w_U, \zeta_U, \theta_U, \xi_{'V}$ and we write them simply w, ζ, θ, ξ. We write their components w^i, ξ^α, etc. with respect to the local coordinates $x^i, y^\alpha : i, \alpha = 1, 2, \cdots, m$. For convienience, we write w^i for ζ^i sometimes. Let

$$G_{ij} = h_{\alpha\beta}\frac{\partial y^\alpha}{\partial x^i}\frac{\partial y^\beta}{\partial x^j},$$

then (1.6) is written as follows:

$$G_{ij} = \alpha g_{ij} + w_i\theta_j + \theta_i w_j + \beta w_i w_j\,. \tag{4.1}$$

We put $\nu = \theta_i\theta^i - \alpha(\alpha+\beta)$, where $\theta^i = g^{ij}\theta_j$ and g^{ij} is the inverse matrix of g_{ij}. Then the inverse matrix $(G^{-1})^{jk}$ of G_{ij} is given by

$$(G^{-1})^{jk} = \frac{1}{\alpha}g^{jk} + \frac{1}{\nu}(w^j\theta^k + \theta^j w^k) - \frac{1}{\alpha\nu}\theta^j\theta^k + \frac{1}{\nu}\left(\beta - \frac{r}{\alpha}\right)w^j w^k\,, \tag{4.2}$$

where $r = \theta_j\theta^j$. If $\theta=0$, (4.2) reduces to

$$(G^{-1})^{jk} = \frac{1}{\alpha}g^{jk} - \frac{\beta}{\alpha(\alpha+\beta)}w^j w^k\,. \tag{4.3}$$

Denoting by ${}^{\varphi}\left\{ {i \atop jk} \right\}$, $\left\{ {i \atop jk} \right\}$ the Christoffel's symbols with respect to G_{ij}, g_{ij} respectively, generally we see that

$$\nabla_k(\varphi^* h)_{ij} - (\varphi^*('\nabla h))_{kij} = \left({}^{\varphi}\left\{ {r \atop ik} \right\} - \left\{ {r \atop ik} \right\}\right)(\varphi^* h)_{rj} + \left({}^{\varphi}\left\{ {r \atop jk} \right\} - \left\{ {r \atop jk} \right\}\right)(\varphi^* h)_{ir} \tag{4.4}$$

holds good, where ∇ and $'\nabla$ are covariant differentiation with respect to g

and h. The second term of the left hand side of (4.4) vanishes. Making up (4.4) in the simplified form, we get

$$(4.5) \qquad 2\left({}^{\varphi}\left\{{r \atop jk}\right\} - \left\{{r \atop jk}\right\}\right)G_{ri} = \nabla_j G_{ki} + \nabla_k G_{ij} - \nabla_i G_{jk}\,.$$

We calculate ${}^{\varphi}\left\{{i \atop jk}\right\}$ by (4.1), (4.2) and (4.5) and we have

$$(4.6) \qquad 2\,{}^{\varphi}\left\{{i \atop jk}\right\} = 2\left\{{i \atop jk}\right\} + \frac{1}{\alpha}(\alpha_j \delta^i_k + \alpha_k \delta^i_j - \alpha^i g_{jk})$$
$$- \frac{1}{\alpha}\beta^i w_j w_k + \frac{2\beta}{\alpha} w_{(k}(w^i{}_{,j)} - w_{j)}{}^{,i})$$
$$- \frac{w^i}{v}\left\{\left(\beta - \frac{r}{\alpha}\right)(\zeta\alpha)\, g_{jk} + \left(\beta - \frac{r}{\alpha}\right)(\zeta\beta)\, w_j w_k\right.$$
$$- 2\left(\beta - \frac{r}{\alpha}\right)\alpha_{(j}w_{k)} + 2\alpha\beta_{(j}w_{k)} + 2\alpha\left(\beta - \frac{r}{\alpha}\right) w_{(j,k)}$$
$$\left. + 2\beta\left(\beta - \frac{r}{\alpha}\right) w_{(j}w_{k),s}w^s\right\} + ([\theta])\,,$$

where we have used the notations $\nabla_i w_j = w_{j,i}$, $\alpha_i = \partial\alpha/\partial x^i$, $\alpha^k = g^{ki}\alpha_i$, $\beta^k = g^{ki}\beta_i$, $\zeta\alpha = w^i\alpha_i$, and ( ) for indices means half of the sum of two terms interchanged two indices, for example

$$2w_{(k}w^i{}_{,j)} = w_k w^i{}_{,j} + w_j w^i{}_{,k}\,,$$

and finally we have put

$$(4.7) \qquad ([\theta]) = 2\alpha^{-1} g^{hi}\{w_{(k}(\theta_{|h|,j)} - \theta_{j),h}) + \theta_{(k}(w_{|h|,j)} - w_{j),h})\}$$
$$+ v^{-1} w^i\{2\alpha_{(j}\theta_{k)} - (\theta\alpha)g_{jk} - (\theta\beta)w_j w_k - 2\alpha\theta_{(j,k)}$$
$$- 2\beta w_{(j}(w_{k),s} - w_{|s|,k)})\theta^s$$
$$- 2w_{(j}(\theta_{k),s} - \theta_{|s|,k)})\theta^s$$
$$- 2\theta_{(j}(w_{k),s} - w_{|s|,k)})\theta^s$$
$$- 2(\beta - r\alpha^{-1})(\theta_{(j}w_{k),s}w^s + w_{(j}(\theta_{k),s} - \theta_{|s|,k)})\,w^s)\}$$
$$+ v^{-1}\theta^i\{(\alpha^{-1}\theta\alpha - \zeta\alpha)\, g_{jk} + (\alpha^{-1}\theta\beta - \zeta\beta)\, w_j w_k$$
$$+ 2\theta_{(j,k)} - 2\alpha w_{(j,k)} + 2\alpha_{(j}w_{k)} + 2\beta_{(j}w_{k)}$$

$$- 2\alpha^{-1}\alpha_{(j}\theta_{k)} - 2\beta w_{(j}w_{k),s}w^s$$
$$- 2(w_{(j}(\theta_{k),s} - \theta_{|s|,k)})w^s + \theta_{(j}w_{k),s}w^s)$$
$$+ 2\alpha^{-1}\beta w_{(j}(w_{k),s} - w_{|s|,k)})\theta^s$$
$$+ 2\alpha^{-1}(w_{(j}(\theta_{k),s} - \theta_{|s|,k)})\theta^s + \theta_{(j}(w_{k),s} - w_{|s|,k)})\theta^s)\} .$$

Contracting with respect to i and k, we have

$$2\,{}^{\varphi}\left\{ {k \atop jk} \right\} = 2\left\{ {k \atop jk} \right\} + (m\alpha^{-1} + \beta\nu^{-1} - 2r\alpha^{-1}\nu^{-1})\alpha_j - \alpha\nu^{-1}\beta_j + \nu^{-1}r_j . \tag{4.8}$$

THEOREM 4.1. *Let φ be an $(m-1)^s$-conformal transformation of M to N. If φ is an affine transformation, we have*
(1) *α and β are constant.*
And as a necessary condition that M and N admit such φ satisfying $\beta \neq 0$, we have
(2) *${}^e\zeta$ is a parallel field.*

PROOF. By the assumption the last term of the right hand side of (4.7) vanishes and ${}^{\varphi}\left\{ {i \atop jk} \right\} = \left\{ {i \atop jk} \right\}$ holds good. Transvecting (4.6) with $w^j w_i$, we get

$$\alpha_k + \beta_k = 0 . \tag{4.9}$$

And from (4.8), it follows that

$$(m(\alpha+\beta) - \beta)\alpha_k + \alpha\beta_k = 0 . \tag{4.10}$$

(4.9) and (4.10) give the following relation

$$(m-1)(\alpha+\beta)\alpha_k = 0 .$$

Thus α is constant and, by (4.9), β is also constant.

In the next place we prove (2). Transvecting (4.6) with $w^j w^k$ and using the fact that α and β are constant, we get

$$\beta w^t{}_{,j} w^j = 0 . \tag{4.11}$$

If $\beta \neq 0$, by (4.11) we get $w_{i,j}w^j = 0$. Transvecting (4.6) with w_i, we have

$$\beta(w_{j,k} + w_{k,j}) = 0 . \tag{4.12}$$

Thus ζ_U has a property of a Killing vector field. Transvecting (4. 6) with w^k we have

$$\beta(w_{i,j} - w_{j,i}) = 0 . \tag{4.13}$$

Hence ζ is a parallel field. q.e.d.

REMARK. In (2) of the Theorem 4.1, the assumption $\beta \neq 0$ means that φ is an essentially $(m-1)^s$-conformal transformation.

As a converse, next theorem follows from (4. 6) immediately.

THEOREM 4.2. *Suppose that φ is an $(m-1)^s$-homothetic transformation of M to N such that β is constant. If ${}^{\varepsilon}\zeta$ is a parallel field. Then φ is an affine transformation.*

5. $(m-1)^s$-conformal and projective transformation.

By definition a projective transformation φ of M to N is one which transforms the system of geodesics in M into the same system in N. Namely as a necessary and sufficient condition that φ is a projective transformation we have

$$2\,{}^{\varphi}\left\{ {i \atop jk} \right\} - 2\left\{ {i \atop jk} \right\} = 2\delta^i_j \psi_k + 2\delta^i_k \psi_j \tag{5.1}$$

where ψ is a differentiable function on M.

Suppose that φ is an $(m-1)^s$-conformal and at the same time projective transformation. Contracting with respect to i and j in (5. 1) and using (4. 8), we get

$$2(m+1)\psi_k = \frac{m(\alpha+\beta)-\beta}{\alpha(\alpha+\beta)}\alpha_k + \frac{1}{\alpha+\beta}\beta_k . \tag{5.2}$$

Thus, if α and β are constant, $\psi_k = 0$ holds good and we see that φ is an affine transformation.

PROPOSITION 5.1. *Let φ be an $(m-1)^s$-conformal and at the same time projective transformation. Then we have*

$$2d\psi = d\log\alpha . \tag{5.3}$$

PROOF. Transvecting (4. 6) with $w^j w_i$ and utilizing (5. 1), we get

$$(5.4) \qquad 2\zeta\psi\cdot w_k + 2\psi_k = \frac{1}{\alpha+\beta}(\alpha_k+\beta_k)\,.$$

Transvecting (5.2) and (5.4) with w^k, we obtain

$$(5.5) \qquad 2(m+1)\zeta\psi = \frac{m(\alpha+\beta)-\beta}{\alpha(\alpha+\beta)}\zeta\alpha + \frac{1}{\alpha+\beta}\zeta\beta\,,$$

$$(5.6) \qquad 4\zeta\psi = \frac{1}{\alpha+\beta}(\zeta\alpha+\zeta\beta)\,.$$

Eliminating $\zeta\psi$ from (5.5) and (5.6), we get

$$(5.7) \qquad (\alpha+2\beta)\zeta\alpha = \alpha\zeta\beta\,.$$

By (5.6) and (5.7), we get $2\zeta\psi = \frac{1}{\alpha}\zeta\alpha$. And so, by virtue of (5.2), (5.4) is written as

$$(5.8) \qquad ((m-1)\beta-\alpha)\alpha_k - m\alpha\beta_k + (m+1)(\alpha+\beta)\zeta\alpha\cdot w_k = 0\,.$$

Transvecting (4.6) with $g^{jk}w_i$ and using (5.1), (5.7) and $2\alpha\zeta\psi = \zeta\alpha$, we get

$$(5.9) \qquad (m-1)\zeta\alpha = 2\beta w^i{}_{,i}\,.$$

Transvecting (4.6) with $w^j w^k$ and g^{jk} respectively and using (5.1), (5.2), (5.7) and (5.9), we have

$$(5.10) \qquad 2(\alpha+\beta)\zeta\alpha\cdot w_k - \alpha(\alpha_k+\beta_k) + 2\alpha\beta w_{k,i}w^i = 0\,,$$

$$(5.11) \qquad \frac{(m+1)\alpha+2\beta}{\alpha}\zeta\alpha\cdot w_k + \frac{(\alpha+\beta)(2-m-m^2)+2\beta}{(\alpha+\beta)(m+1)}\alpha_k - \frac{(\alpha+\beta)(m+1)+2\alpha}{(\alpha+\beta)(m+1)}\beta_k + 2\beta w_{k,i}w^i = 0\,.$$

Eliminating $w_{k,i}w^i$ from the two above equations

$$(5.12) \qquad ((3-m^2)(\alpha+\beta)+2\beta)\alpha_k - 2\alpha\beta_k + (m-1)(m+1)(\alpha+\beta)\zeta\alpha\cdot w_k = 0\,.$$

Eliminating w_k from (5.8) and (5.12), as $m>2$, we get

$$(5.13) \qquad (\alpha+2\beta)\alpha_k - \alpha\beta_k = 0\,.$$

From (5.2) and (5.13), we deduce (5.3).

THEOREM 5.2. *Let φ be an $(m-1)^s$-homothety and at the same time projective transformation. Then φ is an affine transformation. Further β is constant and ${}^\varepsilon\zeta$ is a parallel field.*

PROOF. As α is constant, we have $\psi_k=0$ by Proposition 5.1, hence φ is an affine transformation. Then we can apply Theorem 4.1.

THEOREM 5.3. *Let φ be an $(m-1)^s$-conformal transformation and at the same time projective transformation such that $\alpha+\beta$ is constant. Then we see that α and β are constant and φ is an affine transformation.*

PROOF. This is an immediate consequence of (5.13).

COROLLARY 5.4. *Let φ be an $(m-1)^s$-conformal transformation of M onto itself and at the same time projective transformation such that $\varphi^\varepsilon\zeta = \pm{}^\varepsilon\zeta$. Then φ is an affine transformation.*

6. The Riemannian curvature, Ricci curvature and scalar curvature. Let φ be an $(m-1)$-conformal transformation of M to N. We denote by $R^i{}_{jkl}$, R_{jk}, R and ${}^\varphi R^i{}_{jkl}$, ${}^\varphi R_{jk}$, ${}^\varphi R$ the Riemannian curvatures, Ricci curvatures, scalar curvatures with respect to g and $\varphi^*h=(G_{ij})$ respectively. First we have

$$(6.1)\qquad {}^\varphi R^i{}_{jkl} = R^i{}_{jkl} + W^i_{jk,l} - W^i_{jl,k} + W^i_{rl}W^r_{jk} - W^i_{rk}W^r_{jl}\,,$$

where covariant derivative (,) is the one with respect to $\left\{ {i \atop jk} \right\}$ and

$$(6.2)\qquad W^i_{jk} = {}^\varphi\left\{ {i \atop jk} \right\} - \left\{ {i \atop jk} \right\}.$$

The verification of (6.1) is as follows: When we calculate ${}^\varphi R^i{}_{jkl}$ by $\left\{ {i \atop jk} \right\} + W^i_{jk}$, we have the right side of (6.1) and the terms which contain $\left\{ {i \atop jk} \right\}$'s. At any point x of M, we can find local coordinates x^i such that $\left\{ {i \atop jk} \right\}_x = 0$ holds good. Then we have (6.1) at x, and as (6.1) is a tensor equation, we get (6.1).

Contracting with respect to i and l, we have the relation of the Ricci curvatures by;

$$(6.3)\qquad {}^\varphi R_{jk} = R_{jk} + W^i_{jk,i} - W^i_{ji.k} + W^i_{ri}W^r_{jk} - W^i_{rk}W^r_{ji}\,.$$

Transvecting (6.3) with (4.2), we get

$$(6.4)\quad {}^{\varphi}R = \frac{1}{\alpha}R + \frac{1}{\alpha}(W^i_{jk}g^{jk})_{,i} - \frac{1}{\alpha}(W^i_{ji})^{,j} + \frac{1}{\alpha}W^i_{ir}W^r_{jk}g^{jk} - \frac{1}{\alpha}W^i_{kr}W^r_{ji}g^{jk}$$
$$+ \frac{1}{\nu}\left(\beta - \frac{r}{\alpha}\right)\{R_1(\zeta,\zeta) + W^i_{jk,i}w^jw^k - W^i_{ji,k}w^jw^k$$
$$+ W^i_{ir}W^r_{jk}w^jw^k - W^i_{kr}W^r_{ji}w^kw^j\}$$
$$+ \frac{1}{\nu}\{2R_1(\theta,\zeta) + 2W^i_{jk,i}\theta^jw^k - W^i_{ji,k}(w^j\theta^k + w^k\theta^j)$$
$$+ 2W^i_{ir}W^r_{jk}w^j\theta^k - 2W^i_{kr}W^r_{ji}\theta^kw^j\}$$
$$+ \frac{1}{\alpha\nu}\{-R_1(\theta,\theta) + (-W^i_{jk,i} + W^i_{ji,k} - W^i_{ir}W^r_{jk} + W^i_{kr}W^r_{ji})\theta^j\theta^k\},$$

where R_1 denotes the Ricci curvature.

As a special case, we consider an $(m-1)^s$-homothetic transformation of M to N assuming that ζ is a parallel field. By (4.6), we have

$$(6.5)\qquad 2W^i_{jk} = -\frac{1}{\alpha}\beta^iw_jw_k + \frac{w^i}{\alpha(\alpha+\beta)}(\beta(\zeta\beta)w_jw_k + 2\alpha\beta_{(j}w_{k)}).$$

Then by (6.3) we have

$$(6.6)\qquad 4{}^{\varphi}R_{jk} = 4R_{jk} + \frac{2}{\alpha+\beta}(\beta_{j,l}w^lw_k + \beta_{k,l}w^lw_j - \beta_{j,k})$$
$$+ \frac{1}{\alpha(\alpha+\beta)^2}w_jw_k\{-2(\alpha+\beta)^2\beta^i{}_{,i} + 2\beta(\alpha+\beta)\beta_{i,l}w^iw^l$$
$$+ (\alpha+\beta)\beta_r\beta^r - \beta(\zeta\beta)^2\}$$
$$+ \frac{1}{(\alpha+\beta)^2}(\beta_j\beta_k - 2(\zeta\beta)\beta_{(j}w_{k)}).$$

And finally from (4.3), we have

$$(6.7)\quad 4{}^{\varphi}R = \frac{4}{\alpha}R + \frac{4}{\alpha(\alpha+\beta)}(\beta_{j,l}w^jw^l - \beta^i{}_{,i}) + \frac{2}{\alpha(\alpha+\beta)^2}(\beta_r\beta^r - (\zeta\beta)^2),$$

where we have used $R_{ij}w^iw^j = 0$, which follows from the fact that w^i is a parallel field.

Now, next Proposition is evident:

PROPOSITION 6.1. *Suppose that* ${}^{\varepsilon}\zeta$ *is a parallel field and* φ *is an* $(m-1)^s$-*homothety of* M *to* N *such that* β *is constant. Then the scalar curvatures are in the relation* $'R_{\varphi x}=\frac{1}{\alpha}R_x$, $x\in M$. *Particularly if both scalar curvatures are constant and equal to* $R\neq 0$, *then* φ *is an* $(m-1)^s$-*isometry.*

As usual, we denote by δ the dual of d (i.e. codifferentiation), then we have $\beta^i{}_{,i}=-\delta d\beta$.

THEOREM 6.2. *Suppose that* ${}^{\varepsilon}\zeta$ *is a parallel field and* φ *is an* $(m-1)^s$-*homothety of* M *onto itself such that* $\delta d\beta=0$ *and* $\zeta\beta=0$ *hold good.*

(i) *If* $R=0$, β *is constant.*

(ii) *If* $R=constant<0$, *then* $\alpha\leqq 1$.

(iii) *If* $R=constant>0$, *then* $\alpha\geqq 1$.

In (ii) *and* (iii) *equality holds if and only if* β *is constant.*

PROOF. If R is constant by (6.7) we get

$$\frac{1}{(\alpha+\beta)^2}\beta_r\beta^r=2(\alpha-1)R, \tag{6.8}$$

where we have used $\beta_{j,l}w^jw^l=(\beta_jw^j)_{,l}w^l=0$. From (6.8), (i), (ii) and (iii) follow. If $\alpha=1$, then $\beta_r\beta^r=0$ holds, hence β is constant,

THEOREM 6.3. *Suppose that* ${}^{\varepsilon}\zeta$ *is a parallel field and* φ *is an* $(m-1)^s$-*homothety of* M *onto itself such that* β *is constant. If the scalar curvature is bounded and not equal to* 0 *somewhere, then* φ *is an* $(m-1)^s$-*isometry.*

PROOF. By (6.7) we have ${}^{\varphi}R=\frac{1}{\alpha}R$ namely $R_{\varphi x}=\frac{1}{\alpha}R_x$, x being a point at which R_x is not zero. Then by iteration, we have $R_{\varphi^k x}=\left(\frac{1}{\alpha}\right)^k R_x$. As R is bounded, we can conclude that $\alpha=1$.

THEOREM 6.4. *Suppose that* ${}^{\varepsilon}\zeta$ *is a parallel field and* φ *is an* $(m-1)^s$-*homothety of* M *onto itself which preserves the Ricci curvature.*

(i) *If* $\delta d\beta=0$ *and* $\zeta\beta=0$, *then* β *is constant.*

(ii) *If* M *is compact orientable, then* $d\beta$ *is proportional to* w.

(iii) *If* M *is compact and there exists a point* x *such that* $R_x\neq 0$, *then* $\alpha=1$.

PROOF. Noticing that ${}^{\varphi}R_{jk}=R_{jk}$, we transvect (6.6) with $w^j w^k$, and get

$$2\beta_{j,l}w^j w^l - 2\beta^i{}_{,i} + \frac{1}{\alpha+\beta}\beta_r\beta^r - \frac{1}{\alpha+\beta}(\zeta\beta)^2 = 0\,. \tag{6.9}$$

Then (i) is clear. To prove (ii), if we integrate (6.9) over M, we have

$$\int_M \frac{1}{\alpha+\beta}(\beta_r\beta^r - (\zeta\beta)^2)\,d\sigma = 0\,,$$

where we have used $\int_M \beta_{j,l}w^j w^l\,d\sigma = \int_M (\beta_j w^j w^l)_{,l}d\sigma = 0$, and $d\sigma$ denotes the volume element of M. As $\alpha+\beta$ is positive and $(\beta_r\beta^r-(\zeta\beta)^2)=(\beta_r-(\zeta\beta)w_r)\times(\beta^r-(\zeta\beta)w^r)$ is non-negative, we have $\beta_r=(\zeta\beta)w_r$, namely $d\beta$ is proportional to w. On the other hand, transvecting ${}^{\varphi}R_{jk}=R_{jk}$ with $(G^{-1})^{jk}$ in (6.6), we have ${}^{\varphi}R=\frac{1}{\alpha}R$. As M is compact, R is bounded, so we have $\alpha=1$.

REMARK. Assume that ζ is a parallel field and φ is an $[m-1]^s$-homothety of M onto itself, then we have (i) and (ii). Because ${}^{\varphi}R_1(\zeta,\zeta)=\mu^2R_1(\zeta,\zeta)\cdot\varphi=0$.

7. The sectional curvatures in the case where ${}^{\varepsilon}\zeta$ is parallel along D. We say that ${}^{\varepsilon}\zeta$ is parallel along D, if $\nabla_u\zeta=0$ holds for any vector field u which belongs to the distribution D i.e. $w(u)=0$. First we prove

LEMMA 7.1. *If ${}^{\varepsilon}\zeta$ is parallel along D, then D is completely integrable.*

PROOF. Suppose that u and v belong to D, then we have

$$w(\nabla_v u) = \nabla_v(w(u)) - \nabla_v w\cdot u = 0\,, \tag{7.1}$$

from which we have

$$w([u,v]) = w(\nabla_u v - \nabla_v u) = 0\,. \tag{7.2}$$

This completes the proof.

LEMMA 7.2. *If ${}^{\varepsilon}\zeta$ is parallel along D and if φ is an $(m-1)^s$-homothety of M to N. Let u, v be vector fields which belong to D, then we have*

$$'\nabla_{\varphi v}\varphi u = \varphi\nabla_v u\,. \tag{7.3}$$

PROOF. If u and v belong to D and φ is an $(m-1)^s$-homothety we have (2.4), equivalently

$$h('\nabla_{\varphi v}\varphi u, Y) = \alpha(\varphi^{-1*}g)(\varphi\nabla_v u, Y) \tag{7.4}$$

for any vector field Y on N. By (1.16), we have

$$\varphi^{-1*}g = \frac{1}{\alpha}h - \left(\frac{\beta}{\alpha(\alpha+\beta)}\cdot\varphi^{-1}\right)\eta\otimes\eta\,. \tag{7.5}$$

And we get

$$(\eta\otimes\eta)(\varphi\nabla_v u, Y) = \eta(\varphi\nabla_v u)\cdot\eta(Y) = \gamma w(\nabla_v u)\varphi^{-1}\cdot\eta(Y) = 0\,. \tag{7.6}$$

Then by virtue of (7.4), (7.5) and (7.6), we get

$$h('\nabla_{\varphi v}\varphi u, Y) = h(\varphi\nabla_v u, Y)\,. \tag{7.7}$$

As (7.7) holds for any Y, we have (7.3).

PROPOSITION 7.3. *Let φ be an $(m-1)^s$-homothety of M to N such that β is constant, then two following conditions are equivalent*:

(i) ${}^\varepsilon\zeta$ *is parallel* (*along D resp.*).

(ii) ${}^\delta\xi$ *is parallel* (*along φD resp.*).

PROOF. (i)$\to$(ii). We use (2.7). If ζ is parallel along D, $\nabla_v\zeta = 0$ holds good provided that v belongs to D, and we have $'\nabla_{\varphi v}\varphi\zeta = 0$. Then $\sqrt{\alpha+\beta}\,\xi$, and so ξ, is parallel along φD. If ζ is parallel, each trajectory of ζ is a geodesic. By Theorem 3.1 we see that each trajectory of ξ is also a geodesic. Then ξ is parallel field. The case (ii)$\to$(i) reduces to the first case by taking the inverse φ^{-1}.

THEOREM 7.4. *Suppose that ${}^\varepsilon\zeta$ is parallel along D and φ is an $(m-1)^s$-homothety of M to N. Let u, v, r be vector fields which belong to D, then we have*

$$'R(\varphi u, \varphi v)\,\varphi r = \varphi(R(u, v)r)\,, \tag{7.8}$$

where R and $'R$ denote the Riemannian curvature tensors with respect to g and h.

PROOF. The expression of the Riemannian curvature tensor is as follows:

$$-R(u, v)r = \nabla_u \nabla_v r - \nabla_v \nabla_u r - \nabla_{[u,v]} r \,. \tag{7.9}$$

Thus, if u, v, r belong to D, by Lemma 7.2, we have

$$\begin{aligned} -'R(\varphi u, \varphi v)\varphi r &= '\nabla_{\varphi u}{}'\nabla_{\varphi v}\varphi r - '\nabla_{\varphi v}{}'\nabla_{\varphi u}\varphi r - '\nabla_{[\varphi u, \varphi v]}\varphi r \\ &= '\nabla_{\varphi u}\varphi(\nabla_v r) - '\nabla_{\varphi v}\varphi(\nabla_u r) - '\nabla_{\varphi[u,v]}\varphi r \\ &= -\varphi \cdot R(u, v)r \,, \end{aligned}$$

completing the proof.

We denote by $K_x(u, v)$ the sectional curvature defined by the tangent vectors u and v at a point x, then

$$K_x(u, v) = \frac{g_x(R(u, v)\, u, v)}{|u \wedge v|^2}$$

where $|u \wedge v|$ is the area of the parallelogram with u and v as adjacent sides:

$$|u \wedge v|^2 = |u|^2 |v|^2 - (g(u, v))^2 \,. \tag{7.10}$$

REMARK 1. If ${}^{\varepsilon}\zeta$ is a parallel field, we have $R^i{}_{jkl} w^j = 0$. Therefore the sectional curvature $K(\zeta, u)$ determined by ζ and any other vector u is equal to zero.

THEOREM 7.5. *Assume that ${}^{\varepsilon}\zeta$ is parallel along D and φ is an $(m-1)^s$-homothety. Let u, v be tangent vectors at $x \in M$ which belong to D_x, then we have*

$$K_x(u, v) = \alpha' K_{\varphi x}(\varphi u, \varphi v) \,, \tag{7.11}$$

where $'K(\varphi u, \varphi v)$ is the sectional curvature determined by φu and φv with respect to h.

PROOF. By Theorem 7.4, we have

$$\begin{aligned} h('R(\varphi u, \varphi v)\, \varphi u, \varphi v) &= h(\varphi \cdot R(u, v)\, u, \varphi v) \\ &= \alpha\, g(R(u, v)\, u, v) \cdot \varphi^{-1} \,. \end{aligned} \tag{7.12}$$

From (7.10), it follows that

$$|\varphi u \wedge \varphi v|^2 = \alpha^2 |u \wedge v|^2 \cdot \varphi^{-1} \,. \tag{7.13}$$

By (7.12) and (7.13), we get (7.11). q.e.d.

Let u_x, v_x be two tangent vectors at x and let u, v be their extension to vector fields. Then the value of the function $K(u, v)$ at x is equal to $K_x(u_x, v_x)$. Now we prove

THEOREM 7.6. *Assume that* ${}^{\varepsilon}\zeta$ *and* ${}^{\delta}\xi$ *are parallel fields and* φ *is an* $(m-1)^s$*-homothety. Then we have*

$$K(u,v) = \left(\alpha + \frac{Q(\varphi, u, v)}{|u \wedge v|^2}\right) {}'K(\varphi u, \varphi v) \cdot \varphi \,, \tag{7.14}$$

for any vector fields u and v, where we have put

$$Q(\varphi, u, v) = \beta(a^2 g(v, v) + b^2 g(u, u) - 2ab g(u, v)) \,, \tag{7.15}$$

a and b denoting ${}^{\varepsilon}w(u)$ *and* ${}^{\varepsilon}w(v)$ *respectively.*

PROOF. We decompose u and v as follows:

$$u = u_0 + a\zeta \,, \quad v = v_0 + b\zeta \,, \tag{7.16}$$

where u_0 and v_0 belong to D and $a = {}^{\varepsilon}a = {}^{\varepsilon}w(u)$, $b = {}^{\varepsilon}b = {}^{\varepsilon}w(v)$. Then $\varphi u = \varphi u_0 + ({}^{\varepsilon}a\varphi^{-1})\varphi\zeta = \varphi u_0 + ({}^{\varepsilon}a\varphi^{-1})^{\varepsilon\delta}\mu\, {}^{\delta}\xi$, $\mu^2 \cdot \varphi = \alpha + \beta$, and as ζ and ξ are parallel fields, we have

$$\begin{aligned} h({}'R(\varphi u, \varphi v)\varphi u, \varphi v) &= h({}'R(\varphi u_0, \varphi v_0)\, \varphi u_0, \varphi v_0) \\ &= \alpha\, g(R(u_0, v_0)\, u_0, v_0) \cdot \varphi^{-1} \\ &= \alpha\, g(R(u, v)\, u, v) \cdot \varphi^{-1} \,. \end{aligned} \tag{7.17}$$

On the other hand, using (7.10) and $\varphi\zeta = \mu\xi$, $\mu^2 \cdot \varphi = \alpha + \beta$, we can show the following relation

$$|\varphi u \wedge \varphi v|^2 \cdot \varphi = \alpha^2 |u \wedge v|^2 + \alpha\beta(a^2 g(v, v) + b^2 g(u, u) - 2ab g(u, v)) \,.$$

Thus we have

$$\begin{aligned} K(u, v) &= \left(\frac{1}{\alpha}\right) \frac{h({}'R(\varphi u, \varphi v)\, \varphi u, \varphi v) \cdot \varphi}{|u \wedge v|^2} \\ &= \frac{1}{\alpha} \cdot \frac{\alpha^2 |u \wedge v|^2 + \alpha Q(\varphi, u, v)}{|u \wedge v|^2} ({}'K(\varphi u, \varphi v) \cdot \varphi) \,. \end{aligned}$$

Now we have (7.14).

THEOREM 7.7. *Assume that* ${}^{\varepsilon}\zeta$ *and* ${}^{\delta}\xi$ *are parallel fields and* φ *is an* $(m-1)^s$*-homothety. If* M *is of non-negative curvature* (*non-positive curvature respectively*), *then* N *is of non-negative curvature* (*non-positive curvature respectively*).

PROOF. By (7.17) we see that $K(u,v)$ and $'K(\varphi u, \varphi v)$ have the same sign $+$ or $-$.

REMARK 2. In Theorem 7.6 and 7.7, if φ is an $(m-1)^s$-homothety such that β is constant, then the assumption that ${}^{\delta}\xi$ is a parallel field may be removed by Proposition 7.3.

8. $(m-1)$-Einstein spaces.

Let M and R_1 be an m-dimensional Riemannian manifold and Ricci curvature.

DEFINITION. If M admits an $(m-1)$-dimensional distribution D such that $R_1(u,v)=eg(u,v)$ holds good for $u, v \in D$, e denoting a scalar field, we say that M is an $(m-1)$-Einstein space with respect to the distribution D.

Let ${}^{\varepsilon}\zeta$, ${}^{\varepsilon}w$ be ones defined in §1. By the similar argument we see that R_1 is written as follows

$$R_1 = eg + {}^{\varepsilon}w \otimes {}^{\varepsilon}K + {}^{\varepsilon}K \otimes {}^{\varepsilon}w + f\,{}^{\varepsilon}w \otimes {}^{\varepsilon}w\,, \tag{8.1}$$

where f is a scalar field on M and ${}^{\varepsilon}K$ defines a 1-form K_U in each U in §1. Namely in U, we have

$$R_{ij} = eg_{ij} + w_i K_j + K_i w_j + f w_i w_j\,, \tag{8.2}$$

where $K_i w^i = 0$. Transvecting (8.2) with g^{ij}, we get

$$R = me + f\,. \tag{8.3}$$

By the same letter K we denote the contravariant vector: $K^i = g^{ij}K_j$. Using δ, we have $(\delta w) = -w^i{}_{,i}$.

Now we prove

THEOREM 8.1. *Suppose that* M *is an* $(m-1)$*-Einstein space* $(m>3)$ *with respect to* D. *If in* (8.1), *the three conditions*:

(1) $\delta w_U = 0\,, \quad \nabla_{\zeta_U}\zeta_U = 0\,,$

(2) $R_1(\zeta_U, \zeta_U) =$ *constant*,
(3) $\delta K_U = 0\,, \quad \nabla_{\zeta_U} K_U + \nabla_{K_U}\zeta_U = 0$

are satisfied for each U, *then* e *and* f *are constant on* M. *Further the scalar curvature* R *is constant.*

PROOF. Multiply $w^i w^j$ to (8.2) and contract with respect to i and j, then we have

$$R_{ij}\, w^i w^j = e + f\,. \tag{8.4}$$

Hence, by (2) we have

$$e_k + f_k = 0\,. \tag{8.5}$$

Differentiating covariantly (8.3) we have

$$R_{,k} = m\, e_k + f_k\,. \tag{8.6}$$

And from (8.2), we get

$$g^{is} R_{ik,s} = R_{ik,}{}^{i} = e_k + f_i\, w^i w_k\,, \tag{8.7}$$

where we have used $w^i{}_{,i}=0$, $w_{k,i}w^i=0$, $K^i{}_{,i}=0$ and $K_{k,i}w^i+w_{k,i}K^i=0$. Using the well-known identity $R_{,k}=2R_{ik,}{}^{i}$, (8.6) and (8.7) show that

$$(m-2)\, e_k = 2\zeta f \cdot w_k - f_k\,. \tag{8.8}$$

Eliminating f_k from (8.5) and (8.8), we have

$$(m-3)\, e_k = 2\zeta f \cdot w_k\,. \tag{8.9}$$

Transvecting (8.5) and (8.9) with w^k, we get

$$\zeta e + \zeta f = 0\,, \quad (m-3)\zeta e = 2\zeta f\,.$$

Thus we get $(m-1)\zeta e = 0$ and $\zeta e=0$, $\zeta f=0$. Then from (8.5), (8.6) and (8.9) it follows that $e_k = f_k = 0$ and $R_{,k}=0$.

COROLLARY 8.2. *In an* $(m-1)$*-Einstein space* $(m>3)$, *if* ${}^{\varepsilon}\zeta$ *is a parallel field and if* $\delta K_U=0$, $\nabla_{\zeta_U} K_U = 0$ (*in particular if* $K=0$) *hold good. Then* e, f *and* R *are constant.*

PROOF. (1) of the Theorem holds good. By Ricci's indentity, we have $R_{ij}w^i w^j=0$, satisfying (2).

PROPOSITION 8.3. *In the above Theorem, if* $m=3$, *i.e., M is a* (3−1)-*Einstein space satisfying* (1), (2) *and* (3). *Then*

$$\zeta e = \zeta f = 0 \quad \textit{and} \quad \zeta R = 0\,.$$

PPOOF. By (8.9) we have $\zeta f=0$. And so $\zeta e=0$ and $\zeta R=0$ follow from (8.5) and (8.6).

DEFINITION. We call M a ${}^{\varepsilon}w$-Einstein space if M is an $(m-1)$-Einstein space with respect to D and satisfies ${}^{\varepsilon}K=0$ in (8.1).

REMARK 1. In the study of contact manifolds, some authors treated with η-Einstein spaces, η denoting a contact form ([11], [12]).

REMARK 2. In the Theorem 8.1 and Proposition 8.3, if M is a ${}^{\varepsilon}w$-Einstein space, the condition (3) is satisfied always.

If M is an Einstein space ($R\neq 0$), a transformation which preserves the Ricci curvature is an isometry of M. So there is no essentially $[m-1]$-conformal transformation of M which preserves the Ricci curvature. This is one of the reasons why we consider $(m-1)$-Einstein spaces.

THEOREM 8.4. *Let M be an* $(m-1)$-*Einstein space. If a transformation* φ *of M preserves the Ricci curvature and the distribution* ${}^{\varepsilon}w=0$, *then* φ *is an* $[m-1]$-*conformal transformation.*

PROOF. By assumption we have

$$R_1(\varphi u, \varphi v) = e\, g(\varphi u, \varphi v) + w(\varphi u)K(\varphi v) + K(\varphi u)w(\varphi v) + f\, w(\varphi u)\, w(\varphi v)$$

for any vector fields u, v on M. And we have a family ${}^{\varepsilon\varepsilon'}\gamma = \{\gamma_{UV}\}$ of scalar fields such that $\varphi^* w=\gamma w$. As $R_1(\varphi u, \varphi v)\cdot\varphi = R_1(u, v)$, we have

$$\varphi^* g = \frac{e}{e\cdot\varphi}\, g + \frac{1}{e\cdot\varphi}\, w\otimes(K-\gamma\varphi^* K) + \frac{1}{e\cdot\varphi}(K-\gamma\varphi^* K)\otimes w + \frac{1}{e\cdot\varphi}(f-\gamma^2(f\cdot\varphi))\, w\otimes w\,. \tag{8.10}$$

Though this is not a canonical form of an $(m-1)$-conformal transformation, we see that φ is an $[m-1]$-conformal transformation. If $K-\gamma\varphi^*K$ is proportional to w, φ is an $[m-1]^s$-conformal transformation. And if e is constant, φ is an $[m-1]^s$-isometry.

COROLLARY 8.5. *Let M be a ${}^{\varepsilon}w$-Einstein space $(m>3)$ and suppose that φ preserves the Ricci curvature and the distribution ${}^{\varepsilon}w=0$. If* (1) $\delta w_U=0$, $\nabla_{\zeta_U}\zeta_U=0$ *and* (2) $R_1(\zeta_U, \zeta_U)$ *is constant, then φ is an $[m-1]^s$-isometry.*

PROOF. As $K=0$, by Theorem 8.1, we see that e and f are constant. Thus by (8.10), we get

$$\varphi^*g = g + \frac{f}{e}(1-\gamma^2)\, w\otimes w\,. \tag{8.11}$$

COROLLARY 8.6. *In Corollary* 8.5, *in particular if $R_1(\zeta_U, \zeta_U)$=non-zero constant. Then φ is an isometry.*

PROOF. From $\varphi^*w = \gamma w$, $\varphi\zeta = (\gamma\cdot\varphi^{-1})\zeta$ follows. By contraction (8.11) with ζ, we get

$$\gamma^2 = 1 + \frac{f}{e}(1-\gamma^2)\,.$$

As $e+f \neq 0$, we have $\gamma^2=1$, and hence $\varphi^*g=g$.

9. The group of $[m-1]$-conformal transformations. In m-dimensional manifold M, let D be an $(m-1)$-dimensional distribution of class C^∞. By Π we denote the set of all $[m-1]$-conformal transformation of M on itself with respect to the distribution D. Let φ_1, φ_2 and φ_3 be elements of Π, then

$$\varphi_\lambda{}^*w = \gamma_\lambda w\,, \tag{9.1}$$

$$(\varphi_\lambda{}^*g)_x = \alpha_\lambda(x)\, g_x + w_x\otimes(\theta_\lambda)_x + (\theta_\lambda)_x\otimes w_x + \beta_\lambda(x)\, w_x\otimes w_x\,, \tag{9.2}$$

$\lambda=1,2,3$, where $\gamma_\lambda, \alpha_\lambda, \beta_\lambda$ are scalar fields and θ_λ defines 1-form in each local neighborhood. Then the composition $\varphi_2\cdot\varphi_1$ satisfies

$$((\varphi_2\cdot\varphi_1)^*w)_x = (\varphi_1{}^*\varphi_2{}^*w)_x = \gamma_1(x)\gamma_2(\varphi_1 x)\, w_x\,, \tag{9.3}$$

$$((\varphi_2\cdot\varphi_1)^*g)_x = \alpha_1(x)\,\alpha_2(\varphi_1 x)\, g_x \tag{9.4}$$

$$+ w_x \otimes [\alpha_2(\varphi_1 x)(\theta_1)_x + \gamma_1(x)(\varphi_1{}^*\theta_2)_x]$$
$$+ [\alpha_2(\varphi_1 x)(\theta_1)_x + \gamma_1(x)(\varphi_1{}^*\theta_2)_x] \otimes w_x$$
$$+ [\alpha_2(\varphi_1 x)\beta_1(x) + \beta_2(\varphi_1 x)\gamma_1^2(x)]\, w_x \otimes w_x ,$$

$$((\varphi_3 \cdot \varphi_2 \cdot \varphi_1)^* g) = \alpha_1(x)\alpha_2(\varphi_1 x)\alpha_3(\varphi_2\varphi_1 x)\, g_x + (\cdots), \tag{9.5}$$

where $(\cdots)$ means three terms corresponding to the 2, 3, 4th term in the right hand side of (9.4). The inverse transformation of φ satisfies

$$(\varphi^{-1*}w)_{\varphi x} = \left(\frac{1}{\gamma}\right)(x)\, w_{\varphi x} , \tag{9.6}$$

$$(\varphi^{-1*}g)_{\varphi x} = \left(\frac{1}{\alpha}\right)(x)\, g_{\varphi x} + w_{\varphi x} \otimes \left(-\frac{1}{\alpha\gamma}(x)\right)(\varphi^{-1*}\theta)_{\varphi x} + \left(-\frac{1}{\alpha\gamma}(x)\right)(\varphi^{-1*}\theta)_{\varphi x} \otimes w_{\varphi x} - \left(\frac{\beta}{\alpha\gamma^2}\right)(x)\, w_{\varphi x} \otimes w_{\varphi x} . \tag{9.7}$$

Here we notice that (9.4) and (9.7) are not canonical expression of $(m-1)$-conformal transformations.

We use the notations for the subgroups of the transformation group Π as follows:

Π^s : The totality of $[m-1]^s$-conformal transformations.
Θ : The totality of $[m-1]$-homotheties.
Φ : The totality of $[m-1]$-isometries.
Θ^s : $\Theta \cap \Pi^s$, Φ^s : $\Phi \cap \Pi^s$.

Next theorem is an immediate consequence of (9.5) and (9.7).

THEOREM 9.1. *Φ and Φ^s are normal subgroups of Θ and Θ^s respectively.*

THEOREM 9.2. *Any finite subgroup of Θ is a subgroup of Φ.*

PROOF. Let $\varphi \in \Theta$, then by (9.4) we have

$$(\varphi^{2*}g)_x = \alpha^2 g_x + (*),$$

and by k times iterations we have

$$(\varphi^{k*}g)_x = \alpha^k g_x + (**),$$

where (*) and (**) denote the terms which contain w_x. So, as α^k is not bounded unless $\alpha=1$, the assertion is true. q.e.d.

Some answers to the question "Under what conditions does certain subgroup of Π^s make a Lie group?" are given in §15.

10. $(m-1)$-conformal transformations of complete or compact M. Let φ be an $(m-1)$-conformal transformation of M onto itself satisfying (1. 6) or (4. 1). We take an arbitrary point x of M and take suitable local coordinates x^i in a local coordinate neighborhood U about x such that $(g_{ij})_x=\delta_{ij}$, $w_x=(0, \cdots, 0, 1)$. This is possible as ζ_U is a unit vector field. And let $(\theta_1, \cdots, \theta_{n-1}, 0)$ be components of θ_U, where we have used $\theta_m=0$ as $\theta(\zeta)=0$. Then we have

$$|G_{ij}| = \begin{vmatrix} \alpha & 0 & \cdots & \cdots & 0 & \theta_1 \\ 0 & \alpha & & & \vdots & \vdots \\ \vdots & & \ddots & & \vdots & \vdots \\ \vdots & & & \alpha & 0 & \theta_{m-2} \\ 0 & \cdots & \cdots & 0 & \alpha & \theta_{m-1} \\ \theta_1 & \cdots & \cdots & \theta_{m-2} & \theta_{m-1} & \alpha+\beta \end{vmatrix}$$

where $|G_{ij}|$ denotes the determinant of the matrix G_{ij}. Thus

$$|G_{ij}| = \alpha^{m-1}\{(\alpha+\beta) - \frac{1}{\alpha}\sum_{i=1}^{m-1}\theta_i^2\}$$

holds at x. As $\sum_{i=1}^{m-1}\theta_i^2 = g(\theta, \theta)$ and $|G_{ij}|$, $|g_{ij}|$ are positive, we get

$$\sqrt{|G_{ij}|} = [\alpha^{m-1}(\alpha+\beta-\alpha^{-1}g(\theta, \theta))]^{\frac{1}{2}}\sqrt{|g_{ij}|}\ . \tag{10.1}$$

If M is compact and orientable, we can integrate (10. 1) over M, denoting $\int_M d\sigma = |M|$, we have

THEOREM 10.1. *Suppose that φ is an $(m-1)$-conformal transformation of a compact and orientable manifold M onto itself. Then the following equation is valid:*

$$\frac{1}{|M|}\int_M [\alpha^{m-1}(\alpha+\beta-\alpha^{-1}g(\theta, \theta))]^{\frac{1}{2}}\, d\sigma = 1\,.$$

As an immediate consequence of Theorem 10.1, we get

THEOREM 10.2. *Let φ be an $(m-1)^s$-homothety of a compact and orientable manifold onto itself such that β is constant. Then φ is an isometry, except the case $\alpha \neq 1$ and $\beta = \alpha^{1-m} - \alpha$.*

As a corollary we have

COROLLARY 10.3. *In a compact orientable manifold,* (i) *if φ is an $(m-1)^s$-homothety such that $\varphi^\varepsilon\zeta = \pm^\varepsilon\zeta$, then φ is an isometry.* (ii) *if φ is an $(m-1)^s$-isometry such that β is constant, then φ is an isometry.*

Next we prove

THEOREM 10.4. *Let φ be an $(m-1)^s$-conformal transformation of a complete Riemannian manifold M. If $\alpha < \alpha_0 < 1$ and $\alpha + \beta < \alpha_0 < 1$ on M (or $\alpha > \alpha_0 > 1$ and $\alpha + \beta > \alpha_0 > 1$) for some constant α_0, then there exists a unique fixed point of φ in M.*

PROOF. Let x be an arbitrary point of M and $l = x(t)$ $(0 \leqq t \leqq 1)$ be any differentiable curve which joins $x = x(0)$ and $x(1) = \varphi x$. We denote by $|l|$ the length of l. Now we have

$$g_{\varphi x(t)}\left(\varphi \frac{dx}{dt}, \varphi \frac{dx}{dt}\right) = \alpha_{x(t)} g\left(\frac{dx}{dt}, \frac{dx}{dt}\right) + \beta_{x(t)}\left(w\left(\frac{dx}{dt}\right)\right)^2.$$

We decompose $\frac{dx}{dt}$ as $\frac{dx}{dt} = v_t + r_t\zeta$, $v_t \in D_{x(t)}$, then

$$\begin{aligned} g_{\varphi x(t)}\left(\varphi \frac{dx}{dt}, \varphi \frac{dx}{dt}\right) &= \alpha_{x(t)}(g(v_t, v_t) + r_t^2) + \beta_{x(t)} r_t^2 \\ &= \alpha_{x(t)}(v_t, v_t) + (\alpha + \beta)_{x(t)} r_t^2 \\ &< \alpha_0 g\left(\frac{dx}{dt}, \frac{dx}{dt}\right). \end{aligned}$$

Thus the length $|\varphi l|$ of φl is smaller than $|l|$. By iteration we get $|\varphi^k l| < \alpha_0^k |l|$ for any integer k. Therefore $(x, \varphi x, \cdots, \varphi^k x, \cdots)$ is a Cauchy sequence. By completeness of M, we have limit point $\bar{x} : \varphi\bar{x} = \bar{x}$.

In the case $\alpha > \alpha_0 > 1$ and $\alpha + \beta > \alpha_0 > 1$, we have $|\varphi^k l| > \alpha^k |l|$. Thus this case reduces to the first case by consideration of φ^{-1}. Uniqueness of $\bar{x}$ is seen as follows: If there exist two fixed points $\bar{x}, \bar{x}'$ of φ, we can join $\bar{x}$

and $\bar{x}'$ by the shortest curve l', then $\varphi l'$ is of the smaller length than $|l'|$ which is a contradiction.

COROLLARY 10.5. *Suppose that* ${}^{\varepsilon}\zeta$ *is a parallel field and let* φ *be an* $(m-1)^s$*-homothety of a complete Riemannian manifold* M *such that* β *is constant satisfying* $\alpha<1$ *and* $\alpha+\beta<1$ (*or* $\alpha>1$ *and* $\alpha+\beta>1$). *Then* M *is* (*locally*) *Euclidean.*

PROOF. By Theorem 4.2, φ is an affine transformation of M. And by Proposition 10.4, φ has a fixed point $\bar{x}$. Then by [2], or [5], M is locally Euclidean.

11. Supplimentary results. (i) *Space of constant curvature.* A manifold M is said to be constant curvature if the Riemannian curvature R satisfies

$$R(u,v)z=\kappa\{g(v,z)\cdot u-g(u,z)\cdot v\} \tag{11.1}$$

for any vector fields u, v, z on M, where κ is constant.

THEOREM 11.1. *Let* M *be of constant curvature and* φ *be* $(m-1)^s$*-conformal transformation of* M *onto* M. *If* ${}^{\varepsilon}w(z)=0$, *then we have*

$$R(\varphi u,\varphi v)\varphi z=\varphi(\alpha R(u,v)z)\,. \tag{11.2}$$

PROOF. By (11.1), we get

$$\begin{aligned}R(\varphi u,\varphi v)\varphi z&=\kappa\{g(\varphi v,\varphi z)\varphi u-g(\varphi u,\varphi z)\varphi v\}\\&=\varphi(\alpha R(u,v)z)\,,\end{aligned}$$

because $g(\varphi v,\varphi z)=\alpha g(v,z)\cdot\varphi^{-1}$.

(ii)

THEOREM 11.2. *We assume that* $R_1({}^{\varepsilon}\zeta,{}^{\varepsilon}\zeta)=T$ *and the scalar curvature* R *are constant and* $R\neq T$. *If* $\varphi\in\Pi^s$ *leaves* R_1 *invariant, then* $\varphi\in\Phi^s$. *Further if* $T\neq 0$, φ *is an isometry of* M.

PROOF. As T is constant and $\varphi\zeta=\mu\zeta$, $\mu^2\cdot\varphi=\alpha+\beta$, we have

$$T=R_1(\varphi\zeta,\varphi\zeta)=(\alpha+\beta)T\,. \tag{11.5}$$

On the other hand, we have

$$R = {}^{\varphi}R = (G^{-1})^{ij} R_{ij} = \frac{1}{\alpha} R - \frac{\beta}{\alpha(\alpha+\beta)} T.$$

Namely, we have

$$(11.6) \qquad (\alpha+\beta)(\alpha-1) R + \beta T = 0 .$$

We add $-(\alpha+\beta-1)T=0$ to the last equation, getting

$$(\alpha+\beta)(\alpha-1)R - (\alpha-1)T = 0 .$$

If we use again (11.5), the last equation turns to $(\alpha+\beta)(\alpha-1)(R-T)=0$. So $\alpha=1$ follows. Furthermore, if $T\neq 0$, $\beta=0$ follows from (11.6).

Chapter II

12. Infinitesimal $(m-1)$-conformal and $[m-1]$-conformal transformations. Let D be an $(m-1)$-dimensional distribution and $\varphi_t(|t|<q$; for some positive number $q)$ be a local 1-parameter subgroup of II, then we have

$$(12.1) \qquad \varphi_t{}^{*\,\varepsilon}w = \gamma_t{}^{\varepsilon}w ,$$

$$(12.2) \qquad \varphi_t{}^{*}g = \alpha_t g + {}^{\varepsilon}w \otimes {}^{\varepsilon}\theta_t + {}^{\varepsilon}\theta_t \otimes {}^{\varepsilon}w + \beta_t{}^{\varepsilon}w \otimes {}^{\varepsilon}w,$$

for t: $|t|<q$. In this section too, we abbreviate frequently ε in ${}^{\varepsilon}w$ or ${}^{\varepsilon}\theta$. As $\varphi_0(t=0)$ is an identical transformation of M, we have

$$(12.3) \qquad L(v)\,w = \lim_{t\to 0} \frac{\gamma_t - 1}{t}\, w,$$

$$(12.4) \quad L(v)\,g = \lim_{t\to 0} \frac{\alpha_t-1}{t}\, g + w \otimes \left(\lim_{t\to 0} \frac{\theta_t}{t}\right) + \left(\lim_{t\to 0} \frac{\theta_t}{t}\right) \otimes w + \lim_{t\to 0} \frac{\beta_t}{t}\, w \otimes w ,$$

where v is a vector field on M defined by φ_t. From these, we define that an infinitesimal transformation u is an infinitesimal $[m-1]$-conformal transformation if it satisfies

$$(12.5) \qquad L(u)\,w = cw ,$$

$$(12.6) \qquad (L(u)\,g)(r,s) = 0$$

for any vector fields r, s which belong to D. In (12.5), c does not depend on the choice of U, so c is a scalar field.

By the similar fashion to §1, we see that $L(u)g$ is written as

$$L(u)g = ag + w\otimes F + F\otimes w + bw\otimes w\,, \tag{12.7}$$

where a and b are scalar fields, and $F = ({}^{\varepsilon}F)$ defines a 1-form F_U in each neighborhood U in such a way that w_U and F_U are orthogonal. When we use the local coordinates x^i in U, w and F are treated as covariant tensors. If $F=0$, v is called an *infinitesimal* $[m-1]^s$-*conformal transformation.* If a is constant, v is called an *infinitesimal* $[m-1]$-*homothetic transformation*, etc.. But in many cases, we consider infinitesimal transformations which satisfy only (12.6), and we denote them by *infinitesimal* $(m-1)$-*conformal transformation.*

THEOREM 12.1. *Let u be an infinitesimal $[m-1]$-conformal transformation. Then $L(u)^{\varepsilon}\zeta = p^{\varepsilon}\zeta$ holds good for some scalar field p if and only if $F=0$, i.e., u in an infinitesimal $[m-1]^s$-conformal transformation. And we have $-2p = 2c = a+b$.*

PROOF. Operating Lie differentiation to $w_j=w^i g_{ij}$ with respect to u, we get

$$cw_j = (L(u)w^i)g_{ij} + (a+b)w_j + F_j\,.$$

If $L(u)w^i = pw^i$, we get $F_j = 0$. Conversely if $F_j = 0$, transvecting the last equation with g^{jk}, we obtain

$$L(u)w^i = (c-a-b)w^i\,.$$

THEOREM 12.2. *If ζ_U is an infinitesimal $[m-1]$-conformal transformation in each U. Then it is an infinitesimal $[m-1]^s$-conformal transformation in each U and each trajectory of ${}^{\varepsilon}\zeta$ is a geodesic.*

PROOF. By the equation $L(\zeta)w_i=cw_i$, we see that

$$w_{i,j}w^j = cw_i\,.$$

Transvecting the last equation with w^i, we get $c=0$ and $w_{i,j}w^j = 0$. This means that each trajectory of ζ is a geodesic. Next as $L(\zeta)g = w_{i,j}+w_{j,i}$, we get

$$w_{i,j} + w_{j,i} = ag_{ij} + w_iF_j + F_iw_j + bw_iw_j\,. \tag{12.8}$$

Multiplying (12. 8) by $w^i w^j$ and contracting, we have $a+b=0$. If we transvect (12. 8) with w^j and use $w_{i,j}w^j=0$, we have $(a+b)w_i+F_i=0$. Thus $F_i=0$. q.e.d.

In the above proof, we see also the following

THEOREM 12.3. *If ζ_U is an infinitesimal $(m-1)$-conformal transformation in each U. And if each trajectory of ${}^e\zeta$ is a geodesic, then ζ_U is an infinitesimal $[m-1]^s$-conformal transformation and satisfie $a+b=0$.*

Furthermore we have

THEOREM 12.4. *If ζ_U is an infinitesimal $(m-1)$-conformal transformation and satisfies $\delta w_U = 0$. Then it is an infinitesimal $(m-1)$-isometry in each U.*

PROOF. Transvecting (12. 8) with $w^i w^j$ and g^{ij}, we have $a+b=0$ and $ma+b=0$. Thus $a=0$ and $b=0$ hold good.

THEOREM 12.5. *If ζ_U is an infinitesimal $(m-1)$-conformal transformation in each U. And if each trajecoty of ζ_U is a geodesic and $\delta w_U = 0$. Then ζ_U is an infinitesimal isometry.*

PROOF. By Theorem 12.4, we have $a=b=0$. On the other hand by Theorem 12.3, we have $F_j=0$ completing the proof.

THEOREM 12.6. *Suppose that ζ_U be an infinitesimal $(m-1)$-conformal transformation, then $\rho\zeta_U$ is also an infinitesimal $(m-1)$-conformal transformation for any scalar field ρ.*

PROOF. First we have

$$(\rho w_i)_{,j} + (\rho w_j)_{,i} = \rho(w_{i,j}+w_{j,i}) + \rho_i w_j + \rho_j w_i\,.$$

On the other hand, ρ_i is written as

$$\rho_j = (\rho_j - \zeta\rho\cdot w_j) + \zeta\rho\cdot w_j\,.$$

Therefore, from (12. 8) we get

$$L(\rho\zeta)g_{ij} = a\rho g_{ij} + w_i(\rho F_j+\rho_j-\zeta\rho\cdot w_j) + (\rho F_i+\rho_i-\zeta\rho\cdot w_i)w_j + (b\rho+2\zeta\rho)w_i w_j\,.$$

This completes the proof.

Conversely, we have

THEOREM 12.7. *If $\rho\zeta_U$ is an infinitesimal $(m-1)$-conformal transformation for some non-vanishing scalar ρ. Then ζ_U is also an infinitesimal $(m-1)$-conformal transformation.*

PROOF. We refer to the proof of Theorem 12.6.

Now let u and ρu be two infinitesimal $(m-1)$-conformal transformations, then we have

$$u_{i,j}+u_{j,i}=ag_{ij}+w_iF_j+F_iw_j+b\,w_iw_j\,, \tag{12.9}$$

$$(\rho u_i)_{,j}+(\rho u_j)_{,i}=a'g_{ij}+w_iF'_j+F'_iw_j+b'\,w_iw_j\,, \tag{12.10}$$

where a, a', b, b', are scalar fields. Subtracting (12.9) multiplied by ρ from (12.10), we get

$$\rho_ju_i+\rho_iu_j=(a'-\rho a)g_{ij}+w_i(F'_j-\rho F_j)+(F'_i-\rho F_i)w_j+(b'-\rho b)w_iw_j\,. \tag{12.11}$$

If $m>2$, there exists a vector field which is orthogonal to u and ζ, thus $a'-\rho a=0$ follows form (12.11). Transvecting (12.11) with w^iw^j and g^{ij} respectively, we get

$$2w(u)\zeta\rho=b'-\rho b\,,$$

$$2u\rho=b'-\rho b\,,$$

from which we get $u\rho=w(u)\zeta\rho$. Next transvecting (12.11) with u^i and w^i respectively, we have

$$\begin{aligned} u\rho\cdot u_j+(u_iu^i)\rho_j &= w(u)(F'_j-\rho F_j) \\ &\quad +(F'_i-\rho F_i)u^iw_j+(b'-\rho b)w(u)w_j\,, \end{aligned} \tag{12.12}$$

$$\zeta\rho\cdot u_j+w(u)\rho_j=(F'_j-\rho F_j)+(b'-\rho b)w_j\,. \tag{12.13}$$

Subtracting (12.13) multiplied $w(u)$ from (12.12) and using $u\rho=w(u)\zeta\rho$, we get

$$(u_iu^i-w^2(u))\rho_j=(F'_i-\rho F_i)u^iw_j\,. \tag{12.14}$$

THEOREM 12.8. *Two different infinitesimal $(m-1)^s$-conformal transformations, both of which are not proportional to ${}^{\varepsilon}\zeta$ almost everywhere in M, cannot have the same streamlines, if $m \geqq 3$.*

PROOF. In (12.14), as F and F' vanish, we have $(u_i u^i - w^2(u))\rho_j = 0$. That u^i is not proportional to w^i almost everywhere means, as usual, that the set of the point where u^i is proportional to w^i is of measure zero. And u^i is proportional to w^i at a point x of M if and only if $u_i u^i = w^2(u)$ at x. Thus we have $\rho_j = 0$ almost everywhere, and hence everywhere on M. This means that ρ is constant.

13. Lie derivative of the Christoffel's symbols by an infinitesimal $(m-1)$-conformal transformation and relations with an infinitesimal affine transformation and projective transformation. Let u be an infinitesimal $(m-1)$-conformal transformation:

$$L(u)g_{ij} = ag_{ij} + w_i F_j + F_i w_j + b w_i w_j . \tag{13.1}$$

Into the following formula (see [22], p. 52)

$$2L(u)\left\{ {i \atop jk} \right\} = g^{ir}(\nabla_j L(u) g_{rk} + \nabla_k L(u) g_{rj} - \nabla_r L(u) g_{jk}), \tag{13.2}$$

we substitute (13.1), then we have

$$\begin{aligned} 2L(u)\left\{ {i \atop jk} \right\} = {} & a_j \delta^i_k + a_k \delta^i_j - a^i g_{jk} + b_j w^i w_k + b_k w^i w_j - b^i w_j w_k \\ & + b\{w_k(w^i{}_{,j} - w_j{}^{,i}) + w_j(w^i{}_{,k} - w_k{}^{,i}) + w^i(w_{j,k} + w_{k,j})\} \\ & + w^i(F_{j,k} + F_{k,j}) + (F^i{}_{,j} - F_j{}^{,i}) w_k + (F^i{}_{,k} - F_k{}^{,i}) w_j \\ & + F^i(w_{j,k} + w_{k,j}) + (w^i{}_{,j} - w_j{}^{,i}) F_k + (w^i{}_{,k} - w_k{}^{,i}) F_j . \end{aligned} \tag{13.3}$$

Analogously to Theorem 4.1, we prove

THEOREM 13.1. *Let u be an infinitesimal $(m-1)^s$-conformal transformation. If u is an infinitesimal affine transformation, then we have*

(1) *a and b are constant.*

And as a necessary condition that M admits such u satisfying $b \neq 0$, we have

(2) *${}^{\varepsilon}\zeta$ is a parallel field.*

PROOF. Transvecting (13.3) with $w^k w_i$, δ^k_i respectively and utilizing $F=0$, we get

$$(13.4)\qquad a_j + b_j = 0\,,$$

$$(13.5)\qquad ma_j + b_j = 0\,.$$

Then we see that a and b are constant. Next we transvect (13.3) with $w^j w^k$ and, noticing $a_j = b_j = 0$, we get $b\ w^i{}_{,j} w^j = 0$. Transvecting (13.3) with w^j and w_i respectively, we have $b(w_{i,k} - w_{k,i}) = 0$ and $b(w_{i,k} + w_{k,i}) = 0$. Thus w_i is a parallel field, if $b \neq 0$.

Conversely the following Theorem is obvious by (13.3).

THEOREM 13.2. *If ${}^e\zeta$ is a parallel field and u is an infintiesimal $(m-1)^s$-homothetic transformation such that b is constant. Then u is an infinitesimal affine transformation.*

An infinitesimal projective transformation u is characterized by

$$(13.6)\qquad 2L(u)\left\{ {i \atop jk} \right\} = 2\delta^i_j \overline{\psi}_k + 2\delta^i_k \overline{\psi}_j\,,$$

where $\overline{\psi}$ is a scalar field on M.

Analogously to Theorem 5.2, we prove

THEOREM 13.3. *If u is an infinitesimal $(m-1)^s$-homothetic transformation and at the same time infinitesimal projective transformation. Then u is an infinitesimal affine transformation. Further b is constant and ${}^e\zeta$ is a parallel field.*

PROOF. Transvecting (13.6) with δ^k_i, $w^k w_i$ and using (13.3) with $F = 0$, we have

$$(13.7)\qquad 2(m+1)\overline{\psi}_j = b_j\,,$$

$$(13.8)\qquad 2\zeta\overline{\psi}\cdot w_j + 2\overline{\psi}_j = b_j\,.$$

From (13.7) and (13.8) we deduce the relations $\zeta b = 0$ and $\zeta\overline{\psi} = 0$. Then it is easy to see that $\overline{\psi}_j$ vanishes. That ζ is a parallel field follows from Theorem 13.1.

14. Lie derivative of the Riemannian curvature tensor by an infinitesimal $(m-1)$-conformal transformation. If we substitute (13.3) into the following formula ([22], p. 17)

$$2L(u)R^i_{jkl} = 2\nabla_l L(u)\left\{{i \atop jk}\right\} - 2\nabla_k L(u)\left\{{i \atop jl}\right\}, \tag{14.1}$$

we get

$$\begin{aligned}2L(u)R^i_{jkl} &= 2a_{j,[l}\delta^i_{k]} - 2a^i_{,[l}g_{k]j} + 2w^i b_{j,[l}w_{k]} + 2w_j w_{[l}b^i_{,k]} \\ &+ 2b_j(w^i_{,[l}w_{k]} + w^i w_{[k,l]}) + 2b^i(w_{j,[k}w_{l]} + w_j w_{[l,k]}) \\ &+ 2b_{[l}\{w_{k]}(w^i_{,j} - w_j{}^{,i}) - w_{k]}{}^{,i}w_j + w_{k],j}w^i)\} \\ &+ b\{2w_{[k,l]}(w^i_{,j} - w_j{}^{,i}) + 2(w^i_{,j[l} - w_j{}^{,i}{}_{[l})w_{k]} \\ &+ 2w_{j,[k}w_{l]}{}^{,i} - 2w_{[k}{}^{,i}{}_{l]}w_j + 2w^i{}_{[l}w_{k],j} + 2w^i w_{[k,|j|l]} \\ &+ R^i_{rkl}w^r w_j - w^i R^r_{jkl}w_r\} \\ &+ 2\{w_{[k,l]}(F^i_{,j} - F_j{}^{,i}) + (F^i_{,j[l} - F_j{}^{,i}{}_{[l})w_{k]} \\ &+ w_{j,[l}(F^i_{,k]} - F_{k]}{}^{,i}) + w_j(F^i_{,[kl]} - F_{[k}{}^{,i}{}_{l]}) \\ &+ w^i{}_{[l}(F_{k],j} + F_{|j|,k]}) + w^i(F_{[k,|j|l]} + F_{j,[kl]}) \\ &+ F_{[k,l]}(w^i_{,j} - w_j{}^{,i}) + (w^i_{,j[l} - w_j{}^{,i}{}_{[l})F_{k]} \\ &+ F_{j,[l}(w^i_{,k]} - w_{k]}{}^{,i}) + F_j(w^i_{,[kl]} - w_{[k}{}^{,i}{}_{l]}) \\ &+ F^i_{,[l}(w_{k],j} + w_{|j|,k]}) + F^i(w_{[k,|j|l]} + w_{j,[kl]})\}\,.\end{aligned} \tag{14.2}$$

Contracting with respect to i and l, we have

$$\begin{aligned}2L(u)R_{jk} &= (2-m)a_{j,k} - a^r_{,r}g_{jk} + 2\zeta b\cdot w_{(j,k)} \\ &+ 2w^r_{,r}b_{(j}w_{k)} + 2w^r(b_{r,(j}w_{k)} + b_{(j}w_{k),r}) \\ &- 2b^r(2w_{(j}w_{k),r} - w_{r,(j}w_{k)}) - b_{j,k} - b^r_{,r}w_j w_k \\ &+ 2b\{w^r_{,r}w_{(j,k)} + w_{(j}{}^{,r}(w_{|r|,k)} - w_{k),r}) \\ &+ w_{(j}(w^r_{,k)r} - w_{k)}{}^{,r}{}_r) + w_{(j,k)r}w^r\} \\ &+ 2F^r_{,r}w_{(j,k)} + 2w^r_{,r}F_{(j,k)} + 2w^r_{,(k}(F_{j),r} - 2F_{|r|,j)}) \\ &+ 2(F^r_{,(j} - 2F_{(j}{}^{,r})w_{k),r} + 2w_{(k}(F^r_{,j)r} - F_{j)}{}^{,r}{}_r) \\ &+ 2F_{(k}(w^r_{,j)r} - w_{j)}{}^{,r}{}_r) + 2w_{(j,k)r}F^r + 2F_{(j,k)r}w^r + 4w^r_{,(j}F_{|r|,k)}\,.\end{aligned} \tag{14.3}$$

On the other hand, we have

$$L(u)g^{jk} = -ag^{jk} - w^j F^k - F^j w^k - bw^j w^k\,, \tag{14.4}$$

$$(14.5)\qquad L(u)R = L(u)g^{jk}\cdot R_{jk} + g^{jk}L(u)R_{jk}\,.$$

Transvecting (14.3) with g^{jk}, and substituting the result into (14.5), we have after calculation

$$\begin{aligned}(14.6)\qquad L(u)R = & -aR - 2R_{jk}w^jF^k - bR_{jk}w^jw^k + (1-m)a^r{}_{,r} \\ & - b^r{}_{,r} + \{2\zeta b\cdot w^r{}_{,r} + \zeta(\zeta b) + w_{j,k}b^jw^k + b(w^r{}_{,r})^2 \\ & + b(w^{r,l}w_{l,r} + w^rw^l{}_{,rl} + w^rw^l{}_{,lr}) \\ & + 2F^r{}_{,r}w^l{}_{,l} + 2w^{l,r}F_{r,l} + w^r(F^l{}_{,rl}+F^l{}_{,lr})+F^r(w^l{}_{,rl} + w^l{}_{l,r})\}\,,\end{aligned}$$

where we have used $0 = (w^kF_k)_{,r}{}^r = w^{k,r}{}_rF_k + 2w^{k,r}F_{k,r} + w^kF_{k,r}{}^r$.

We sometimes write F_U to denote not only 1-form but also for a contravariant vector field on U associated with it. And ${}^{\varepsilon}F=\{F_U\}$.

PROPOSITION 14.1. *Suppose that u is an infinitesimal $(m-1)$-conformal transformation on M. Then we have*

$$(14.7)\qquad L(u)R + aR + 2R_1({}^{\varepsilon}\zeta, {}^{\varepsilon}F) + bR_1({}^{\varepsilon}\zeta, {}^{\varepsilon}\zeta) = \delta(u, {}^{\varepsilon}\zeta)\,,$$

where $(u, {}^{\varepsilon}\zeta)$ denotes a certain 1-form on M.

PROOF. The sixth term indicated by $\{\ \ \}$ of the right hand side of (14.6) is equal to

$$\begin{aligned}(*)\qquad & (\zeta b\cdot w^r)_{,r} + (b\,w^rw^l{}_{,r})_{,l} + (b\,w^lw^r{}_{,r})_{,l} \\ & + (w^lF^r{}_{,r})_{,l} + (F^rw^l{}_{,l})_{,r} + (w^rF^l{}_{,r})_{,l} + (F^rw^l{}_{,r})_{,l}\,.\end{aligned}$$

Although ζ, F and w are generally neither globally defined vector field nor 1-forms, each term of the above (*) contains two of ζ, F, w. Thus each term can be considered as a δ-image of a globally defined 1-form. As a and b are scalar fields, we have (14.7) from (14.6).

PROPOSITION 14.2. *Suppose that ζ_U is an incompressible vector field on each U and each trajectory of ζ_U is a geodesic. Then an infinitesimal $(m-1)^s$-conformal transformation u on M satisfies*

$$L(u)R = -aR - bR_1(\zeta_U, \zeta_U) - (1-m)\delta da + \zeta_U(\zeta_Ub) + \delta db\,.$$

PROOF. In (14.6), we put $F=0$ and use the relation $w^i{}_{,j}w^j = 0$, $w^i{}_{,i}=0$. Then Proposition 14.2 follows.

COROLLARY 14.3. *Besides the assumptions on* ζ_U *as in Proposition* 14.2, *we suppose that* M *is of constant scalar curvature and* u *is an infinitesimal* $(m-1)^s$*-homothety such that* b *is also constant. Then we have*

$$aR + bR_1(\zeta_U, \zeta_U) = 0 .$$

Particularly,

(1) *If* M *is an Einstein space, we have* $(am+b)R = 0$. *So if* $R \neq 0$, *we get* $am+b=0$.

(2) *If* ${}^e\zeta$ *is a parallel field and* $R \neq 0$, *then* u *is an infinitesimal* $(m-1)^s$*-isometry.*

Propositions 14.1 and 14.2 are useful in §16.

The properties of an infinitesimal (m−1)-conformal transformation, which leaves R, R_{jk}, or $R^i{}_{jkl}$ invariant respectively, will be studied in other papers.

15. Lie algebras of infinitesimal $(m-1)$-conformal transformations and Lie transformation groups. In this section, we prove that the groups of certain $[m-1]$-conformal transformations are Lie groups, if the Riemannian manifold satisfies some conditions.

Let u be an infinitesimal $[m-1]$-conformal transformation:

$$L(u)g = ag + w\otimes F + F\otimes w + bw\otimes w ,$$

$$L(u)w = cw .$$

Then we have a local 1-parameter group φ_t $(|t| < q(x))$ of local transformations of M:

$$u_x = \lim_{t\to 0} \frac{\varphi_t x - x}{t} ,$$

where q is a positive function on M. We fix a point x_0, a positive number q_0 and neighborhoods U and V of x_0 satisfying $\varphi_t V \subset U$, for any $t: |t| < q_0 < q(x_0)$. As a first step, we consider maps $\varphi_t: V \to \varphi_t V$.

LEMMA 15.1. *There exists a family of differentiable functions* γ_t $(|t| < q_0)$ *on* V *such that* $\varphi_t{}^* w = \gamma_t w$.

Proof is standard and similarly done as the proof of Lemma 15.2, so we shall omit it.

LEMMA 15.2. *Each* φ_t $(|t| < q_0)$ *is an* $(m-1)$*-conformal transformation of* V *onto* $\varphi_t V$.

PROOF. First let X, Y, A, B be any tangent vectors at x_0 which belong to the distribution D_{x_0}, such that the inner products of X, Y and A, B are not zero. Then we have a real number λ such that $g_{x_0}(X, Y) = \lambda g_{x_0}(A, B)$. We prove $g_{\varphi_t x_0}(\varphi_t X, \varphi_t Y) = \lambda g_{\varphi_t x_0}(\varphi_t A, \varphi_t B)$, for this purpose we put

$$\Xi(t) = (\varphi_t{}^* g)(X, Y) - \lambda(\varphi_t{}^* g)(A, B)\,. \tag{15.1}$$

It is clear by definition that $\Xi(0) = 0$. As Ξ is a function of t ($\Xi : (-q_0, q_0) \to R$), we can differentiate it and get

$$\begin{aligned}\frac{d\Xi}{dt} &= \lim_{s\to 0} \frac{\varphi_{t+s}{}^* g - \varphi_t{}^* g}{s}(X, Y) - \lambda \lim_{s\to 0} \frac{\varphi_{t+s}{}^* g - \varphi_t{}^* g}{s}(A, B) \\ &= \lim_{s\to 0} \frac{\varphi_s{}^* g - g}{s}(\varphi_t X, \varphi_t Y) - \lambda \lim_{s\to 0} \frac{\varphi_s{}^* g - g}{s}(\varphi_t A, \varphi_t B) \\ &= (L(u)g)(\varphi_t X, \varphi_t Y) - \lambda(L(u)g)(\varphi_t A, \varphi_t B) \\ &= ag(\varphi_t X, \varphi_t Y) - \lambda\, ag(\varphi_t A, \varphi_t B)\,,\end{aligned}$$

since $\varphi_t X, \varphi_t Y, \varphi_t A$ and $\varphi_t B$ belong to $D_{\varphi x_0}$ by Lemma 15.1. Therefore we get

$$\frac{d\Xi}{dt} = a(\varphi_t x_0)\, \Xi(t)\,. \tag{15.2}$$

This means that Ξ is of the form $pe^{\int a dt}$, p denoting a constant. By $\Xi(0) = 0$, we have $\Xi(t) = 0$ identically. Thus we get

$$\frac{(\varphi_t{}^* g)(X, Y)}{g(X, Y)} = \frac{(\varphi_t{}^* g)(A, B)}{g(A, B)} \tag{15.3}$$

for all $X, Y, A, B \in D_{x_0}$, $g(X, Y) \neq 0$, $g(A, B) \neq 0$. And φ_t is an $(m-1)$-conformal transformation.

LEMMA 15.3. *If u is an infinitesimal $[m-1]^s$-conformal transformation, then φ_t is an $(m-1)^s$-conformal transformation of V_t onto $\varphi_t V$.*

PROOF. By Lemma 15.2, we have α_t, β_t and θ_t of functions and 1-forms on V such that

$$\varphi_t{}^* g = \alpha_t g + w \otimes \theta_t + \theta_t \otimes w + \beta_t w \otimes w\,. \tag{15.4}$$

We prove $\theta_t = 0$. Let X be any tangent vector x_0 belonging to D_{x_0}. Then

$t \to \theta_t(X)$ defines a function $\Xi' : (-q_0, q_0) \to R$. As $\theta_t(X) = (\varphi_t{}^* g)(\zeta, X)$, we have

$$\begin{aligned} \frac{d\Xi'}{dt} &= \lim_{s\to 0} \frac{\varphi_s{}^* g - g}{s} (\varphi_t \zeta, \varphi_t X) \\ &= a(\varphi_t x_0)\, \Xi'(t) . \end{aligned}$$

Thereby $\Xi'(t)=0$ holds and so $\theta_t = 0$ follows.

LEMMA 15.4. *If u is an infinitesimal $[m-1]$-homothety, then φ_t is an $(m-1)$-homothety. In particular, if u is an infinitesimal $[m-1]$-isometry, then φ_t is an $(m-1)$-isometry.*

PROOF. We put

$$\Xi''(t, x) = \alpha_t(x) \qquad |t| < q_0,\ x \in M .$$

Then we have

$$\frac{\partial \Xi''}{\partial t}(t, x) = \lim_{s\to 0} \frac{\alpha_s(\varphi_t x)\alpha_t(x) - \alpha_t(x)}{s} ,$$

since $\alpha_{t+s}(x) = \alpha_s(\varphi_t x)\alpha_t(x)$ by (9. 4). Thus we get

$$\begin{aligned} \frac{\partial \Xi''}{\partial t}(t, x) &= \alpha_t(x) \frac{\partial \Xi''}{\partial t}(0, \varphi_t x) \\ &= a\Xi''(t, x) , \end{aligned} \tag{15. 5}$$

because by assumption, $\frac{\partial \Xi''}{\partial t}(0, x) = a =$ constant. And as a solution of (15. 5), we have

$$\Xi''(t, x) = f(x) e^{at} , \tag{15. 6}$$

where f is a function on V independent of t. On the other hand $\Xi''(0, x) = \alpha_0(x) = 1$, and so $f(x)=1$. This shows that $\Xi''(t, x) = \Xi''(t)$ is constant e^{at} on V for each $t : |t| > q_0$. In particular, if $a=0$, then $\alpha_t = 1$.

Similarly we can prove

LEMMA 15.5. *If c is constant, then γ_t in Lemma 15.1 is constant.*

We use the notations:

$\mathfrak{P} = \{u:$ infinitesimal $[m-1]$-conformal transformation$\}$,
$\mathfrak{H} = \{u:$ infinitesimal $[m-1]$-homothety$\}$,
$\mathfrak{J} = \{u:$ infinitesimal $[m-1]$-isometry$\}$,
$\mathfrak{P}^s = \{u:$ infinitesimal $[m-1]^s$-conformal transformation$\}$.

And we put

$$\mathfrak{H}^s = \mathfrak{H} \cap \mathfrak{P}^s, \quad \mathfrak{J}^s = \mathfrak{J} \cap \mathfrak{P}^s.$$

By definition we have $\mathfrak{P} \supset \mathfrak{H} \supset \mathfrak{J}$, concerning a bracket operation, we have

PROPOSITION 15.6.

$$[\mathfrak{P}, \mathfrak{P}] \subset \mathfrak{P}, \quad [\mathfrak{P}^s, \mathfrak{P}^s] \subset \mathfrak{P}^s. \tag{15.7}$$

$$[\mathfrak{H}, \mathfrak{H}] \subset \mathfrak{J}, \quad [\mathfrak{H}^s, \mathfrak{H}^s] \subset \mathfrak{J}^s. \tag{15.8}$$

$$[\mathfrak{J}, \mathfrak{J}] \subset \mathfrak{J}, \quad [\mathfrak{J}^s, \mathfrak{J}^s] \subset \mathfrak{J}^s. \tag{15.9}$$

By preceding Lemmas, we have

PROPOSITION 15.7. *If u is an element of $\mathfrak{P}$, $\mathfrak{H}$, $\mathfrak{J}$, $\mathfrak{P}^s$, $\mathfrak{H}^s$ or $\mathfrak{J}^s$ and generates a 1-parameter group φ_t $(t \in R)$ of global transformations of M, then each φ_t belongs to Π, Θ, Φ, Π^s, Θ^s or Φ^s respectively.*

LEMMA 15.8. *Let u be an infinitesimal transformation such that $L(u)g = ag + bw \otimes w$, where b is a constant. Then the set of all such u is finite demensional.*

PROOF. Let u be an element of the set such that b is not zero. We define $\bar{u}$ by $\bar{u} = (1/b)u$, then

$$L(\bar{u})g = a_0 g + w \otimes w. \tag{15.10}$$

where a_0 is a differentiable function on M. Then for any element v of the set:

$$L(v)g = a'g + b'w \otimes w, \tag{15.11}$$

we have

$$v = (v - b'\bar{u}) + b'\bar{u} \tag{15.12}$$

where $v-b'\bar{u}$ is an infinitesimal conformal transformation. Thus the set of such $v-b'\bar{u}$ is finite dimensional, whence the set of such v is also finite demensional.

THEOREM 15.9. *The subgroup of* Θ^s, *whose element satisfies* $\varphi^* w_V = \gamma_{VU} w_U$ *for some constant* γ_{VU}, *is a Lie group.*

PROOF. By R. S. Palais' theorem [13], it is enough to prove finite dimensionality of the Lie subalgebra of $\mathfrak{H}^s$, whose element generates a 1-parameter group of global $[m-1]^s$-homotheties which satisfy $\varphi_t^* w_V = \gamma_{tVU} w_U$ for some constant γ_{tVU} for each $t \in R$.

Any element u of the Lie subalgebra satisfies $L(u)w=cw$ and $L(u)g=ag+bw\otimes w$, where c and a are constant. Then b is also constant. Thus by Lemma 15.8, the Lie subalgebra is finite dimensional.

LEMMA 15.10. *If* ζ_U *is a Killing vector field for each* U *and if* u *is an element of* $\mathfrak{P}^s$. *Then* $L(\zeta_U)a = 0$.

PROOF. Taking the Lie derivative of $L(u)g$ with respect to ζ we have

$$L(\zeta)L(u)g = \zeta a\cdot g + \zeta b\cdot w\otimes w\,, \tag{15.13}$$

where we have used $L(\zeta)g=0$ and $L(\zeta)w=0$. And as

$$L(c\zeta) = -L([u,\zeta]) = -L(u)\,L(\zeta) + L(\zeta)\,L(u)$$

and $L(c\zeta)g=dc\otimes w+w\otimes dc$, we have

$$dc\otimes w + w\otimes dc = \zeta a\cdot g + \zeta b\cdot w\otimes w\,. \tag{15.14}$$

In the above equation each term excepting $\zeta a\cdot g$ contains w, so we see that $\zeta a=0$.

LEMMA 15.11. *Suppose that the distribution defined by* ${}^\varepsilon\zeta$ *is regular,* ζ_U *is a Killing vector field, and each trajectory of* ${}^\varepsilon\zeta$ *is complete. Then the set* $M/{}^\varepsilon\zeta = \tilde{M}$ *of all trajectories of* ${}^\varepsilon\zeta$ *becomes a Riemannian manifold and each* $u\in\mathfrak{P}^s$ *induces an infinitesimal conformal transformation* $\tilde{u}$ *on* $\tilde{M}$.

PROOF. Following [21], first we assume that there exists a point x of M such that the trajectory $l(x)$ which passes through x is closed. Then we have the length $s=|l(x)|$ of $l(x)$, and we take a sufficiently small tubular neighborhood $W=W(l(x))$ of $l(x)$.

On W we can define a vector field $\bar{\zeta}$ such that $\bar{\zeta}|_U=\zeta_U$ or $-\zeta_U$ for each U if $U\cap W$ is non-empty. Then $\bar{\zeta}$ is a Killing vector field on W and generates a 1-parameter group $\phi_t(t\in R)$ of isometries of W. And as $\bar{\zeta}$ is also regular, we can conclude that ϕ_s is an identity transformation of W and each trajectory of $\bar{\zeta}$ and hence ${}^{\varepsilon}\zeta$ is of constant length s ([21]).

Therefore either all trajectories of ${}^{\varepsilon}\zeta$ are homeomorphic to a circle, or all trajectories are homeomorphic to the real line R. By [13] or [21], $M/{}^{\varepsilon}\zeta$ is a differentiable manifold which has Riemannian metric h such that $g=\pi^*h+w\otimes w$, π denoting the natural projection: $M\to M/{}^{\varepsilon}\zeta=\tilde{M}$.

It may be remarked that if ${}^{\varepsilon}\zeta$ is a globally defined vector field, then $M/{}^{\varepsilon}\zeta$ is a principal fiber bundle.

Now let $u\in\mathfrak{P}^s$. As $L(\zeta)u=-L(u)\zeta=c\zeta$, by the differential π of π, $\pi u=\tilde{u}$ is a vector field on $\tilde{M}$. Denote by φ_t and $\tilde{\varphi}_t$ the (local) 1-parameter groups of (local) transformations generated by u and $\tilde{u}$, then they satisfy $\pi\varphi_t=\tilde{\varphi}_t\pi$.

Using the fact that φ_t is an $[m-1]^s$-conformal transformation, we have

$$
\begin{aligned}
(\tilde{\varphi}_t{}^*h)(X,Y) &= h(\tilde{\varphi}_tX,\tilde{\varphi}_tY)\\
&= h(\pi\varphi_t\pi^{-1}X,\pi\varphi_t\pi^{-1}Y)\\
&= (\varphi_t{}^*(\pi^*h))(\pi^{-1}X,\pi^{-1}Y)
\end{aligned}
$$

for any tangent vectors X,Y at $\tilde{x}\in\tilde{M}$, where we consider $\pi^{-1}X$ as a tangent vector at $x\in\tilde{x}$ such that $w(\pi^{-1}X)=0$ and $\pi(\pi^{-1}X)=X$. Of course $\pi^{-1}X$ at x is uniquely determined and we can prove that the value of $h_{\tilde{\varphi}_t\tilde{x}}(\pi_{\varphi x}\varphi_{tx}\pi^{-1}X_{\tilde{x}}, \pi_{\varphi x}\varphi_{tx}\pi^{-1}Y_{\tilde{x}})$ does not depend on the choice of $x\in\tilde{x}$ and the choice of $\pi^{-1}X$ or $\pi^{-1}Y$ so far as $\pi(\pi^{-1}X)=X$ and $\pi(\pi^{-1}Y)=Y$ are satisfied, because the difference is of the form $k\zeta$, for some real number k. Then, as

$$
\begin{aligned}
\varphi_t{}^*(\pi^*h) &= \varphi_t{}^*(g-w\otimes w)\\
&= \alpha_tg+\beta_t w\otimes w-\gamma_t^2 w\otimes w\,,
\end{aligned}
$$

we obtain

$$
\begin{aligned}
(\tilde{\varphi}_t{}^*h)(X,Y) &= (\alpha_tg)(\pi^{-1}X,\pi^{-1}Y)\\
&= (\alpha_t\pi^*h+\alpha_tw\otimes w)(\pi^{-1}X,\pi^{-1}Y)\\
&= \alpha_th(X,Y)\,.
\end{aligned}
$$

Notice here that α_t is constant on each trajectory of ${}^{\varepsilon}\zeta$. Namely by Lemma 15.10, we have $\zeta a=0$ and by the almost similar method in Lemma 15.4, we can show that $\zeta\alpha_t=0$. Therefore $\tilde{u}$ is an infinitesimal conformal transformation on $\tilde{M}$.

THEOREM 15.12. *Suppose that the* 1*-dimensional distribution by* ${}^{\varepsilon}\zeta$ *is regular,* ζ_U *is a Killing vector field and each trajectory of* ${}^{\varepsilon}\zeta$ *is complete. Then the subgroup* Π^{sc} *of* Π^s *which consists of the element* φ *satisfying* $\varphi^* w_V = \gamma_{VU} w_U$ *on* $U \cap \varphi^{-1}V$ *for some constant* γ_{VU} *is a Lie group.*

PROOF. We devide the proof into two parts.

(1°) *The case*: ${}^{\varepsilon}\zeta$ is not a parallel field.

Denote by $\mathfrak{P}^{sc}$ the Lie subalgebra of $\mathfrak{P}^s$ which consists of all u satisfying $L(u)w=cw$ for some constant c. The map $\pi: u \to \tilde{u}$ gives a homomorphism of $\mathfrak{P}^s$, also of $\mathfrak{P}^{sc}$, into the set of all infinitesimal conformal transformations on $\tilde{M}$ as Lie algebras. The kernel of π is the set of the form ${}^{\varepsilon}f\,{}^{\varepsilon}\zeta$ for some ${}^{\varepsilon}f$, f_U is a scalar field on each U. As ζ_U is a Killing vector field, ${}^{\varepsilon}f\,{}^{\varepsilon}\zeta$ belongs to $\mathfrak{P}^{sc}$ if and only if $L(f\zeta)w = df = rw$ for some constant r. Taking the exterior differentiation of $df_U = rw_U$, we have $rdw_U=0$ on each U. However, ${}^{\varepsilon}\zeta$ is a Killing vector field and not a parallel field, there exists U on which $dw_U \neq 0$. Consequently we get $r=0$. Then we have $df_U=0$ for each U. So $|{}^{\varepsilon}f|$ of ${}^{\varepsilon}f\,{}^{\varepsilon}\zeta$ is constant. Of course as ${}^{\varepsilon}f\,{}^{\varepsilon}\zeta$ must be a vector field, if M admits non-trivial ${}^{\varepsilon}f$ we can assume that ζ is a globally defined vector field by suitable choice of $\zeta_U, -\zeta_U$. And so the kernel is given by $\{t\zeta, t \in R\}$. Thus, as the set of all infinitesimal conformal transformations on $\tilde{M}$ is finite dimensional, $\mathfrak{P}^{sc}$ is also finite dimensional.

(2°) *The case*: ${}^{\varepsilon}\zeta$ is a parallel field.

In this case, we take the universal covering manifold $\bar{M}$ of M and define $\bar{g}, {}^{\varepsilon}\bar{\zeta}, {}^{\varepsilon}\bar{w}$ and $\bar{u}$ for $u \in \mathfrak{P}^{sc}$ naturally on $\bar{M}$ by the local diffeomorphisms. Then $\bar{u}$ is also an infinitesimal $[m-1]^s$-conformal transformation on $\bar{M}$. So it suffices to show the finite dimensionality of the set $\{\bar{u}\}$ in $\bar{M}$. As ${}^{\varepsilon}\bar{\zeta}$ is parallel and $\bar{M}$ is simply connected we may assume that $\bar{\zeta}$ is a globally defined vector field on $\bar{M}$. And it is easy to see that $\bar{\zeta}$ is also regular, so we have $\bar{M}/\bar{\zeta}$ and a Riemannian metric $\bar{h}$ on $\bar{M}/\bar{\zeta}$. $\bar{M}/\bar{\zeta}$ is also simply connected, and $\bar{M}$ is a principal fiber bundle over $\bar{M}/\bar{\zeta}$. $\bar{w}$ defines an infinitesimal connection on $\bar{M}$, and $w=0$ is completely integrable. Similarly to the case (1°), we consider the projection of $\bar{u}$ by the projection $\bar{M} \to \bar{M}/\bar{\zeta}$, and study its kernel. Then any element of the kernel is of the form $\bar{f}\,\bar{\zeta}$ for some scalar field $\bar{f}$ satisfying $d\bar{f}=r\bar{w}$ for some constant r. As a special case we take $r=1$, then the solution $\bar{f}_0$ of $\bar{w}=d\bar{f}$ is uniquely determined, if we fix a horizontal global section S in $\bar{M}$ and give the initial condition $\bar{f}_0=1$ on S, because $\bar{f}$ is constant on each horizontal section. So general solution of $d\bar{f}=r\bar{w}$ is $\bar{f}=r\bar{f}_0+s$ for constant r and s. That is to say the kernel is $\{r(\bar{f}_0\bar{\zeta})+s\bar{\zeta}: r, s \in R\}$ and at most 2-dimensional, Thus $\mathfrak{P}^{sc}$ is finite dimensional. Therefore in both cases (1°), (2°), Π^{sc} makes a Lie group.

REMARK 1. If M admits an element u in $\mathfrak{P}^{sc}$ such that $L(u)w = cw$ for a constant $c \neq 0$. Then ${}^{e}\zeta$ is necessarily regular (see [18], §4).

PEMARK 2. In (1°) above, if ${}^{e}\zeta$ cannot define a globally difined vector field by any choice of $\zeta_U, -\zeta_U$. Then the dimension of $\mathfrak{P}^{sc}$ is not greater than that of the set of all infinitesimal conformal transformations on $\tilde{M}$.

PEMARK 3. In the above Theorem, if M is complete, then each trajectory of ${}^{e}\zeta$ is complete.

16. The cases where *M* is compact and the scalar curvature *R* is constant. In the first place, we prove general theorems.

THEOREM 16.1. *Suppose that M is compact and orientable and u is an infinitesimal $(m-1)$-conformal transformation, then we have*

$$\int_M (am+b)\, d\sigma = 0\,. \tag{16.1}$$

PROOF. Contracting (13. 1) with g^{ij} and noticing that $L(u)g_{ij} = u_{i,j}+u_{j,i}$, we get

$$2u^i{}_{,i} = am + b\,.$$

Integration of the last equation over M is (16. 1).

In the following, we denote the left hand side of (16. 1) by a global inner product $\langle am+b, 1\rangle$.

DEFINITION. We call M a *ζ-space*, if

(i) $\delta w_U=0$, (i.e. ζ_U: volume-preserving),

(ii) $\nabla_{\zeta_U}\zeta_U=0$, (i.e. each trajectory of ζ_U is a geodesic),

(iii) $R_1(\zeta_U, \zeta_U)=T=$constant

for each U. If M satisfies only (i) and (ii), then we say that M has properties (i) and (ii).

EXAMPLES. (1°) K-contact manifold is a ζ-space such that $T = \dfrac{m-1}{4}$ ([17], p. 329).

(2°) If a manifold M admits a parallel direction field ${}^{\varepsilon}\zeta$. Then M is a ζ-space with $T=0$.

THEOREM 16.2. *Suppose that M is compact and orientable and has properties* (i) *and* (ii), *if* $L(u)w_U = cw_U$, *then*

$$<c,1> = 0\,. \tag{16.2}$$

Further if u is an infinitesimal $[m-1]^s$-conformal transformation then we have

$$<a,1> = 0\,, \quad <b,1> = 0\,. \tag{16.3}$$

PROOF. Expression of $L(u)w=cw$ by local coordinates is as follows:

$$w_{i,r}u^r + w_r u^r{}_{,i} = cw_i\,. \tag{16.4}$$

Transvecting (16.4) with w^i, we have

$$c = w_r u^r{}_{,i}\, w^i = (w_r u^r w^i)_{,i} \tag{16.5}$$

because $w^i{}_{,i}=0$ and $w_{r,i}w^i=0$. Although w is not a globally defined tensor, $(w_r u^r w^i)$ is a globally defined vector field. So if we integrate (16.5) over M, we have (16.2). By Theorem 12.1, if u is an $[m-1]^s$-conformal transformation, we have $2c=a+b$. Thus

$$<a+b,1> = 0\,. \tag{16.6}$$

Then (16.1) and (16.6) yield (16.3).

COROLLARY 16.3. *In a compact orientable M with properties* (i) *and* (ii), *if the scalar field c in $L(u)w_U=cw_U$ is constant, then $c=0$.*

THEOREM 16.4. *In a compact orientable M with properties* (i) *and* (ii), *every infinitesimal $[m-1]^s$-homothety is an infinitesimal $[m-1]^s$-isometry.*

PROOF. This is an immediate consequence of Theorem 16.2.

LEMMA 16.5. *If M is compact and orientable, and if the scalar curvature R is constant, we have*

$$<aR+bT,1> = 0 \tag{16.7}$$

for any infinitesimal $(m-1)^s$-*conformal transformation* u.

PROOF. (16.7) follows from Proposition 14.1.

LEMMA 16.6. *In a compact orientable* M *with properties* (i) *and* (ii), *we have*

$$(16.8) \qquad (m-1)<da, da> - <a, aR+bT> - <a, L(u)R> + <da-(\zeta a)w, db-(\zeta b)w> = 0$$

for any infinitesimal $(m-1)^s$-*conformal transformation* u.

PROOF. As δ is dual to d, we have

$$(16.9) \qquad <da, da> - <a, \delta da> = 0\,.$$

On the other hand, by virtue of Proposition 14.2, we get

$$(16.10) \qquad (m-1)\delta da = aR + bT + L(u)R - \delta db - \zeta(\zeta b)\,.$$

And we get

$$<a, \delta db> = <da, db>\,,$$
$$<a, \zeta(\zeta b)> = <aw, d(\zeta b)>$$
$$= -<\zeta a, \zeta b>\,,$$

since $\delta(aw)=a\delta w-\zeta a$ and $\delta w=0$. Moreover

$$(16.11) \qquad <da-(\zeta a)w, db-(\zeta b)w> = <da, db> - <\zeta a, \zeta b>\,.$$

Substitution δda of (16.10) into (16.9) using above relations yields (16.8).

LEMMA 16.7. *In a compact orientable* M, *if* w_U *is a closed form, then we have*

$$(16.12) \qquad <da-(\zeta a)w, db-(\zeta b)w> = -<da, da> + <\zeta a, \zeta a>$$

for any infinitesimal $[m-1]^s$-*conformal transformation.*

PROOF. (16.12) is valid always with respect to the (local) inner product which we denote by (,). So we prove here (16.12) for the inner product.

As $L(u)w = cw$ for some scalar field c, we have $dL(u)w = dc\wedge w + cdw$ by exterior differentiation, where $\wedge$ denotes the exterior product. Since d and $L(u)$ are commutative and $dw=0$, $dc\wedge w=0$ follows. Thus dc is proportional to w and $dc=\zeta c\cdot w$. By Theorem 12.1, we have $da+db = \zeta(a+b)\cdot w$. And so we consider the inner product with da, and get

$$(da, da+db) = (\zeta a, \zeta a+\zeta b)\,,$$

from which we have

$$(16.13)\qquad (da, db) - (\zeta a, \zeta b) = -(da, da) + (\zeta a, \zeta a)\,.$$

Here we notice that (16.11) holds also with respect to the inner product. Then, from (16.11) and (16.13), relation (16.12) for the inner product follows.

LEMMA 16.8. *As for T we have;*
If w_U is a harmonic form,

$$(16.14)\qquad T = -2(\nabla w, \nabla w) \leqq 0\,.$$

If ζ_U is a Killing vector field,

$$(16.15)\qquad T = 2(\nabla w, \nabla w) \geqq 0\,.$$

Proof is easy, since ζ_U is a unit vector field.
As a general statement, we have

PROPOSITION 16.9. *In a compact orientable M, we assume that w_U is a harmonic form for each U. Then an infinitesimal $[m-1]^s$-conformal transformation u is an infinitesimal $[m-1]^s$-isometry if and only if it satisfies*

$$(16.16)\qquad < a, aR + bT + L(u)R > \leqq 0\,.$$

PROOF. If w is a harmonic form, we have $dw=0$ and $\delta w=0$. The length of w being equal to 1, $\nabla_\zeta\zeta=0$ follows from $dw=0$. Then, by Lemma 16.6 and 16.7, we have

$$(16.17)\qquad (m-2)< da, da > - < a, aR+bT+L(u)R > + < \zeta a, \zeta a > = 0\,.$$

If (16.16) holds, (16.17) means that each term is zero. So $da=0$ follows, that is a is constant. Moreover, by $(16.3)_1$ in Theorem 16.2, a is equal to zero.
q.e.d.

If ζ is a parallel field, M is a ζ-space with $T=0$. Therefore we get

THEOREM 16.10. *In a compact orientable M, if ${}^{\varepsilon}\zeta$ is a parallel field and $R=constant \leqq 0$. Then any infinitesimal $[m-1]^s$-conformal transformation is an infinitesimal $[m-1]^s$-isometry.*

REMARK 1. In the above Theorem, essencially we need the condition that a compact orientable M is a ζ-space satisfying $T=0$, $dw=0$ and $R=$constant $\leqq 0$. However, if ζ-space satisfies $dw=0$, w_U is a harmonic form. So T is non-positive by Lemma 16.8. Thus if $T=0$, w_U is necessarily a parallel field.

Next we consider the case where an infinitesimal $[m-1]^s$-conformal transformation u satisfies $c=0$. Of course, the only possible case of $c=$constant is the case $c=0$ in the manifold with properties (i) and (ii) by Theorem 16.2. Now as $2c=a+b$, we have $da=-db$. On the other hand

$$<da-(\zeta a)w, da-(\zeta a)w> \leqq <da, da> . \tag{16.18}$$

If we utilize (16.8) and (16.18), we get

$$(m-2)<da, da> - <a, aR-aT+L(u)R> \leqq 0 . \tag{16.19}$$

So, if the second term is non-negative, we have $da=0$ and $a=0$. Consequently we have also $b=0$ and u is a Killing vector field. Thus we have

PROPOSITION 16.11. *In a compact orientable M with properties* (i) *and* (ii), *an infinitesimal $[m-1]^s$-conformal transformation u such that $L(u)w=0$ is an infinitesimal isometry if and only if it satisfies*

$$<a, aR-aT+L(u)R> \leqq 0 . \tag{16.20}$$

THEOREM 16.12. *In a compact orientable M, if ζ_U is a Killing vector field and $R=constant \leqq 0$. Then any infinitesimal $[m-1]^s$-conformal transformation u satisfying $L(u)w=cw$ for some constant c is a Killing vector filed.*

PROOF. As ζ is a unit and Killing vector field, M has properties (i) and (ii). $c=0$ follows from Corollary 16.3. By (16.15) in Lemma 16.8, T is non-negative. And R is a non-positive constant, (16.20) holds good. Then by Proposition 16.11, u is a Killing vector field.

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