Tôhoku Math. Journ.
Vol. 18, No. 1, 1966

GIBBS' PHENOMENON FOR A FAMILY OF SUMMABILITY METHODS

KYUHEI IKENO

(Received December 14, 1965)

1. Introduction. The Gibbs' phenomenon of Fourier series $\sum_{n=1}^{\infty}\frac{\sin nx}{n}$ was found by W. Gibbs and O. Szasz [7], L. Lorch [4], C. L. Miracle [6], K. Ishiguro [1], [2] and many other authors have investigated this phenomenon for various summability methods.

In a recent paper A. Meir has introduced in [5] a family of summability methods $F(a, q(p))$ which is defined by two parameters a and $q(p)$ and has shown that this family contains Borel, Valiron, Euler, Taylor, and S_α-transformation.

In this paper we shall study the Gibbs' phenomenon of the Fourier series $\sum_{n=1}^{\infty}\frac{\sin nx}{n}$ for this family of summability methods. If we define $s_n(x)$ by the partial sum of the Fourier series $\sum_{n=1}^{\infty}\frac{\sin nx}{n}$ which is equal to $\frac{1}{2}(\pi - x)$ for $0 < x < 2\pi$, and define $\sigma_p(x)$ by the linear transformation of $s_n(x)$ by means of a family of summability methods whose matrix belongs to $F(a, q(p))$, then we obtain the following result:

If $\{x_p\}$ satisfies the condition that for given τ, in the case $0 \leqq \tau < \infty$, $x_p \to +0$, $q(p)x_p \to \tau$ and in the case $\tau = \infty$, $q(p)x_p \to \infty$, $q(p)x_p^2 \to +0$ as p tends to infinity, where $q = q(p)$ is the parameter of $F(a, q(p))$, then we get

$$\lim_{p\to\infty}\sigma_p(x_p) = \int_0^\tau \frac{\sin u}{u}\,du. \tag{1.1}$$

We can prove the above formula (1.1) by the same calculation as the one which we use in order to obtain Lebesgue constant for a family of summability methods (see [3]). Gibbs' phenomenon for this family is independent of the parameter a of $F(a, q(p))$. Since $s_n(x)$ is odd function of x, a similar phenomenon occurs in the left-hand neighbourhood of $x=0$.

From this result we shall show that we can obtain the Gibbs' phenomenon for Borel, Valiron, Euler, Taylor and S_α-transformation which are contained in this family of summability methods.

2. The Family F($a, q(p)$) of summability methods. Following A. Meir [5], let us say that the summability matrix $[c_{pk}]$ belongs to $F(a, q(p))$ if it satisfies the following conditions: p is a discrete or continuous parameter; a is a positive constant; $q=q(p)$ is a positive increasing function which tends to infinity as $p\to\infty$; for every δ: $\frac{1}{2}<\delta<\frac{2}{3}$,

$$c_{pk}=\sqrt{\frac{a}{\pi q}}e^{-\frac{a}{q}(k-q)^2}\left\{1+O\left(\frac{|k-q|+1}{q}\right)+O\left(\frac{|k-q|^3}{q^2}\right)\right\} \tag{2.1}$$

as $p\to\infty$ uniformly in k for $|k-q|\leqq q^\delta$,

$$c_{po}+\sum_{|k-q|>q^\delta}k\,c_{pk}=O(\exp(-q^\eta)) \tag{2.2}$$

where η is some positive number independent of p, and

$$c_{pk}\geqq 0. \tag{2.3}$$

It is known that the family $F(a, q(p))$ with appropriate a and $q(p)$ contains such summability methods as Borel, Valiron, Euler, Taylor and S_α-transformation (see A. Meir [5]).

From the definition of $\sigma_p(x)$, we have

$$\sigma_p(x)=\sum_{k=1}^{\infty}c_{pk}s_k(x)=\sum_{k=0}^{\infty}c_{pk}\left(-\frac{x}{2}+\int_0^x\frac{\sin\left(n+\frac{1}{2}\right)u}{2\sin\frac{u}{2}}du\right). \tag{2.4}$$

We shall now investigate the behaviour of $\sigma_p(x)$ in the neighbourhood of $x=0$.

3. Two lemmas. In order to prove the formula (1.1), we require the following two lemmas.

LEMMA 3.1. *If the summability matrix $[c_{pk}]$ belongs to $F(a, q(p))$, we have*

$$\sum_{k=0}^{\infty} c_{pk} = 1 + o(q^{-\frac{1}{2}}) \quad as \quad p \to \infty . \tag{3.1}$$

The proof follows from (2.1) and (2.2) by a simple calculation.

LEMMA 3.2. *If $\{x_p\}$ satisfies the condition which is mentioned in section* 1 *and p tends to infinity, then we have*

$$\int_0^{x_p} \frac{1}{\sin u} \sum_{|k-q|\leqq q^\delta} \sqrt{\frac{a}{\pi q}} e^{-\frac{a}{q}(k-q)^2} \frac{|k-q|+1}{q} |\sin(2k+1)u|\, du = o(1) \tag{3.2}$$

and

$$\int_0^{x_p} \frac{1}{\sin u} \sum_{|k-q|\leqq q^\delta} \sqrt{\frac{a}{\pi q}} e^{-\frac{a}{q}(k-q)^2} \frac{|k-q|^3}{q^2} |\sin(2k+1)u|\, du = o(1) . \tag{3.3}$$

PROOF. From the condition on $\{x_p\}$, for sufficiently large p, we get

$$\int_0^{x_p} \frac{1}{\sin u} \sum_{|k-q|\leqq q^\delta} \sqrt{\frac{a}{\pi q}} e^{-\frac{a}{q}(k-q)^2} \frac{|k-q|+1}{q} |\sin(2k+1)u|\, du$$

$$= O\left(\int_0^{x_p} \frac{1}{\sin u} \sum_{|k-q|\leqq q^\delta} \sqrt{\frac{a}{\pi q}} e^{-\frac{a}{q}(k-q)^2} \frac{|k-q|+1}{q} (2|k-q|+2q+1) u\, du\right)$$

$$= O\left(\int_0^{x_p} \sqrt{q}\, du\right) = o(1) .$$

Similary, we get for sufficiently large p,

$$\int_0^{x_p} \frac{1}{\sin u} \sum_{|k-q|\leqq q^\delta} \sqrt{\frac{a}{\pi q}} e^{-\frac{a}{q}(k-q)^2} \frac{|k-q|^3}{q^2} |\sin(2k+1)u|\, du$$

$$= O\left(\int_0^{x_p} \frac{1}{\sin u} \sum_{|k-q|\leqq q^\delta} \sqrt{\frac{a}{\pi q}} e^{-\frac{a}{q}(k-q)^2} \left(\frac{(k-q)^4}{q^2} + \frac{|k-q|^3}{q}\right) u\, du\right)$$

$$= O\left(\int_0^{x_p} \sqrt{q}\, du\right) = o(1) .$$

4. Gibbs' phenomenon. In this section we consider the Gibbs' phenomenon for a family of summability methods whose matrix $[c_{pk}]$ belongs to $F(a, q(p))$.

THEOREM. *Let $\sigma_p(x)$ denote the linear transformation of $s_n(x)$ by* *means of a family of summability methods whose matrix belongs to* $F(a, q(p))$.

If $\{x_p\}$ *satisfies the following condition that for given* τ, *in the case* $0 \leqq \tau < \infty$, $x_p \to +0$, $q(p)x_p \to \tau$ *and in the case* $\tau = \infty$, $q(p)x_p \to \infty$, $q(p)x_p^2 \to +0$ *as* $p \to \infty$, *where* $q = q(p)$ *is mentioned in section* 2, *then we have*

$$\lim_{p\to\infty} \sigma_p(x_p) = \int_0^\tau \frac{\sin u}{u}\, du\,. \tag{1.1}$$

PROOF. From lemma 3.1, (2.2) and (2.4), we have

$$\sigma_p(x_p) = \int_0^{x_p/2} \frac{1}{\sin u} \sum_{k=0}^{\infty} c_{pk} \sin(2k+1)u du + o(1)\,.$$

We put $I_1(x_p)$ and $I_2(x_p)$ as follows:

$$\begin{aligned}\sigma_p(x_p) &= \int_0^{x_p/2} \frac{1}{\sin u}\left(\sum_{|k-q|\leqq q^\delta} + \sum_{|k-q|>q^\delta}\right) c_{pk} \sin(2k+1)u\, du + o(1)\\ &= I_1(x_p) + I_2(x_p) + o(1)\,.\end{aligned}$$

Applying lemma 3.2 and (2.2) to $I_1(x_p)$ and $I_2(x_p)$, we get for sufficiently large p,

$$\begin{aligned}I_1(x_p) &= \int_0^{x_p/2} \frac{1}{\sin u} \sum_{|k-q|\leqq q^\delta} \sqrt{\frac{a}{\pi q}}\, e^{-\frac{a}{q}(k-q)^2}\\ &\qquad \times \left(1 + O\left(\frac{|k-q|+1}{q}\right) + O\left(\frac{|k-q|^3}{q^2}\right)\right) \sin(2k+1)u\, du\\ &= \int_0^{x_p/2} \frac{1}{\sin u} \sum_{|k-q|\leqq q^\delta} \sqrt{\frac{a}{\pi q}}\, e^{-\frac{a}{q}(k-q)^2} \sin(2k+1)u\, du + o(1)\end{aligned}$$

and

$$\begin{aligned}I_2(x_p) &= \int_0^{x_p/2} \frac{1}{\sin u} \sum_{|k-q|>q^\delta} c_{pk} \sin(2k+1)u\, du\\ &= O\left(\int_0^{x_p/2} \frac{1}{\sin u}\left((c_{po} + \sum_{|k-q|>q^\delta} k\, c_{pk}\right) u\, du\right)\\ &= O(x_p \exp(-q^\eta)) = o(1)\,,\end{aligned}$$

Then we obtain

$$(4.1)\qquad \sigma_p(x_p)=\int_0^{x_p/2}\frac{1}{\sin u}\sum_{|k-q|\leqq q^\delta}\sqrt{\frac{a}{\pi q}}\,e^{-\frac{a}{q}(k-q)^2}\sin(2k+1)u\,du+o(1).$$

1°) The case where $q=q(p)$ is integer.
When we put $n=k-q(p)$, we have

$$\sum_{|k-q|\leqq q^\delta}\sqrt{\frac{a}{\pi q}}\,e^{-\frac{a}{q}(k-q)^2}\sin(2k+1)u$$

$$=\Im\left\{e^{i(2q+1)u}\sqrt{\frac{a}{\pi q}}\sum_{|n|\leqq q^\delta}e^{-\frac{a}{q}n^2+2uni}\right\}$$

$$=\Im\left\{e^{i(2q+1)u}\sqrt{\frac{a}{\pi q}}\left(\sum_{n=-\infty}^{+\infty}-\sum_{|n|>q^\delta}\right)e^{-\frac{a}{q}n^2+2uni}\right\}.$$

Using the property of Theta function [8], we get

$$\sqrt{\frac{a}{\pi q}}\sum_{n=-\infty}^{+\infty}e^{-\frac{a}{q}n^2+2uni}=\sum_{n=-\infty}^{+\infty}e^{-\frac{q}{a}(u-n\pi)^2}$$

and consequently for $0\leqq u\leqq x_p$,

$$(4.2)\qquad \sum_{|k-q|\leqq q^\delta}\sqrt{\frac{a}{\pi q}}\,e^{-\frac{a}{q}(k-q)^2}\sin(2k+1)u$$

$$=\Im\left\{e^{i(2q+1)u}\sum_{n=-\infty}^{+\infty}e^{-\frac{a}{q}(u-n\pi)^2}\right\}+O\left(\sum_{|n|>q^\delta}\sqrt{\frac{a}{\pi q}}\,e^{-\frac{a}{q}n^2}\,|\sin(2n+2q+1)u|\right)$$

$$=e^{-\frac{q}{a}u^2}\sin(2q+1)u+O(qe^{-aq^{2\delta-1}}u).$$

From (4.1) and (4.2), we get

$$(4.3)\qquad \sigma_p(x_p)=\int_0^{x_p/2}\frac{1}{\sin u}\left\{e^{-\frac{q}{a}u^2}\sin(2q+1)u+O(qe^{-aq^{2\delta-1}}u)\right\}du+o(1)$$

$$=\int_0^{x_p/2}\frac{e^{-\frac{q}{a}u^2}\sin(2q+1)u}{u}\,du+o(1).$$

We put $f(u,p)$ and $D_p(x_p)$ as follows:

$$
(4.4)\qquad \begin{aligned} f(u, p) &= \frac{1}{u}(1-e^{-\frac{q}{a}u^2}) \\ D_p(x_p) &= \int_0^{x_p/2} f(u, p) \sin(2q+1)u\, du\,. \end{aligned}
$$

Applying integration by parts to $D_p(x_p)$, we get

$$
(4.5)\quad D_p(x_p) = f\left(\frac{x_p}{2}, p\right)\cdot \frac{-\cos(2q+1)\frac{x_p}{2}}{2q+1} + \int_0^{x_p/2} f'(u, p) \frac{\cos(2q+1)u}{2q+1} du\,.
$$

From $f'(u, p) > 0$ for $0 < u \leqq x_p/2$ and $f'(+0, p) = \dfrac{2q}{a} > 0$,

$$
(4.6)\qquad \int_0^{x_p/2} |f'(u, p)|\, du = O(qx_p)\,.
$$

From (4.5) and (4.6), we get

$$
(4.7)\qquad D_p(x_p) = O(x_p) + O\left(\frac{1}{q}\int^{x_p/2} |f(u, p)|\, du\right) = O(x_p) = o(1)\,.
$$

Consequently we get from (4.3), (4.4) and (4.7) for sufficiently large p

$$
(4.8)\qquad \begin{aligned} \sigma_p(x_p) &= \int_0^{x_p/2} \frac{\sin(2q+1)u}{u} du + o(1) \\ &= \int_0^{\tau} \frac{\sin u}{u} du + o(1) \end{aligned}
$$

Thus the theorem has been proved when $q = q(p)$ is integer. Next we shall consider the other case.

2°) The case where $q = q(p)$ is not integer.

Let $[q]$ denote the integral part of $q = q(p)$ and $q_0 = [q]+1$. We put $D_1(x_p)$, $D_2(x_p)$, $D_3(x_p)$ and $D_4(x_p)$ as follows:

$$
(4.9)\qquad \Bigg| \int_0^{x_p/2} \frac{1}{\sin u} \sum_{|k-q|\leqq q^\delta} \sqrt{\frac{a}{\pi q}}\, e^{-\frac{a}{q}(k-q)^2} \sin(2k+1)u\, du - \int_0^{x_p/2} \frac{1}{\sin u} \sum_{|k-q_0|\leqq q_0^\delta} \sqrt{\frac{a}{\pi q_0}}\, e^{-\frac{a}{q_0}(k-q_0)^2} \sin(2k+1)u\, du \Bigg|
$$

$$\leqq \int_0^{x_p/2} \frac{1}{\sin u}\left(\sum_{q<k\leqq q+q^\delta} + \sum_{q-q^\delta\leqq k<q}\right)\left|\left(\sqrt{\frac{a}{\pi q}}\,e^{-\frac{a}{q}(k-q)^2} - \sqrt{\frac{a}{\pi q_0}}\,e^{-\frac{a}{q_0}(k-q_0)^2}\right)\sin(2k+1)u\right| du$$

$$+ \int_0^{x_p/2} \frac{1}{\sin u} \sum_{q+q^\delta<k\leqq q_0+q_0^\delta} \sqrt{\frac{a}{\pi q_0}}\,e^{-\frac{a}{q_0}(k-q_0)^2} |\sin(2k+1)u|\,du$$

$$+ \int_0^{x_p/2} \frac{1}{\sin u} \sum_{q-q^\delta<k\leqq q_0-q_0^\delta} \sqrt{\frac{a}{\pi q}}\,e^{-\frac{a}{q}(k-q)^2} |\sin(2k+1)u|\,du$$

$$= D_1(x_p) + D_2(x_p) + D_3(x_p) + D_4(x_p).$$

i) In the case where $q < q_0 \leqq k \leqq q+q^\delta$, we have

$$0 \leqq (k-q_0)/\sqrt{q_0} < (k-q)/\sqrt{q} < (k-[q])/\sqrt{[q]}\,.$$

Hence the following estimation results:

$$\left|\sqrt{\frac{a}{\pi q}}\,e^{-\frac{a}{q}(k-q)^2} - \sqrt{\frac{a}{\pi q_0}}\,e^{-\frac{a}{q_0}(k-q_0)^2}\right|$$

$$= O\left(\frac{1}{\sqrt{q_0}} \int_{(k-q_0)/\sqrt{q_0}}^{(k-[q])/\sqrt{[q]}} |xe^{-ax^2}|\,dx\right)$$

$$= O\left(\frac{1}{\sqrt{q_0}}\,e^{-\frac{a}{q_0}(k-q_0)^2}\left(\frac{(k-q_0)^2}{q_0^2} + \frac{|k-q_0|}{q_0} + \frac{1}{q_0}\right)\right).$$

Then we get

$$D_1(x_p) = O\left(\int_0^{x_p/2} \frac{1}{\sin u} \sum_{q<k\leqq q+q^\delta} \frac{1}{\sqrt{q_0}}\,e^{-\frac{a}{q_0}(k-q_0)^2}\right. \tag{4.10}$$

$$\left. \times \left(\frac{(k-q_0)^2}{q_0^2} + \frac{|k-q_0|}{q_0} + \frac{1}{q_0}\right)(2|k-q|+2q+1)\,u\,du\right)$$

$$= O\left(\sqrt{q} \int_0^{x_p/2} \frac{u}{\sin u}\,du\right) = o(1).$$

ii) In the case where $k \leqq [q] < q < q_0$, we have

$$(k-q_0)/\sqrt{q_0} < (k-q)/\sqrt{q} < (k-[q])/\sqrt{[q]} \leqq 0.$$

Hence the following estimation results just as in the case of i):

$$\left|\sqrt{\frac{a}{\pi q}}\,e^{-\frac{a}{q}(k-q)^2}-\sqrt{\frac{a}{\pi q_0}}\,e^{-\frac{a}{q_0}(k-q_0)^2}\right|$$

$$=O\left(\frac{1}{\sqrt{q}}\int_{(k-q_0)/\sqrt{q_0}}^{(k-[q])/\sqrt{[q]}}|xe^{-ax^2}|\,dx\right)$$

$$=O\left(\frac{1}{\sqrt{[q]}}\,e^{-\frac{a}{[q]}(k-[q])^2}\left(\frac{(k-[q])^2}{[q]^2}+\frac{|k-[q]|}{[q]}+\frac{1}{[q]}\right)\right).$$

Then we get

$$D_2(x_p)=O\left(\int_0^{x_p/2}\frac{1}{\sin u}\sum_{q-q^\delta\leqq k<q}\frac{1}{\sqrt{[q]}}\,e^{-\frac{a}{[q]}(k-[q])^2}\right. \tag{4.11}$$

$$\left.\times\left(\frac{(k-[q])^2}{[q]^2}+\frac{|k-[q]|}{[q]}+\frac{1}{[q]}\right)(2|k-q|+2q+1)\,u\,du\right)$$

$$=O\left(\sqrt{q}\int_0^{x_p/2}\frac{u}{\sin u}\,du\right)=o(1).$$

Next we shall estimate $D_3(x_p)$, $D_4(x_p)$ and we get

$$D_3(x_p)=\int_0^{x_p/2}\frac{1}{\sin u}\sum_{q+q^\delta<k\leqq q_0+q_0^\delta}\sqrt{\frac{a}{\pi q_0}}\,e^{-\frac{a}{q_0}(k-q_0)^2}\,|\sin(2k+1)u|\,du \tag{4.12}$$

$$=O\left(\sqrt{q}\,e^{-aq^{2\delta-1}}\int_0^{x_p/2}\frac{u}{\sin u}\,du\right)=o(1),$$

$$D_4(x_p)=\int_0^{x_p/2}\frac{1}{\sin u}\sum_{q-q^\delta\leqq k<q_0-q_0^\delta}\sqrt{\frac{a}{\pi q}}\,e^{-\frac{a}{q}(k-q)^2}\,|\sin(2k+1)u|\,du \tag{4.13}$$

$$=O\left(\sqrt{q}\,e^{-aq^{2\delta-1}}\int_0^{x_p/2}\frac{u}{\sin u}\,du\right)=o(1).$$

From (4.1), (4.9), (4.10), (4.11), (4.12), (4.13) and the result of 1°), we have

$$\sigma_p(x_p)=\int_0^{x_p/2}\frac{1}{\sin u}\sum_{|k-q|\leqq q^\delta}\sqrt{\frac{a}{\pi q}}\,e^{-\frac{a}{q}(k-q)^2}\sin(2k+1)u\,du+o(1)$$

$$=\int_0^{x_p/2}\frac{1}{\sin u}\sum_{|k-q_0|\leqq q_0^\delta}\sqrt{\frac{a}{\pi q_0}}\,e^{-\frac{a}{q_0}(k-q_0)^2}\sin(2k+1)u\,du+o(1)$$

$$= \int_0^{x_p/2} \frac{\sin(2q_0+1)u}{u} du + o(1)$$

$$= \int_0^{T} \frac{\sin u}{u} du + o(1).$$

Thus we have obtained Gibbs' phenomenon for a family of summability methods whose matrix $[c_{pk}]$ belongs to $F(a, q(p))$.

5. Gibbs' phenomenon for Borel, Valiron, Euler, Taylor and S_α-transformation. In this section, we suppose that $\{x_p\}$ satisfies the same condition as in the theorem of section 4.

From the theorem, we get the following results:

i) Borel-transformation (see L. Lorch [4]).

The summability matrix of Borel-transformation is defined by

$$c_{pk} = e^{-p}\frac{p^k}{k!} \qquad (k=0, 1, 2, \cdots),$$

where $p > 0$, $a = \frac{1}{2}$ and $q(p) = p$ (see A. Meir [5]).

If we define $B_p(x)$ by the linear transformation of $s_n(x)$ by means of Borel-transformation, we have from (1. 1)

$$\lim_{p\to\infty} B_p(x_p) = \lim_{p\to\infty} e^{-p}\sum_{k=0}^{\infty}\frac{p^k}{k!}s_k(x_p) = \int_0^T \frac{\sin u}{u} du$$

ii) Valiron-transformation.

The summability matrix of Valiron-transformation is defined by

$$c_{pk} = \sqrt{\frac{\alpha}{\pi p}}\, e^{-\frac{\alpha}{p}(k-p)^2} \qquad (p=1, 2, \cdots, \ k=0, 1, 2, \cdots)`$$

where $\alpha > 0$, $a = \alpha$ and $q(p) = p$.

If we define $V_p(x)$ by the linear transformation of $s_n(x)$ by means of Valiron-transformation, we have

$$\lim_{p\to\infty} V_p(x_p) = \lim_{p\to\infty}\sqrt{\frac{\alpha}{\pi p}}\sum_{k=0}^{\infty} e^{-\frac{\alpha}{p}(k-p)^2} s_k(x_p) = \int_0^T \frac{\sin u}{u} du.$$

iii) Euler-transformation (see O. Szasz [7]).

The summability matrix of Euler-transformation is defined by

$$c_{pk} = \begin{cases} \binom{p}{k}\alpha^k(1-\alpha)^{p-k} & \text{for} \quad 0 \leqq k \leqq p, \\ 0 & \text{for} \quad p+1 \leqq k, \end{cases} \qquad (p = 1, 2, 3, \cdots)$$

where $0 < \alpha < 1$, $a = 1/2(1-\alpha)$ and $q = \alpha p$ (see A. Meir [5]). If we define $E_p(x)$ by the linear transformation of $s_n(x)$ by means of Euler-transformation, we have

$$\lim_{p\to\infty} E_p(x_p) = \lim_{p\to\infty} \sum_{k=0}^{p} \binom{p}{k} \alpha^k(1-\alpha)^{p-k} s_k(x_p)$$

$$= \int_0^{\alpha\tau} \frac{\sin u}{u} du.$$

iv) Taylor-transformation (see K. Ishiguro [1]).

The summability matrix of Taylor-transformation is defined by

$$c_{pk} = \begin{cases} 0 & \text{for} \quad 0 \leqq k \leqq p-1, \\ r^{p+1}\binom{k}{p}(1-r)^{k-p} & \text{for} \quad p \leqq k, \end{cases}$$

where $0 < r < 1$, $a = r/2(1-r)$ and $q(p) = p/r$ (see A. Mier [5]). If we define $T_p(x)$ by the linear transformation of $s_n(x)$ by means of Taylor-transformation, we have

$$\lim_{p\to\infty} T_p(x_p) = \lim_{p\to\infty} \sum_{k=p}^{\infty} r^{k+1} \binom{k}{p} (1-r)^{k-p} s_k(x_p)$$

$$= \int_0^{\tau/r} \frac{\sin u}{u} du.$$

v) S_α-transformation (see K. Ishiguro [2]).

The summability matrix of S_α-transformation is defined by

$$c_{pk} = (1-\alpha)^{p+1}\binom{p+k}{k}\alpha^k \qquad (k=0, 1, 2, \cdots, \quad p=1, 2, \cdots)$$

where $0 < \alpha < 1$, $a=(1-\alpha)/2$ and $q(p) = \alpha p/(1-2)$ (see A. Meir [5]). If we define $\sigma_p(x)$ by the linear transformation of $s_n(x)$ by means of S_α-transformation, we have

$$\lim_{p\to\infty} \sigma_p(x_p) = \lim_{p\to\infty} \sum_{k=0} (1-\alpha)^{p+1} \binom{p+k}{k} \alpha^k s_k(x_p)$$

$$= \int_0^{\alpha\tau/(1-\alpha)} \frac{\sin u}{u} du .$$

REFERENCES

[1] K. ISHIGURO, Zur Gibbsschen Erscheinung für das Kreisverfahren, Math. Zeitschr., 76 (1961), 288-294.

[2] K. ISHIGURO, Über das S_α-Verfahren bei Fourier-Reihen, Math. Zeitschr., 80(1962), 4-11.

[3] K. IKENO, Lebesgue constants for a family of summability methods, Tôhoku Math. Journ., 17(1965), 250-265.

[4] L. LORCH, The Gibbs phenomenon for Borel means, Proc. Amer. Math. Soc., 8(1957), 81-84.

[5] A. MEIR, Tanberian constants for a family of transformations, Ann. of Math., 78 (1963), 594-599.

[6] C. L. MIRACLE, The Gibbs phenomenon for Taylor means and for $[F, d_n]$ means, Canad. Journ. Math., 12(1960), 367-370.

[7] O. SZASZ, On the Gibbs' phenomenon for Euler means, Acta Univ. Szeged, 12(1950), 107-111.

[8] E. T. WHITTAKER AND G. N. WATSON, A Course of Modern Analysis, Cambridge University Press, 1935.

AKITA UNIVERSITY.