GENERALIZED JAMES PRODUCT AND THE HOPF CONSTRUCTION

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1. Introduction. I. M. James [6] discussed the homotopy theory of maps into an H-space and defined a product $<\alpha,\beta> \in \pi_{p+q}(X)$ for $\alpha \in \pi_p(X)$ and $\beta \in \pi_q(X)$ which induces a bilinear pairing of $\pi_p(X)$ with $\pi_q(X)$ to $\pi_{p+q}(X)$. We generalize these results to the generalized homotopy groups of an H-space. First, by means of the mapping cone, we generalize the notion of sepration elements in [6].

In his paper 'The generalized Whitehead product', M. Arkowitz gave a generalization of Whitehead product and then introduced a homotopy equivalence between the product space of suspension spaces and some mapping cone. As a main tool of our generalization of James product, we shall use this homotopy equivalence.

In §5 we give an alternative definition of the Hopf construction and we give its characterization. Theorem 5.5 is a generalization of Lemma 8.2 in [5] and our definition of the Hopf construction is a generalization of Definition 8.3 in [4].

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2. Preliminaries. Throughout this paper all spaces have base points denoted by $*$ and respected by maps and homotopies. Here we list some definitions and notations which we shall use throughout.

Following Eckmann and Hilton [2], we shall say that $X \xrightarrow{f} Y \xrightarrow{p} Y/f(X)$ is a cofibration if, for any space Z and maps $g: X \to Z$, $G: Y \to Z$ with $g = G\circ f$, each homotopy of g can be obtained by composing f with some homotopy of G.

The (reduced) suspension ΣX of X is the space obtained from $X \times I$ by identifying $X\times \dot{I} \cup * \times I$ to a point. We denote the point of ΣX by $<x, t>$.

The (reduced) cone CX of X is the space obtained from $X\times I$ by identifying $X \times 0 \cup * \times I$ to a point. We denote the point of CX by (x, t).

Given a map $f: X \to Y$, the mapping cone C_f of f is the space obtained from $CX \cup Y$ by identifying $(x, 1)$ with $f(x)$.

We denote by $X \vee Y$ the subspace $X \times * \cup * \times Y$ of $X \times Y$. The collapsed product $X\# Y$ of X and Y is the space from $X\times Y$ by identifying $X\vee Y$ to a point. An H-space is a pair consisting of a space X and a map $\mu: X\times X \to X$ such that $\mu | X\times * = \text{identity} = \mu | *\times X$. The map μ is called multiplication.

For the notational convenience we abbreviate $\mu(x, y)$ to $x\cdot y$. Following I. M. James, we refer to countable CW-complexes with one vertex as special complexes. Any connected countable CW-complex can be deformed into a special complex without altering its homotopy type. Let X_∞ denote the reduced product space of X, as defined in [4]. Then it is well known ([4]) that, if X is a special complex, then X_∞ is a special complex which contains X as a subcomplex, and that X_∞ is an associative H-space with $*$ as the unit and multiplication by juxtaposition. Let X and Y be special complexes and let $f: X\to Y$ be a map. Then the induced map $f_\infty : X\to Y$, as defined in §1 in [4], is multiplicative.

3. Separation elements. Let $f: A\to X$ be a map. Then we have a cofibration $X\to C_f \to \Sigma A$. Let Y be any space and let $u, v: C_f \to Y$ be maps such that $u | X = v | X$. Then a map $w: \Sigma A\to Y$ is defined by

$$w\langle a, t\rangle = \begin{cases} u(a, 2t) & 9 \leqslant t \leqslant 1/2 \\ v(a, 2-2t) & 1/2 \leqslant t \leqslant 1 . \end{cases}$$

We denote the homotopy class $[w]$ of w by $d(u, w)$ and we say it a separation element of u and v. $d(u, v)$ generalizes one defined in [6]. The following relations are easily verified and we shall omit the proofs except (3.3).

THEOREM 3.1. *Let* $u, v: C_f \to Y$ *be maps such that* $u | X = v | X$. Then $u \simeq v$ *rel* X *if and only if* $d(u, v) = 0$.

COROLLARY. *If* $u: C_f \to Y$ *is a map, then* $d(u, u) = 0$.

THEOREM 3.2. *Let* $u, v, w: C_f \to Y$ *be maps such that* $u | X = v | X = w | X$. *Then* $d(u, w) = d(u, v) + d(v, w)$.

COROLLARY. *Let* u, v *be maps such that* $u | X = v | X$. *Then*
$$d(u, v) + d(v, u) = 0 .$$

THEOREM 3.3. *Let* $\delta \in \pi(\Sigma A, Y)$ *and* $u: C_f \to Y$ *be a map. Then there exists a map* $v: C_f \to Y$ *such that* $v | X = u | X$ *and* $d(u, v) = \delta$.

PROOF. Let δ be represented by a map $d: \Sigma A\to Y$. Then we define a map $v: C_f \to Y$ as follows:

$$v(x) = u(x) \qquad x \in X,$$

$$v(a, t) = \begin{cases} d<a, 1-2t> & 0 \leqslant t \leqslant 1/2 \\ u(a, 2t-1) & 1/2 \leqslant t \leqslant 1. \end{cases}$$

Now $d(u, v)$ is represented by a map $c: \Sigma A \to Y$ given by

$$c<a, t> = \begin{cases} u(a, 2t) & 0 \leqslant t \leqslant 1/2 \\ u(a, 3-4t) & 1/2 \leqslant t \leqslant 3/4 \\ d<a, 4t-3> & 3/4 \leqslant t \leqslant 1. \end{cases}$$

Thus we have $d(u, v) = \delta$.

THEOREM 3.4. *Let $u_t, v_t: C_f \to Y$ be homotopies such that $u_t|X = v_t|X$. Then* $$d(u_0, v_0) = d(u_1, v_1).$$

THEOREM 3.5. *Let $u, v: C_f \to Y$ be maps such that $u|X = v|X$ and $h: Y \to Z$ any map. Then* $d(hu, hv) = h_* d(u, v)$.

4. Generalized James product. Throughout §4, §5 of this paper, we shall work in the category of connected countable *CW*-complexes.

Following Arkowitz [1], the following results are known; Let $\widetilde{k}: \Sigma(A_1 \# A_2) \to \Sigma A_1 \vee \Sigma A_2$ be a G.W.P.-map determined by injections $i_1: \Sigma A_1 \to \Sigma A_1 \vee \Sigma A_2$ and $i_2: \Sigma A_2 \to \Sigma A_1 \vee \Sigma A_2$. Then there exists a homotopy equivalence F: $C\Sigma(A_1 \# A_2), \Sigma(A_1 \# A_2) \to \Sigma A_1 \times \Sigma A_2, \Sigma A_1 \vee \Sigma A_2$ such that $F|\Sigma(A_1 \# A_2) = \widetilde{k}$. Also a map G: $C_{\widetilde{k}}, \Sigma A_1 \vee \Sigma A_2 \to \Sigma A_1 \times \Sigma A_2, \Sigma A_1 \vee \Sigma A_2$ with $G|\Sigma A_1 \vee \Sigma A_2$ = identity is defined by F and it is a homotopy equivalence.

Then we have a commutative diagram:

$$\begin{array}{ccc} C_{\widetilde{k}}, \Sigma A_1 \vee \Sigma A_2 & \xrightarrow{G} & \Sigma A_1 \times \Sigma A_2, \Sigma A_1 \vee \Sigma A_2 \\ \downarrow p & & \downarrow q \\ \Sigma^2(A_1 \# A_2) & \xrightarrow{\varphi} & \Sigma A_1 \# \Sigma A_2 \end{array} \tag{4.1}$$

where p and q denote the projection and φ is a map defined by G. φ is a homotopy equivalence and we denote its homotopy inverse by ψ. From (4.1) we have a commutative diagram:

$$
(4.2)\qquad
\begin{array}{ccccc}
\Sigma A_1 \vee \Sigma A_2 & \xrightarrow{\ i\ } & C_{\tilde{k}} & \xrightarrow{\ p\ } & \Sigma^2(A_1 \# A_2) \\
\Big\downarrow id & & \Big\downarrow G & & \Big\downarrow \varphi \\
\Sigma A_1 \vee \Sigma A_2 & \xrightarrow{\ j\ } & \Sigma A_1 \times \Sigma A_2 & \xrightarrow{\ q\ } & \Sigma A_1 \# \Sigma A_2\,.
\end{array}
$$

Let X be any space and let $u, v : \Sigma A_1 \times \Sigma A_2 \to X$ be maps such that $u | \Sigma A_1 \vee \Sigma A_2 = v | \Sigma A_1 \vee \Sigma A_2$. Consider the composite maps $uG, vG : C_{\tilde{k}} \to X$, then evidently $uG | \Sigma A_1 \vee \Sigma A_2 = vG | \Sigma A_1 \vee \Sigma A_2$. Hence from §3 $d(uG, vG) \in \pi(\Sigma^2(A_1 \# A_2), X)$ may be defined. We now define $d(u, v) \in \pi(\Sigma A_1 \# \Sigma A_2, X)$ to be $\psi^* d(uG, vG)$.

Especially let X be an H-space with multiplication μ in the rest of this section. Let $\alpha \in \pi(\Sigma A_1, X)$, $\beta \in \pi(\Sigma A_2, X)$ be represented by $f : \Sigma A_1 \to X$, $g : \Sigma A_2 \to X$ respectively. We define maps $h, k : \Sigma A_1 \times \Sigma A_2 \to X$ by

$$h = \mu \circ (f \times g), \qquad k = \mu \circ (g \times f) \circ T,$$

where T is the transposition in $\Sigma A_1 \times \Sigma A_2$. Then it is clear that $h | \Sigma A_1 \vee \Sigma A_2 = k | \Sigma A_1 \vee \Sigma A_2$. Hence by the above arguments a separation element $d(h, k) \in \pi(\Sigma A_1 \# \Sigma A_2, X)$ may be defined and it depends only on the homotopy classes of f and g. Thus we may write

$$(4.3) \qquad \langle \alpha, \beta \rangle = d(h, k)\,.$$

In case A_1 and A_2 are spheres, $\langle \alpha, \beta \rangle$ reduces to one defined in [6].

If, in the diagram (4.2), we interchange factors ΣA_1 with ΣA_2, then we have a similar diagram:

$$
(4.4)\qquad
\begin{array}{ccccc}
\Sigma A_2 \vee \Sigma A_1 & \xrightarrow{\ i'\ } & C_{\tilde{k}'} & \xrightarrow{\ p'\ } & \Sigma^2(A_2 \# A_1) \\
\Big\downarrow id & & \Big\downarrow G' & & \Big\downarrow \phi' \\
\Sigma A_2 \vee \Sigma A_1 & \xrightarrow{\ j'\ } & \Sigma A_2 \times \Sigma A_1 & \xrightarrow{\ q'\ } & \Sigma A_2 \# \Sigma A_1
\end{array}
$$

where G', ϕ' are homotopy equivalences corresponding to G, ϕ respectively. We denote by ψ' the homotopy inverse ϕ'. We employ the same notation T for the maps induced by transposition T, for example, $T : \Sigma A_1 \# \Sigma A_2 \to \Sigma A_2 \# \Sigma A_1$ etc. Then we have

THEOREM 4.4. $\langle \beta, \alpha \rangle = -T^* \langle \alpha, \beta \rangle\,.$

PROOF. We set $h' = \mu \circ (g \times f)$ and $k' = \mu \circ (f \times g) \circ T$, where T is the transposition in $\Sigma A_2 \times \Sigma A_1$. Then we have $k' = h \circ T$ and $h' = k \circ T$. By definition

$$\begin{aligned} T^* <\alpha, \beta> &= T^* \psi^* d(hG, kG) \\ &= \psi'^* T^* d(hG, kG) \\ &= \psi'^* d(hGT, kGT) \\ &= \psi'^* d(hTG', kTG') \\ &= \psi'^* d(k'G', h'G') . \end{aligned}$$

However, by Corollary to Theorem 3.2, we have $d(k'G', h'G') = -d(h'G', k'G')$. Thus the theorem is proved.

Let Y be a space and let X be an H-space. The product of two maps $u, v : Y \to X$ is the map $u \cdot v : Y \to X$ which is defined by

$$(u \cdot v)(y) = u(y) \cdot v(y) \qquad y \in Y . \tag{4.5}$$

LEMMA 4.6. *Let $h, k, h', k' : \Sigma A_1 \times \Sigma A_2 \to X$ be maps such that h, k have the same section and h', k' have the same section. Then*

$$d(h \cdot h', k \cdot k') = d(h, k) + d(h', k') .$$

PROOF. Evidently $h \cdot h'$ and $k \cdot k'$ have the same section and hence $(h \cdot h') \circ G$ and $(k \cdot k') \circ G$ do so. It is sufficient to prove that $d((h \cdot h') \circ G, (k \cdot k') \circ G) = d(hG, kG) + d(h'G, k'G)$.

We easily see that $(h \cdot h') \circ G = hG \cdot h'G$ and $(k \cdot k') \circ G = kG \cdot k'G$. Now, by definition, $d(hG, kG)$ and $d(h'G, k'G)$ are represented by maps $w, w' : \Sigma^2(A_1 \# A_2) \to X$ which are given as follows;

$$w<x, t> = \begin{cases} hG(x, 2t) & 0 \leqslant t \leqslant 1/2 \\ kG(x, 2-2t) & 1/2 \leqslant t \leqslant 1 \end{cases}$$

$$w'<x, t> = \begin{cases} h'G(x, 2t) & 0 \leqslant t \leqslant 1/2 \\ k'G(x, 2-2t) & 1/2 \leqslant t \leqslant 1 . \end{cases}$$

By [7 ; p. 6, Theorem 1.5] we may regard that $d(hG, kG) + d(h'G, k'G)$ is represented by a map $w \cdot w' : \Sigma^2(A_1 \# A_2) \to X$. However,

$$(w\cdot w')<x,t> = \begin{cases} hG(x,2t)\cdot h'G(x,2t) & 0\leqslant t\leqslant 1/2 \\ kG(x,2-2t)\cdot k'G(x,2-2t) & 1/2\leqslant t\leqslant 1 \end{cases}$$

$$= \begin{cases} (hG\cdot h'G)(x,2t) & 0\leqslant t\leqslant 1/2 \\ (kG\cdot k'G)(x,2-2t) & 1/2\leqslant t\leqslant 1\,. \end{cases}$$

This shows that $d((h\cdot h')\circ G, (k\cdot k')\circ G) = d(hG, hG) + d(h'G, k'G)$.

Let $f: \Sigma A_1 \to X$, $g: \Sigma A_2 \to X$ be the sections of a map $h: \Sigma A_1\times\Sigma A_2 \to X$. We now define $h': \Sigma A_1\times\Sigma A_2\to X$ by $h'(x,y) = f(x)\cdot g(y)(x\in\Sigma A_1, y\in\Sigma A_2)$. Then h and h' have the same section. We write

$$\delta(h) = d(h', h)\,. \tag{4.7}$$

If $k: \Sigma A_1\times\Sigma A_2\to X$ is another map with the same section as h, then k', defined as in h', is equal to h'. It follows from (3.2) that

$$\delta(k) = \delta(h) + d(h,k)\,.$$

THEOREM 4.8. *Let $h, h': \Sigma A_1\times\Sigma A_2\to X$ be given maps. Let $f: \Sigma A_1\to X$, $g: \Sigma A_2\to X$ be the sections of h and $f': \Sigma A_1\to X$, $g': \Sigma A_2\to X$ the sections of h'. Setting $\alpha=[f]$, $\alpha'=[f']$, $\beta=[g]$ and $\beta'=[g']$, then*

$$\delta(h\cdot h') = \delta(h) + \delta(h') + <\alpha', \beta>\,.$$

REMARK. Products $f(x)\cdot f(x)\cdot g(y)\cdot g(y)$ and $f(x)\cdot g(y)\cdot f(x)\cdot g(y)$ do not depend on the order in which products are taken. We prove only the former. We define a homotopy $f_s: \Sigma A_1\to X$ by

$$f_s<a,t> = \begin{cases} f<a, \dfrac{2t}{2-s}> & 0\leqslant t\leqslant 1/2 \\ * & 1/2\leqslant t\leqslant 1\,, \end{cases}$$

and we replace $f: \Sigma A_1\to X$ by

$$f_1<a,t> = \begin{cases} f<a,2t> & 0\leqslant t\leqslant 1/2 \\ * & 1/2\leqslant t\leqslant 1\,. \end{cases}$$

By an analogous homotopy we replace $f': \Sigma A_1\to X$ by

$$f_1'<a,t> = \begin{cases} * & 0\leqslant t\leqslant 1/2 \\ f'<a,2t-1> & 1/2\leqslant t\leqslant 1\,. \end{cases}$$

By the same way we replace g, g' by g_1, g_1'. Then $f_1(x)\cdot f_1'(x)\cdot g_1(y)\cdot g_1'(y)$ does not depened on the order in which the product is taken.

PROOF OF THEOREM 4.8. The proof follows as in [6]. We define $h_1=f\circ p_1: \Sigma A_1\times\Sigma A_2\to X$ and $h=g\circ p_2$ where $p_i: \Sigma A_1\times\Sigma A_2\to\Sigma A_i$ is the projection $i=1,2$. Using f' and g', we define h_1' and h_2' in the same way. Setting $k=h_1\cdot h_2$ and $k'=h_1'\cdot h_2'$, we have $\delta(h)=d(k,h)$ and $\delta(h')=d(k',h')$. By Lemma 4.6 $d(k\cdot k',h\cdot h')=d(k,h)+d(k',h')=\delta(h)+\delta(h')$. Since $H=(h_1\cdot h_1')\cdot(h_2\cdot h_2')$ have the same section as $h\cdot h', \delta(h\cdot h')=d(H,h\cdot h')$. By Theorem 3.2 $d(H,h\cdot h')=d(H,k\cdot k')+d(k\cdot k',h\cdot h')$. On the other hand

$$
\begin{aligned}
<\alpha',\beta> &= d(h_1'\cdot h_2, h_2\cdot h_2') \\
&= d(h_1\cdot h_2)+d(h_1'\cdot h_2, h_2\cdot h_1')+d(h_2', h_2') && \text{by Cor. to 3.1} \\
&= d(h_1\cdot(h_1'\cdot h_2), h_1\cdot(h_2\cdot h_1'))+d(h_2', h_2') && \text{by 4.6} \\
&= d(h_1\cdot(h_1'\cdot h_2)\cdot h_2', h_1\cdot(h_2\cdot h_1')\cdot h_2') && \text{by 4.6} \\
&= d(H, k\cdot k') && \text{by Remark.}
\end{aligned}
$$

Thus the proof of Theorem 2.7 is complete.

THEOREM 4.9. *$(\alpha,\beta)\to<\alpha,\beta>$ is a bilinear pairing of $\pi(\Sigma A_1, X)\times\pi(\Sigma A_2, X)$ into $\pi(\Sigma A_1\#\Sigma A_2, X)$.*

PROOF. The proof is analogous to that of [6; Theorem 3.7]. For $\alpha,\alpha'\in\pi(\Sigma A_1, X)$ and $\beta\in\pi(\Sigma A_2, X)$, we only prove

$$<\alpha+\alpha',\beta> = <\alpha,\beta> + <\alpha',\beta>\,.$$

Let $f: \Sigma A_1\to X$, $f': \Sigma A_2\to X$ be the representatives of α and α' respectively and let $g: \Sigma A_2\to X$ be that of β. We may replace f, f' by f_1, f_1' given by

$$f_1<a,t> = \begin{cases} f<a,2t> & 0\leqslant t\leqslant 1/2 \\ * & 1/2\leqslant t\leqslant 1 \end{cases}$$

$$f_1'<a,t> = \begin{cases} * & 0\leqslant t\leqslant 1/2 \\ f<a,2t-1> & 1/2\leqslant t\leqslant 1\,. \end{cases}$$

We now define $h, h': \Sigma A_1\times\Sigma A_2\to X$ by

$$h(x,y)=g(y)\cdot f_1(x)\,,\qquad h'(x,y)=f_1'(x)\,.$$

Then $(h\cdot h')(x,y)=(g(y)\cdot f_1(x))\cdot f_1'(x)=g(y)\cdot f_1(x)\cdot(f_1'(x))$. Since $\alpha+\alpha'$ is represented by a map $f\cdot f':\Sigma A_1\to X$, we have

$$<\alpha+\alpha',\beta> = \delta(h\cdot h')\,.$$

Hence by Theorem 4.8 $<\alpha+\alpha',\beta> = \delta(h)+\delta(h')+<\alpha',\beta>$. However $\delta(h) = <\alpha,\beta>$ and $\delta(h')=0$. This completes the proof.

5. The Hopf construction. Following Arkowitz [1], we define a map $H\colon C\Sigma(A_1\# A_2),\Sigma(A_1\# A_2)\to C_{\tilde{k}},\Sigma A_1\vee\Sigma A_2$ to be the composition of the injection $C\Sigma(A_1\# A_2)\subset C\Sigma(A_1\# A_2)\cup\Sigma A_1\vee\Sigma A_2$ and the projection $C\Sigma(A_1\# A_2)\cup\Sigma A_1\vee\Sigma A_2\to C_{\tilde{k}}$. Then $H|\Sigma(A_1\# A_2)=\tilde{k}$ and H induces homology isomorphisms. Also we have $F=G\circ H$.

Let $f\colon\Sigma A_1\to X$, $g:\Sigma A_2\to X$ represent $\alpha\in\pi(\Sigma A_1,X)$, $\beta\in\pi(\Sigma A_2,X)$ respectively. Let X_∞ be the reduced product space of X. We now define a map $h:\Sigma A_1\times\Sigma A_2\to X_\infty$ by $h(x,y)=f(x)\cdot g(y)$, where the dot $\cdot$ denotes multiplication in the reduced space X_∞. Then $\bar{h}=h|\Sigma A_1\vee\Sigma A_2:\Sigma A_1\vee\Sigma A_2\to X$. Consider a pair of maps $(\bar{h}\circ\tilde{k},h\circ F)$;

$$\begin{array}{ccc} \Sigma(A_1\# A_2) & \xrightarrow{\ \bar{h}\circ\tilde{k}\ } & X \\ \downarrow{\scriptstyle\iota} & & \downarrow{\scriptstyle i} \\ C\Sigma(A_1\# A_2) & \xrightarrow{\ h\circ F\ } & X_\infty\,, \end{array}$$

where ι and i denote the inclusions.

Since the homotopy class $[(\bar{h}\circ\tilde{k},h\circ F)]\in\pi(C\Sigma(A_1\# A_2),\Sigma(A_1\# A_2);X_\infty,X)$ depends only on α and β, we may write

$$\{\alpha,\beta\} = [(\bar{h}\circ\tilde{k},h\circ F)]\,. \tag{5.1}$$

If $\partial:\pi(C\Sigma(A_1\# A_2),\Sigma(A_1\# A_2);X_\infty,X)\to\pi(\Sigma(A_1\# A_2),X)$ is the boundary homomorphism (cf. [7]), then we have

$$\partial\{\alpha,\beta\} = [\bar{h}\circ\tilde{k}] = [\alpha,\beta]\,,$$

where $[\alpha,\beta]$ is the generalized Whitehead product [1] of α and β.

Let $J\colon\pi(\Sigma^2(A_1\# A_2),X_\infty)\to\pi(C\Sigma(A_1\# A_2),\Sigma(A_1\# A_2);X_\infty,X)$ be a homomorphism defined in [7; §4].

THEOREM 5.2. *Let* $\alpha\in\pi(\Sigma A_1,X)$, $\beta\in\pi(\Sigma A_2,X)$. *Then*

$$\{\alpha, \beta\} - T^*\{\beta, \alpha\} = J\phi^* < i_*\alpha, i_*\beta > ,$$

where i_* denotes the homomorphism induced by the inclusion $i: X \to X_\infty$.

PROOF. By definition of H we have the next commutative diagram;

$$\begin{array}{ccccc} C\Sigma(A_1 \# A_2), \Sigma(A_1 \# A_2) & \xrightarrow{H} & C_{\tilde{k}}, \Sigma A_1 \vee \Sigma A_2 & \xrightarrow{G} & \Sigma A_1 \times \Sigma A_2, \Sigma A_1 \vee \Sigma A_2 \\ & & \xrightarrow{\quad F \quad} & & \\ \downarrow p' & & \downarrow p & & \downarrow q \\ \Sigma^2(A_1 \# A_2) & \xrightarrow{id} & \Sigma^2(A_1 \# A_2) & \xrightarrow{\phi} & \Sigma A_1 \# \Sigma A_2 \ . \end{array}$$

Hence we may regard that $\phi^* < i_*\alpha, i_*\beta >$ is equal to $d(hF, kF)$ where $k: \Sigma A_1 \times \Sigma A_2 \to X_\infty$ is defined by $k(x, y) = g(y) \cdot f(x)$. Then $J\phi^* < i_*\alpha, i_*\beta > \in \pi(C\Sigma(A_1 \# A_2), \Sigma(A_1 \# A_2); X_\infty, X)$ is represented by

$$\begin{array}{ccc} \Sigma(A_1 \# A_2) & \xrightarrow{*} & X \\ \iota \downarrow & & \downarrow i \\ C\Sigma(A_1 \# A_2) & \xrightarrow{w'} & X_\infty \ , \end{array}$$

where w' is induced by $w: \Sigma^2(A_1 \# A_2) \to X$,

$$w < x, t > = \begin{cases} hF(x, 2t) & 0 \leqslant t \leqslant 1/2 \\ kF(x, 2-2t) & 1/2 \leqslant t \leqslant 1 \ . \end{cases}$$

Let $\tilde{k}': \Sigma(A_2 \# A_1) \to \Sigma A_2 \vee \Sigma A_1$, $F': C\Sigma(A_2 \# A_1), \Sigma(A_2 \# A_1) \to \Sigma A_2 \times \Sigma A_1, \Sigma A_2 \vee \Sigma A_1$ be defined as in §4 corresponding to $\tilde{k}, F$ respectively. Define $l: \Sigma A_2 \times \Sigma A_1 \to X_\infty$ by $l(y, x) = g(y) \cdot f(x) (x \in \Sigma A_1, y \in \Sigma A_2)$ and set $\bar{l} = l | \Sigma A_2 \vee \Sigma A_1$. Then $T^*\{\beta, \alpha\}$ is represented by

$$\begin{array}{ccccc} \Sigma(A_1 \# A_2) & \xrightarrow{T} & \Sigma(A_2 \# A_1) & \xrightarrow{\bar{l} \circ \tilde{k}'} & X \\ \downarrow & & & & \downarrow i \\ C\Sigma(A_1 \# A_2) & \xrightarrow{T} & C\Sigma(A_2 \# A_1) & \xrightarrow{l \circ F'} & X_\infty \ . \end{array}$$

However we see that $l \circ F' \circ T = l \circ T \circ F = k \circ F$ and $\bar{l} \circ \tilde{k}' \circ T = \bar{k} \circ \tilde{k}$. Hence $T^*\{\beta, \alpha\}$ is also represented by

$$\begin{array}{ccc} \Sigma(A_1 \# A_2) & \xrightarrow{\overline{k}\circ\widetilde{k}} & X \\ \downarrow{\scriptstyle \iota} & & \downarrow{\scriptstyle i} \\ C\Sigma(A_1 \# A_2) & \xrightarrow{k\circ F} & X_\infty . \end{array}$$

On the other hand $\{\alpha, \beta\}$ is represented by

$$\begin{array}{ccc} \Sigma(A_1 \# A_2) & \xrightarrow{\overline{h}\circ\widetilde{k}} & X \\ \downarrow{\scriptstyle \iota} & & \downarrow{\scriptstyle i} \\ C\Sigma(A_1 \# A_2) & \xrightarrow{h\circ F} & X_\infty . \end{array}$$

Obviously $\overline{k}\circ\widetilde{k} = \overline{h}\circ\widetilde{k}$. Hence applying [3; §1.5 Lemma 4] we may conclude the theorem.

Let $i_j : \Sigma A_j \to \Sigma A_1 \times \Sigma A_2$ be the injection and let $p_j : \Sigma A_1 \times \Sigma A_2 \to \Sigma A_j$ be the projection $j = 1, 2$. Define $h : \Sigma A_1 \times \Sigma A_2 \to (\Sigma A_1 \times \Sigma A_2)_\infty$ by $h(x, y) = i_1(x)\cdot i_2(y)$. Then $\{i_1 i_2\}$ is represented by

$$\begin{array}{ccc} \Sigma(A_1 \# A_2) & \xrightarrow{\overline{h}\circ\widetilde{k}} & \Sigma A_1 \times \Sigma A_2 \\ \downarrow{\scriptstyle \iota} & & \downarrow{\scriptstyle i} \\ C\Sigma(A_1 \# A_2) & \xrightarrow{h\circ F} & (\Sigma A_1 \times \Sigma A_2)_\infty . \end{array}$$

Consider the following exact sequence [7]:

$$\pi(\Sigma^2(A_1 \# A_2), (\Sigma A_1 \times \Sigma A_2)_\infty) \xrightarrow{J} \pi(C\Sigma(A_1 \# A_2), \Sigma(A_1 \# A_2); (\Sigma A_1 \times \Sigma A_2)_\infty, \Sigma A_1 \times \Sigma A_2)$$
$$\xrightarrow{\partial} \pi(\Sigma(A_1 \# A_2), \Sigma A_1 \times \Sigma A_2).$$

By [1, Proposition 5.1] we have $[i_1, i_2] = 0$. Hence there exists $y \in \pi(\Sigma^2(A_1 \# A_2), (\Sigma A_1 \times \Sigma A_2)_\infty$ such that $J(y) = \{i_1 i_2\}$. Define $\rho_j : \Sigma A_1 \times \Sigma A_2 \to \Sigma A_1 \times \Sigma A_2$ by $\rho_j = i_j \circ p_j$ $j=1, 2$, then

(5.3) $\qquad \rho_i \circ \rho_i = \rho_i \quad i = 1, 2, \qquad \rho_i \circ \rho_j = * \;$ for $\; i \neq j$.

Let $(\rho_j)_\infty : (\Sigma A_1 \times \Sigma A_2)_\infty \to (\Sigma A_1 \times \Sigma A_2)_\infty$ be the multiplicative map determined by ρ_j. Then $(\rho_j)_\infty$ induces the endomorphism ρ_{j*} of $\pi(C\Sigma(A_1 \# A_2), \Sigma(A_1 \# A_2); (\Sigma A_1 \times \Sigma A_2)_\infty, \Sigma A_1 \times \Sigma A_2)$. Now consider $\rho_{1*}\{i_1 i_2\}$. $\rho_{1*}\{i_1, i_2\}$ is represented by

$$\begin{array}{ccccc}
\Sigma(A_1\#A_2) & \xrightarrow{\overline{h}\circ\widetilde{k}} & \Sigma A_1\times\Sigma A_2 & \xrightarrow{\rho_1} & \Sigma A_1\times\Sigma A_2 \\
\downarrow\iota & & & & \downarrow i \\
C\Sigma(A_1\#A_2) & \xrightarrow{h\circ F} & (\Sigma A_1\times\Sigma A_2)_\infty & \xrightarrow{\rho_{1\infty}} & (\Sigma A_1\times\Sigma A_2)_\infty\,.
\end{array}$$

However we easily see that $\overline{h}=j:\Sigma A_1\vee\Sigma A_2\to\Sigma A_1\times\Sigma A_2$ and $j\widetilde{k}=F\iota$. Also it is easy to check that $i\rho_1=\rho_{1\infty}h$. Therefore the above diagram is reduced to the diagram

$$\begin{array}{ccccc}
\Sigma(A_1\#A_2) & \xrightarrow{\iota} & C\Sigma(A_1\#A_2) & \xrightarrow{F} & \Sigma A_1\times\Sigma A_2 \\
\downarrow\iota & & & & \downarrow i\rho_1=\rho_{1\infty}h \\
C\Sigma(A_1\#A_2) & \xrightarrow{F} & \Sigma A_1\times\Sigma A_2 & \xrightarrow{\rho_{1\infty}h} & (\Sigma A_1\times\Sigma A_2)_\infty\,.
\end{array}$$

Here we shall remark that the following lemma is easily proved.

LEMMA 5.4. *Consider the commutative diagram*:

$$\begin{array}{ccccc}
A & \xrightarrow{\iota} & CA & \xrightarrow{f} & X \\
\downarrow\iota & & & & \downarrow\beta \\
CA & \xrightarrow{f} & X & \xrightarrow{\beta} & Y\,.
\end{array}$$

Then $[(f\iota,\beta f)]=0$ *in* $\pi_1(A,\beta)$.

Therefore we have $\rho_{1*}\{i_1,i_2\}=0$. Similarly $\rho_{2*}\{i_1,i_2\}=0$. Hence $J\rho_{1*}(y)=\rho_{1*}J(y)=0$ and $J\rho_{2*}(y)=0$. Set $x=y-\rho_{1*}(y)-\rho_{2*}(y)$, then the following conditions are satisfied;

$$(5.5)\qquad \begin{array}{ll} \text{(i)} & J(x)=\{i_1,i_2\}\,, \\ \text{(ii)} & \rho_{1*}(x)=0 \quad\text{and}\quad \rho_{2*}(x)=0\,. \end{array}$$

Let x' be another element satisfying the above conditions (i), (ii). Since

$$\pi(\Sigma^2(A_1\#A_2),\Sigma A_1\times\Sigma A_2)\xrightarrow{i_*}\pi(\Sigma^2(A_1\#A_2),(\Sigma A_1\times\Sigma A_2)_\infty)$$
$$\xrightarrow{J}\pi(C\Sigma(A_1\#A_2),\Sigma(A_1\#A_2);(\Sigma A_1\times\Sigma A_2)_\infty,\Sigma A_1\times\Sigma A_2)$$

is exact, there exists $z\in\pi(\Sigma^2(A_1\#A_2),\Sigma A_1\times\Sigma A_2)$ such that $i_*(z)=x-x'$. By the properties of the direct product (cf. [2]), we have

$$\pi(\Sigma^2(A_1 \# A_2), \Sigma A_1 \times \Sigma A_2) \approx \operatorname{Im} \rho_{1*} + \operatorname{Im} \rho_{2*} \quad \text{(direct sum)}.$$

Hence we may write $z = \rho_{1*}(u) + \rho_{2*}(v)$ for some $u, v \in \pi(\Sigma^2(A_1 \# A_2), \Sigma A_1 \times \Sigma A_2)$. Then

$$x - x' = i_*(z) = i_* \rho_{1*}(u) + i_* \rho_{2*}(v) = \rho_{1*} i_*(u) + \rho_{2*} i_*(v)\,.$$

By (5.3) and (5.5) (ii), $\rho_{1*} i_*(u) = \rho_{1*}(x) - \rho_{1*}(x') = 0$. Similarly $\rho_{2*} i_*(v) = 0$. Hence we can conclude that $x = x'$. Thus we have the next theorem;

THEOREM 5.6. *There exists only one* $x \in \pi(\Sigma^2(A_1 \# A_2); (\Sigma A_1 \times \Sigma A_2)_\infty)$ *such that*

$$J(x) = \{i_1, i_2\} \quad \textit{and} \quad \rho_{1*}(x) = \rho_{2*}(x) = 0\,.$$

DEFINITION 5.7. Let $\alpha \in \pi(\Sigma A_1, X)$ and $\beta \in \pi(\Sigma A_2, X)$. We say that $h : \Sigma A_1 \times \Sigma A_2 \to X$ is of type (α, β) if $[h|\Sigma A_1] = \alpha$ and $[h|\Sigma A_2] = \beta$. Suppose that there exists a map $h : \Sigma A_1 \times \Sigma A_2 \to X$ of type (α, β). Let $f: \Sigma A_1 \to X$, $g : \Sigma A_2 \to X$ represent α, β respectively and let $i : X \to X_\infty$ be the injection. We define $h', h'' : \Sigma A_1 \times \Sigma A_2 \to X_\infty$ by $h' = i \circ h$ and $h''(x, y) = f(x) \cdot g(y)$ respectively. By definition $\phi^* \delta(h') = \phi^* d(h'', h)$ is represented by

$$w\langle x, t\rangle = \begin{cases} h''F(x, 2t) & 0 \leqslant t \leqslant 1/2 \\ h'F(x, 2-2t) & 1/2 \leqslant t \leqslant 1 \end{cases} \qquad \langle x, t\rangle \in \Sigma^2(A_1 \# A_2)\,.$$

Recall that $h''F(x, 1) = \overline{h''}\tilde{k}(x)$. By [1 ; Proposition 5.1] $\overline{h''}\tilde{k} \simeq *$. We denote this nullhomotopy by u_t. We now define $w' : \Sigma^2(A_1 \# A_2) \to X_\infty$ by

$$w'\langle x, t\rangle = \begin{cases} h''F(x, 2t) & 0 \leqslant t \leqslant 1/2 \\ iu_{2t-1}(x) & 1/2 \leqslant t \leqslant 1\,. \end{cases}$$

On the other hand $\{\alpha, \beta\}$ is represented by

$$\begin{array}{ccc} \Sigma(A_1 \# A_2) & \xrightarrow{\overline{h''} \circ \tilde{k}} & X \\ \downarrow \iota & & \downarrow i \\ C\Sigma(A_1 \# A_2) & \xrightarrow{h'' \circ F} & X_\infty \end{array}$$

where $\bar{h}'' = h''|\Sigma A_1 \vee \Sigma A_2$. Since $\iota: \Sigma(A_1 \# A_2) \to C\Sigma(A_1 \# A_2)$ is a cofibration, the deformation u_s of $\bar{h}'' \circ \tilde{k}$ may be extended to the deformation v_s of $h'' \circ F$ given by

$$v_s(x, t) = \begin{cases} h''F(x, (1+s)t) & 0 \leqslant t \leqslant 1/(1+s) \\ iu_{t+st-1}(x) & 1/(1+s) \leqslant t \leqslant 1 . \end{cases}$$

Then

$$v_1(x, t) = \begin{cases} h''F(x, 2t) & 0 \leqslant t \leqslant 1/2 \\ iu_{2t-1}(x) & 1/2 \leqslant t \leqslant 1 . \end{cases}$$

Hence we see that $\{\alpha, \beta\} = J[w']$. Next we define $U: C\Sigma(A_1 \# A_2) \to X_\infty$ by $U(<a, s>, t) = iu_{1-t}<a, s>$ $a \in A_1 \# A_2$. Then $U(<a, s>, 1) = i\bar{h}''\tilde{k}<a, s>$ and $U(<a, s>, 0) = *$. Now we shalll show $h'F \cong U$ rel. $\Sigma(A_1 \# A_2)$. For the simplicity we set $A = A_1 \# A_2$. We define a map $G: A \times I \times I \times \dot{I} \cup A \times I \times 1 \times I \cup A \times \dot{I} \times I \times I \to X_\infty$ by

$$G(a, s, t, 0) = h'F(<a, s>, t),$$
$$G(a, s, t, 1) = U(<a, s>, t),$$
$$G(a, s, 1, u) = i\bar{h}''\tilde{k}<a, s>,$$
$$G(a, \epsilon, t, u) = * \qquad \epsilon = 0, 1 .$$

Using a retraction $I \times I \times I \to I \times 1 \times I \cup I \times I \times \dot{I} \cup \dot{I} \times I \times I$, we may extend G to a whole map $G: A \times I \times I \times I \to X_\infty$. We now define a homotopy $\psi_u: C\Sigma A \to X_\infty$ by $\psi_u(<a, s>, t) = G(a, s, t, u(1+2t)/(t+u(1+t)))$. Then ψ_u is well defined and it provides $h'F \cong U$ rel. $\Sigma(A_1 \# A_2)$. Therefore we obtain the following theorem:

THEOREM 5.8. *Let $\alpha \in \pi(\Sigma A_1, X)$ and $\beta \in \pi(\Sigma A_2, X)$. If there is a map $h: \Sigma A_1 \times \Sigma A_2 \to X$ of type (α, β), then*

$$J\phi^* \delta(h') = \{\alpha, \beta\},$$

where h' denotes the composition of h and the inclusion $i: X \to X_\infty$.

DEFINITION 5.9. Let $f: \Sigma A_1 \times \Sigma A_2 \to X$ be a map and let $x \in \pi(\Sigma^2(A_1 \# A_2), (\Sigma A_1 \times \Sigma A_2)_\infty)$ be an element which is obtained by Theorem 5.6. Moreover let f_* denote the homomorphism induced by multiplicative map $f_\infty: (\Sigma A_1 \times \Sigma A_2)_\infty \to X_\infty$. Then we say that $c(f) = f_*(X)$ is obtained from f by the

Hopf construction.

THEOREM 5.10. *$c(f)$ is characterized uniquely by the following three properties*:

i) *if* $f\colon \Sigma A_1\times\Sigma A_2\to X$ *is of type* (α_1, α_2), *where* $\alpha_i\in\pi(\Sigma A_i, X)$ $i=1, 2$, *then*

$$J(c(f))=\{\alpha_1, \alpha_2\}.$$

ii) *Let* $g\colon X\to Y$ *be a map, then*

$$c(g\circ f)=g_*c(f).$$

iii) *Let* $f\colon \Sigma A_1\times\Sigma A_2\to\Sigma A_1\times\Sigma A_2$ *be either of the projections* $(x, y)\to(x, *)$ *or* $(x, y)\to(*, y)$ *where* $x\in\Sigma A_1, y\in\Sigma A_2$, *then*

$$c(f)=0.$$

PROOF. It follows from Theorem 5.6 and Definition 5.9 that $c(f)$ satisfies the conditions i) and iii). ii) is easily checked. The uniqueness of this characterization follows from the above theorem and definition.

THEOREM 5.11. *Let* $h\colon \Sigma A_1\times\Sigma A_2\to X$ *be a map and let* h' *denote its inclusion into* X_∞. *Let* $c(h)$ *denote the element of* $\pi(\Sigma^2(A_1\# A_2), X_\infty)$ *which is obtained from* h *by the Hopf construction* (*see Definition* 5.9). *Then*

$$c(h)=\phi^*\delta(h').$$

PROOF. We check that $\phi^*\delta(h')$ satisfies conditions i), ii) and iii) in (5.10). i) follows from Theorem 5.8. Let $f\colon X\to Y$ be a map. Then we have $f_*\phi^*\delta(h')=\phi_*\delta(f_\infty h')=\phi^*\delta(ifh)$. This proves ii). Finally let $h\colon \Sigma A_1\times\Sigma A_2\to\Sigma A_1\times\Sigma A_2$ be defined by $h(x, y)=(x, *)$. Denote the sections of h by f, g. Then $h''(x, y)=f(x)\cdot g(y)=h'(x, y)$, and hence $\delta(h')=0$. Thus iii) is proved.

THEOREM 5.12. *A necessary and sufficient condition that* $\alpha\in\pi(\Sigma^2(A_1\# A_2), X_\infty)$ *can be obtained from some map* $\Sigma A_1\times\Sigma A_2\to X$ *of type* (α_1, α_2) *by the Hopf construction, where* $\alpha_i\in\pi(\Sigma A_i, X)$ $i=1, 2$, *is*

$$J(\alpha)=\{\alpha_1, \alpha_2\}.$$

PROOF. Let $f\colon \Sigma A_1\times\Sigma A_2\to X$ be a map of type (α_1, α_2). If α is obtained from f by the Hopf construction, then $\alpha=f_*(x)$ by Definition 5.9. Moreover we have $J(\alpha)=Jf_*(x)=f_*J(x)=f_*\{i_1, i_2\}$ by (5.6). Recall that $\{i_1 i_2\}$ is represented by

$$\begin{array}{ccc} \Sigma(A_1 \# A_2) & \xrightarrow{\bar{h}\circ\tilde{k}} & \Sigma A_1 \times \Sigma A_2 \\ \downarrow \iota & & \downarrow i \\ C\Sigma(A_1 \# A_2) & \xrightarrow{h\circ F} & (\Sigma A_1 \times \Sigma A_2)_\infty \end{array}$$

where $h: \Sigma A_1 \times \Sigma A_2 \to (\Sigma A_1 \times \Sigma A_2)_\infty$ is defined by $h(x, y) = i_1(x)\cdot i_2(y)$ and $\bar{h} = h|\Sigma A_1 \vee \Sigma A_2$. Also $f_*\{i_1, i_2\}$ is represented by

$$\begin{array}{ccccc} \Sigma(A_1 \# A_2) & \xrightarrow{\bar{h}\circ\tilde{k}} & \Sigma A_1 \times \Sigma A_2 & \xrightarrow{f} & X \\ \downarrow \iota & & & & \downarrow i \\ C\Sigma(A_1 \# A_2) & \xrightarrow{h\circ F} & (\Sigma A_1 \times \Sigma A_2)_\infty & \xrightarrow{f_\infty} & X_\infty . \end{array}$$

But $f_\infty h(x, y) = fi_1(y)\cdot fi_2(y)$, $fh(x, *) = fi_1(x)$ and $fh(*, y) = fi_2(y)$ $x \in \Sigma A_1$, $y \in \Sigma A_2$. Since α_j is represented $fi_j: \Sigma A_j \to \Sigma A_1 \times \Sigma A_2 \to X$ $j = 1, 2$, it follows that $f_*\{i_1, i_2\} = \{\alpha_1, \alpha_2\}$.

Conversely let $\alpha \in \pi(\Sigma^2(A_1 \# A_2), X)$ and $\alpha_i \in \pi(\Sigma A_i, X)$ $i = 1, 2$ be given such that $J(\alpha) = \{\alpha_1, \alpha_2\}$. Let α_i be represented by a map $f_i: \Sigma A_i \to X$. By the properties of the wedge product (cf. [2]), there exists a map $f'': \Sigma A_1 \vee \Sigma A_2 \to X$ such that $f''i_1 = f_1$ and $f''i_2 = f_2$. Now we have

$$f''_*[i_1, i_2] = [\alpha_1, \alpha_2] = \partial\{\alpha_1, \alpha_2\} = \partial J(\alpha) = 0 .$$

Hence by [1; Proposition 5.1] there exists a map $f': \Sigma A_1 \times \Sigma A_2 \to X$ such that $f'j \simeq f''$. Without the loss of generality we may assume that $f'j = f''$, since $j: \Sigma A_1 \vee \Sigma A_2 \to \Sigma A_1 \times \Sigma A_2$ is a cofibration (Note that $\Sigma A_1 \times \Sigma A_2$ is a *CW*-complex). Obviously f' is of type (α_1, α_2). Setting $\alpha' = c(if')$, then $J(\alpha') = \{\alpha_1, \alpha_2\}$. Since $\pi(\Sigma^2(A_1 \# A_2), X) \xrightarrow{i_*} \pi(\Sigma^2(A_1 \# A_2), X_\infty) \xrightarrow{J} \pi(C\Sigma(A_1 \# A_2), \Sigma(A_1 \# A_2); X_\infty, X)$ is exact, there exists $\delta \in \pi(\Sigma^2(A_1 \# A_2), X)$ such that $i_*(\delta) = \alpha - \alpha'$. For such a δ, by Theorem 3.3, there exists a map $g'': C_{\tilde{k}} \to X$ such that $g''i' = f''$ and $d(f'G, g'') = \delta$, where $i': \Sigma A_1 \vee \Sigma A_2 \to C_{\tilde{k}}$ is the inclusion. Let G' be a homotopy inverse of the homotopy equivalence $G: C_{\tilde{k}} \to \Sigma A_1 \times \Sigma A_2$ and we set $g = g'' \circ G'$. Then $g \circ G \simeq g''$ and so $g \circ G \circ i' \simeq f''$. But $G \circ i' = j: \Sigma A_1 \vee \Sigma A_2 \to \Sigma A_1 \times \Sigma A_2$ and hence there exists a map $g': \Sigma A_1 \times \Sigma A_2 \to X$ such that $g' \simeq g$ and $g'j = f''$. Thus we have $d(f'G, g'G) = \delta$. We now define $l: \Sigma A_1 \times \Sigma A_2 \to X_\infty$ by $l(x, y) = if'i_1(x)\cdot if'i_2(y)$. Then

$$\begin{aligned} i_* d(f'G, g'G) &= d(if'G, ig'G) \\ &= d(if'G, lG) + d(lG, ig'G) \qquad \text{by (3.2)} \end{aligned}$$

$$= -[d(lG, if'G) - d(lG, ig'G)] \qquad \text{by Cor. to (3.2)}$$

$$= -[\phi^*\delta(if') - \phi^*\delta(ig')] \qquad \text{by Definition}$$

$$= -c(if') + c(ig') \qquad \text{by (5.11).}$$

On the other hand $i_* d(f'G, g'G) = i_*\delta = \alpha - \alpha'$. Hence $c(ig') - c(if') = \alpha - \alpha'$. Thus we have $\alpha = c(ig')$.

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