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REMARKS ON HELSON-SZEGÖ PROBLEMS

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The purpose of this paper is to note that some results considered by Helson-Szegö [2] are valid for a general Dirichlet algebra.

1. Let X be a compact Hausdorff space. Let $C(X)$ $[C_R(X)]$ denote the complex [real] linear algebra of all continuous complex-valued [real-valued] functions on X with the usual supremum norm. Let A be a function algebra on X, i.e. a closed subalgebra of $C(X)$ which separates the points of X and contains the constants.

We say that the function algebra A is a Dirichlet algebra (on X) if the space $\mathrm{Re}\,A$, the set of all real parts of functions in A, is uniformly dense in $C_R(X)$. Throughout this paper A will denote a Dirichlet algebra on X. For a complex homomorphism Φ of A, there exists a unique representing measure[1] m for Φ such that

$$\log|\Phi(f)| = \int \log|f|\,dm \qquad (f \in V)$$

where V denotes the set of all invertible elements of A (see Hoffman [3]). We fix such a representing measure m.

Let μ be a finite (positive Baire) measure on X. For $1 \leqq p \leqq \infty$, let $L^p(d\mu)$ be the usual L^p-space with respect to $d\mu$. If $1 \leqq p < \infty$, $H^p(d\mu)$ shall be the closure $[A]_p$ of A in $L^p(d\mu)$ and $H^\infty(d\mu)$ shall be the weak*-closure of A in $L^\infty(d\mu)$. We put

$$A_0 = \{f \in A; \int f\,dm = 0\} \text{ and } H_0^p(d\mu) = \{f \in H^p(d\mu); \int f\,dm = 0\} \qquad (1 \leqq p \leqq \infty).$$

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1) A representing measure for Φ is a positive regular measure m on X such that $\Phi(f) = \int f\,dm$ $(f \in A)$.

A function f of $H^1(dm)$ is called outer if $[fA]_1 = H^1(dm)$. This is equivalent to saying that

$$\int \log|f|dm = \log\left|\int fdm\right| > -\infty .$$

An outer function is determined by its modulus, up to a constant factor.

PROPOSITION 1. *Let f be a non-negative function in $L^1(dm)$. Then*

$$f(x) = |h(x)|^2 \quad a.e.,$$

where h is an outer function of $H^2(dm)$, if, and only if, $\log f \in L^1(dm)$.

We shall refer to Hoffman [3] for the proofs of above results.

2. It is obvious that each $u \in A+\bar{A}$ is represented uniquely as

$$u = f + \bar{g} + d \qquad f, g \in A_0, \qquad d \in \{1\} \tag{1}$$

where the bar denotes complex conjugation and $\{1\}$ the space spanned by 1. Representation (1) gives rise to a linear operator C:

$$i(\bar{g} - f) = Cu .$$

Bochner [1] has shown that the M. Riesz theorem is valid for C, i.e. for $1 < p < \infty$ there exists a finite constant M_p such that

$$\|Cu\|_p \leqq M_p \|u\|_p \qquad (u \in A+\bar{A})$$

Therefore C can be extended as a bounded linear operator C on $L^p(dm)$ into itself. We call C the conjugate operator and Cu the conjugate of u.

PROPOSITION 2. *If f is real, measurable and $|f| \leqq 1$, then for $0 < \lambda < \pi/2$*

$$\int \exp(\lambda|Cf|)\, dm \leqq N_\lambda < \infty .$$

This is proved along the line [5, p. 257] and we omit the proof.

RROPOSITION 3. *If f is real, measurable and $|f| \leqq \frac{\pi}{2} - \varepsilon$ $(\varepsilon > 0)$, then*

$$\exp(if - Cf) \in H^1(dm)$$

PROOF. By proposition 2, $\exp(if-Cf) \in L^1(dm)$. Since A is a Dirichlet algebra, there exist real $f_n \in A+\overline{A}$ such that $|f_n| \leqq \frac{\pi}{2} - \varepsilon$ and $f_n \to f$ a.e.. We can assume that $\exp(if_n - Cf_n)$ converges to $\exp(if-Cf)$ weakly in $L^1(dm)$. For $g \in A_0$,

$$\int g\cdot\exp(if_n - Cf_n)\,dm = 0, \quad \text{and so} \quad \int g\cdot\exp(if-Cf)\,dm = 0\,.$$

The assertion follows from $H^1(dm) = \{f \in L(dm)\,;\, \int fgdm = 0(\forall g \in A_0)\}$ ([3, p. 298]).

PROPOSITION 4. *If* $f \in A$ *and* $\mathrm{Re} f > 0$, *then* $\log f \in A$.

PROOF. We need the following result (for instance, see [4, p. 78]). If $\mathfrak{A}$ is a commutative semi-simple Banach algebra with unit, $f \in \mathfrak{A}$ and $F(z)$ is analytic in a region of the complex plane which includes the spectrum of f, (the range of the Gelfand transform $\hat{f}$ of f). Then there exists an unique $g \in \mathfrak{A}$ such that $\hat{g}(\Phi) = F(\hat{f}(\Phi))$ for all complex homomorphisms Φ of $\mathfrak{A}$.

Since each $x \in X$ determines a complex homomorphism $f \in A \to f(x)$, A is semi-simple. Now we note that the spectrum of $f \in A$ such that $\mathrm{Re} f > 0$ is contained in the half-plane $\mathrm{Re}\, z > 0$. Indeed the representing measure m is positive and so

$$\mathrm{Re}\widehat{f}(\Phi) = \mathrm{Re}\int f dm = \int \mathrm{Re} f dm > 0$$

for all complex homomorphisms Φ of A. In addition $F(z) = \log z$ is analytic in the half-plane $\mathrm{Re}\, z > 0$. Consequently we can use the above result in this case and we get $F(f) = \log f \in A$.

PROPOSITION 5. *If* $f \in H^1(dm)$ *and* $\mathrm{Re} f \geqq \varepsilon > 0$, *then* $\log f \in H^1(dm)$.

PROOF. It is easy to see that $\{u \in A; \mathrm{Re} u > 0\}$ is L^1-dense in $\{f \in H^1(dm);\ \mathrm{Re} f > 0\}$. Therefore, for $f \in H^1(dm)$ such that $\mathrm{Re} f \geqq \varepsilon$, there exist $u_n \in A$ $(n=1,2,\cdots)$ such that $\mathrm{Re}\, u_n > 0$ $(n=1,2,\cdots)$ and $\|u_n - f\|_1 \to 0$ $(n\to\infty)$. We can assume $\mathrm{Re} u_n \geqq \frac{\varepsilon}{2} > 0$ because $\mathrm{Re} f \geqq \varepsilon$. Since $\{u_n\}$ converges in the mean to f, $\{u_n\}$ converges in measure to f and so a subsequence $\{u_{n,k}\}$ of $\{u_n\}$

converges to f almost everywhere. Consequently we may assume $\{u_n\}$ converges to f almost everywhere from the start.

(i) Since $(\log x)' = 1/x$,

$$\frac{|\log|u_n| - \log|f||}{|u_n - f|} \leqq \frac{|\log|u_n| - \log|f||}{||u_n| - |f||} \leqq \left(\frac{\varepsilon}{2}\right)^{-1}.$$

It follows from this that $\{\log|u_n|\}$ converges to $\log|f|$ in L^1.

(ii) Since $\{u_n\}$ converges to f a.e., $\{\arg u_n\}$ converges to $\arg f$ a.e.. Also $|\arg u_n| < \frac{\pi}{2}$ (from $\operatorname{Re} u_n \geqq \frac{\varepsilon}{2}$). By Lebesgue bounded convergence theorem, $\{\arg u_n\}$ converges to $\arg f$ in L^1.

(iii) From (i) and (ii) we see that $\{\log u_n\}$ converges to $\log f$ in L^1. But $\log u_n \in A$ by proposition 4. This implies $\log f \in H^1(dm)$.

3. Two linear subspaces $\mathfrak{M}$ and $\mathfrak{N}$ in a Hilbert space are said to be at positive angle if

$$\rho = \sup\{|(f, g)|;\ f \in \mathfrak{M},\ g \in \mathfrak{N}.\ \|f\| = \|g\| = 1\} < 1 \tag{2}$$

It is easy to see that $0 \leqq \rho < 1$ if, and only if, there exists a ρ'; $0 \leqq \rho' < 1$ such that for every $f \in \mathfrak{M}$, $g \in \mathfrak{N}$,

$$2(1-\rho')\|f\|\|g\| \leqq \|f+g\|^2 \tag{3}$$

Now let μ be a positive finite measure and we consider the Hilbert space $L^2(d\mu)$. We take $H^2(d\mu)$ and $\overline{H}_0^2(d\mu)$ as $\mathfrak{M}$ and $\mathfrak{N}$, respectively and evaluate ρ of them.

PROPOSITION 6. *The linear subspaces $H^2(d\mu)$ and $\overline{H}_0^2(d\mu)$ are at positive angle in $L^2(d\mu)$ if, and only if, there is a constant M such that $\|Cu\|_\mu \leqq M\|u\|_\mu$ for every $u \in A+\overline{A}$ in the norm of $L^2(d\mu)$.*

PROOF. For $u \in A+\overline{A}$ we write as $u = f+\overline{g}+d, f, g \in A_0$, $d \in \{1\}$. Define a linear operator P by $f+d = Pu$. It is easy to see that C is bounded if, and only if, P is bounded, and the assertion is immediate from (3).

Next we consider the case that μ is absolutely continuous with respect to m i.e. $d\mu = w\,dm$, and $w^{-1} \in L^\infty(dm)$ and so $\log w \in L^1(dm)$. Then by proposition 1 there exists an outer function $h \in H^1(dm)$ such that $w = |h|$. We take $-\pi \leqq \arg z < \pi$, then we have

PROPOSITION 7. *Two linear subspaces* $H^2(wdm)$ *and* $\overline{H}_0^2(wdm)$ *are at positive angle if, and only if, there exist an* $\varepsilon > 0$ *and* $g \in H^\infty(dm)$ *such that* $|g(x)| \geqq \varepsilon$ *a.e. and* $|\arg g(x)h(x)| \leqq \frac{\pi}{2} - \varepsilon$ *a.e..*

The proof follows the same line of reasoning as Helson-Sezegö [2].

THEOREM 1. *Two linear subspaces* $H^2(wdm)$ *and* $\overline{H}_0^2(wdm)$ *are at positive angle if, and only if,*

$$w = \exp(u + Cv)$$

where u, v *are real functions in* $L^\infty(dm)$ *and* $\|v\|_\infty < \pi/2$.

PROOF. By Propositions 7 and 5, $\log gh \in H^1(dm)$ (g, h are the same as in proposition 7). Now

$$\log g(x)h(x) = \log|g(x)h(x)| + i\arg g(x)h(x).$$

Since $\arg gh \in L^\infty(dm)$, $\log|gh|$ is a conjugate of $-\arg gh(=v)$. Also $\|v\|_\infty < \pi/2$. Therefore

$$w(x) = \frac{1}{|g(x)|}|g(x)\,h(x)| = \exp(u(x) + Cv(x)).$$

The proof of the sufficiency is the same as in [2].

COROLLARY. *There is a constant* M *such that for* $u \in A + \overline{A}$

$$\|Cu\|_w \leqq M\|u\|_w$$

if and only if,

$$w = \exp(u + Cv)$$

where u, v *are real functions in* $L^\infty(dm)$ *and* $\|v\|_\infty < \pi/2$.

THEOREM 2. *Two linear subspaces* $H^2(wdm)$ *and* $\overline{H}_0^2(wdm)$ *are at positive angle if, and only if, there exist a* $g \in V_\infty$ *and an* $\varepsilon > 0$ *such that*

$$|\arg gh| \leqq \frac{\pi}{2} - \varepsilon \quad a.e.$$

where V_∞ *is the set of all invertible elements of* $H^\infty(dm)$.

PROOF. It suffices to see that gh is outer where g and h are the same as in Proposition 7. For, if gh is outer, then g is outer since h is outer and $\int ghdm = \int gdm \cdot \int hdm$, and so $g \in V_\infty$ because of $|g| \geqq \varepsilon$ (propositon 7). Now $gh \in H^1(dm)$, $|gh| \geqq \varepsilon'$ and $|\arg gh| \leqq \frac{\pi}{2} - \varepsilon'$ a.e.. We put $\arg gh = f$ and apply Proposition 3 to f. Then $F = \exp(if - Cf)$ is outer. We assert that $F = gh$. Indeed $\arg F = f = \arg gh$ and

$$|F| = \exp(-Cf) = \exp(-C \arg gh) = \exp(\log|gh|) = |gh|$$

from the proof of Theorem 1.

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REFERENCES

[1] S. BOCHNER, Generalized conjugate and analytic functions without expansions, Proc. Nat. Acad. Sci. U. S. A., 45 (1959), 855-857.
[2] H. HELSON AND G, SZEGÖ, A problem in prediction theory, Annali di Matematica, 51 (1960), 107-138.
[3] K. HOFFMAN, Analytic functions and logmodular Banach algebras, Acta Math., 108 (1962), 271-317.
[4] L. LOOMIS, An Introduction to Abstract Harmonic Analysis, Van Nostrand, 1953.
[5] A. ZYGMUND, Trigonometric Series, Vol. 1, Cambridge Univ. Press, 1959.

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