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ON A GENERALISED CESÀRO SUMMABILITY METHOD OF INTEGRAL ORDER

D. BORWEIN

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Let p be a non-negative integer, $\{\lambda_n\}$ a strictly increasing unbounded sequence with $\lambda_0 \geqq 0$, and let $\sum_{n=0}^{\infty} a_n$ be an arbitrary series. Write

$$A^p(w) = \sum_{\lambda_n < w} (w-\lambda_n)^p a_n \qquad (w \geqq 0);$$

$$C_n^0 = \sum_{\nu=0}^{n} a_\nu,\ C_n^p = \sum_{\nu=0}^{n} (\lambda_{n+1}-\lambda_\nu)\cdots(\lambda_{n+p}-\lambda_\nu)\, a_\nu \qquad (p \geqq 1);$$

$$\lambda_{n,0} = 1,\ \lambda_{n,p} = \lambda_{n+1}\cdots\lambda_{n+p} \qquad (p \geqq 1).$$

The series $\sum a_n$ is said to be

(i) summable by the Riesz method (R, λ, p) to s if $w^{-p}A^p(w) \to s$ as $w \to \infty$,

(ii) summable by the generalised Cesàro method (C, λ, p) to s if $C_n^p/\lambda_{n,p} \to s$ as $n \to \infty$.

The relationship between these two summability methods has been investigated by Jurkat [2] and Burkill [1], who independently defined generalised Cesàro methods essentially the same as the above; and by Russell [3]. All three established the inclusions

$$(\mathrm{I}_1):\ (C, \lambda, p) \subseteq (R, \lambda, p),$$

$$(\mathrm{I}_2):\ (R, \lambda, p) \subseteq (C, \lambda, p)$$

under various hypotheses on the sequence $\{\lambda_n\}$. The most general results to date are due to Russell [3], who proved that (I_1) holds without restriction on $\{\lambda_n\}$, and that (I_2) holds provided

$$(C_1):\ \frac{\lambda_n}{\lambda_n-\lambda_{n-1}} = O\left(\frac{\lambda_{n+1}}{\lambda_{n+1}-\lambda_n}\right)$$

when $p \geqq 3$ and unrestrictedly when $p \leqq 2$.

The object of this note is to prove that (I_2) also holds if

$$(C_2): \lambda_{n+1} = O(\lambda_n).$$

Conditions (C_1) and (C_2) are independent (see [3]).

We shall prove the following theorem in which there is no restriction on $\{\lambda_n\}$.

THEOREM. *If $\eta(w)$ is non-negative and monotonic non-decreasing for $w \geqq 0$ and $A^p(w) = o(\eta(w))$ as $w \to \infty$, then $C_n^p = o(\eta(\lambda_{n+p}))$.*

The theorem remains valid if o is replaced by O throughout.

As an immediate corollary we have:

COROLLARY. *If (C_2) and $A^p(w) = o(w^p)$ as $w \to \infty$, then $C_n^p = o(\lambda_{n+1}^p) = o(\lambda_{n,p})$.*

Since the methods (C, λ, p) and (R, λ, p) are both regular (see [3], Cor. 1B), a consequence of the corollary is that (C_2) is a sufficient condition for inclusion (I_2).

PROOF OF THE THEOREM. Since the theorem is trivially true when $p = 0$, we shall assume that $p \geqq 1$. Let $m = m(n)$ be an integer such that

$$\lambda_{m+1} - \lambda_m = \max_{n \leqq i \leqq n+p-1} (\lambda_{i+1} - \lambda_i), \quad n \leqq m \leqq n+p-1,$$

and let

$$b_i = b_{n,i} = (p+1)\frac{\lambda_{n+i} - \lambda_m}{\lambda_{m+1} - \lambda_m} \quad (i = 1, 2, \cdots, p).$$

Then (c.f. Burkill [1], p. 57) there are numbers $y_i = y_{n,i}$ $(i=0, 1, \cdots, p)$ such that

$$(1) \qquad (x+b_1)(x+b_2)\cdots(x+b_p) \equiv y_0(x+1)^p + y_1(x+2)^p + \cdots + y_p(x+p+1)^p,$$

which is equivalent to the system of linear equations

$$(2) \qquad \sum_{j=0}^{p} (j+1)^i y_i = k_i \qquad (i=0, 1, \cdots, p)$$

where

$$k_i = \binom{p}{i}^{-1} \sum_{1\leqq r_1<\cdots<r_i\leqq p} b_{r_1} b_{r_2} \cdots b_{r_i}.$$

The determinant of the system (2) is

$$\Delta = \prod_{1\leqq r<s\leqq p+1} (s-r) \geqq 1 ;$$

and $y_r = \Delta_r/\Delta$ where Δ_r is the determinant of the matrix

$$(c_{i,j})\ (i,j=0,1,\cdots,p) \text{ where } c_{i,r}=k_i \text{ and } c_{i,j} = (j+1)^i\ (j \neq r).$$

Since $|b_r|<(p+1)^2$, we see that $|k_i|<(p+1)^{2p}$ and hence that $|c_{i,j}|<(p+1)^{2p}$. Consequently

(3) $$|y_r| = |y_{n,r}| \leqq |\Delta_r| < (p+1)!\,(p+1)^{2p} \quad (r=0,1,\cdots,p;\ n=0,1,\cdots).$$

Putting $x=(p+1)\dfrac{\lambda_m-\lambda_\nu}{\lambda_{m+1}-\lambda_m}$ in identity (1), we obtain

$$(\lambda_{n+1}-\lambda_\nu)\cdots(\lambda_{n+p}-\lambda_\nu) = y_{n,0}(\mu_{n,0}-\lambda_\nu)^p + \cdots + y_{n,p}(\mu_{n,p}-\lambda_\nu)^p$$

where

(4) $$\lambda_m < \mu_{n,i} = \lambda_m + \frac{i+1}{p+1}(\lambda_{m+1}-\lambda_m) \leqq \lambda_{m+1} \leqq \lambda_{n+p}$$

$$(i=0,1,\cdots,p;\ n=0,1,\cdots).$$

Hence

(5) $$C_n^p = \sum_{\nu=0}^{m} (\lambda_{n+1}-\lambda_\nu)\cdots(\lambda_{n+p}-\lambda_\nu)\,a_\nu = \sum_{i=0}^{p} y_{n,i}\,A^p(\mu_{n,i});$$

and the theorem is an immediate consequence of (3), (4) and (5).

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UNIVERSITY OF WESTERN ONTARIO
LONDON, CANADA.