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ON CERTAIN RIEMANNIAN MANIFOLDS OF POSITIVE CURVATURE

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1. Introduction. Several authors (cf. H. E. Rauch [12], W. Klingenberg [6], [7], [9], M. Berger [1], V. A. Toponogov [14], Y. Tsukamoto [16]) proved the so-called sphere theorem: "*Compact simply connected* δ *pinched* $\left(\delta > \frac{1}{4}\right)$ *Riemannian manifolds are homeomorphic to spheres*". In this paper we deal with another version of the pinching theorem.

We assume that M is a compact simply connected Riemannian manifold of positive curvature K, $0 < K \leqq 1$. Let $d(p, q)$ be the distance between two points p, q of M. K. Hatsuse introduced the following number $L(M)$;

$$L(M) = \operatorname*{Max}_{p,q,r \in M} \{d(p, q) + d(q, r) + d(r, p)\} .$$

And he proved the following theorem.

THEOREM A. *Let* $L(M) < 3\pi$. *Then* M *is hemeomorphic to a sphere. In particular, if* $L(M) = 2\pi$. *then* M *is isometric to the sphere with constant curvature* 1.

REMARK 1. We can easily verify the inequality $L(M) \geqq 2\pi$.

In this paper we prove the following theorems.

THEOREM B. *Let* $L(M) = 3\pi$. *Then* M *has the same integral cohomology ring as the symmetric space of compact type of rank* 1.

THEOREM C. *If* M *is a compact Kaehlerian manifold of positive holomorphic sectional curvature* hol K, $0 <$ hol $K \leqq 1$ *and let* $L(M) = 3\pi$, *then* M *is isometric to the complex projective space with canonical metric, where* $\dim_{c} M \geqq 2$.

THEOREM D. *Let the inequalities* $0 < k \leqq K \leqq 1$ *be satisfied everywhere* *and the inequality* $L(M) > 3\pi/2\sqrt{k}$ *be satisfied, where* k *is a constant. Then* M *is a homological sphere. In particular* M *is homeomorphic to a sphere if* $\dim M \neq 3, 4$.

REMARK 2. The estimation of Theorem D is the best possible. In fact, let M be a symmetric space of compact type of rank 1 with canonical metric which is different from sphere, and the inequalities $1/4 \leqq K \leqq 1$ be satisfied everywhere. Then $L(M) = 3\pi$ and M is not a homological sphere.

REMARK 3. By using Theorem A and D we can immediately obtain the sphere theorem.

REMARK 4. Under the assumption of Theorem D we have the inequality $L(M) \leqq 2\pi/\sqrt{k}$. (cf. [9], [13])

THEOREM E. *Let the inequalities* $0 < k \leqq K$ *be satisfied everywhere and* $L(M) = 2\pi/\sqrt{k}$ *be satisfied. Then* M *is isometric to the sphere with constant curvature* k.

REMARK 5. If M is a compact simply connected Riemannian manifold with sectional curvature K, $4/9 \leqq k \leqq K \leqq 1$, then the closed geodesic of length $= 2\pi/\sqrt{k}$ can be regarded as a geodesic triangle, because of the inequality $2\pi/\sqrt{k} \leqq 3\pi$. Hence we have the following proposition from Theorem E.

PROPOSITION. *Let* M *be a compact simply connected Riemannian manifold with sectional curvature* K, $4/9 \leqq k \leqq K \leqq 1$. *If* M *admits a closed geodesic of length* $2\pi/\sqrt{k}$, *then* M *is isometric to the sphere with constant curvature* k.

2. Notations and definitions. Let M be a Riemannian manifold of dimension n ($n \geqq 2$). We denote by $<\ ,\ >$ (resp. $\|\ \ \|$) the scalar product (resp. norm) which defines the Riemannian structure of M. All the geodesics considered on M are parametrized by the arc-length measured from their origin. If $\Lambda = \{\lambda(s)\}$ ($0 \leqq s \leqq s_0$) is such a geodesic, then $\lambda'(s)$ denotes its tangent vector at $\lambda(s)$ and we have $\|\lambda'(s)\| = 1$ for all s. We denote by $d(p, q)$ the distance between two points p and q of M, with respect to the structure of metric space associated canonically to its Riemannian structure. If the manifold M is compact, we denote by $d(M)$ its diameter, that is the least upper bound of $d(p, q)$ when p and q vary on M. We denote by $G(p, q)$ the set of geodesics on M each of which join p to q and whose length is equal to

$d(p, q)$. By a geodesic triangle here we always mean a geodesic triangle composed of three shortest geodesic arcs.

3. Review of the known results. The following results are necessary from now on.

THEOREM 1. (Klingenberg [6], Toponogov [15]) *If the sectional curvature* K *of a compact simply connected Riemannian manifold satisfies the inequalities* $0 < K \leqq 1$, *the inequality* $d(p, C(p)) \geqq \pi$ *is satisfied for all points* p *of* M, *where* $C(p)$ *denotes the cutlocus of* p *on* M. *In particular we have the inequality* $d(M) \geqq \pi$.

THEOREM 2. (Klingenberg [8], Bishop and Goldberg [4]) *If the holomorphic sectional curvature* $\mathrm{hol}\, K$ *of a compact Kaehlerian manifold satisfies the inequalities* $0 < \mathrm{hol}\, K \leqq 1$, *the inequality* $d(p, C(p)) \geqq \pi$ *is satisfied for all points* p *of* M.

THEOREM 3. (Klingenberg [6]) *Assume that there exist two points* p *and* q *of* M *and a positive number* ρ *with the following properties*: (1) $d(p, q) \geqq \rho$, (2) *for any point* $r \in M$ *we have* $d(p, r) < \rho$ *or* $d(q, r) < \rho$, (3) *if* r, s *are any two points of* M *such that* $d(r, s) < \rho$, *then the shortest geodesic arc joining* r *to* s *is exactly one. Then* M *is homeomorphic to a sphere.*

THEOREM 4. (Berger [2]) *Suppose that* M *satisfies the conditions of Theorem* 1 *and the equality* $d(M) = \pi$. *Then all geodesics of* M *are simply closed and of length* 2π.

THEOREM 5. (Bott [5], Milnor [10]) *Let all geodesics of a complete simply connected Riemannian manifold* M *be simply closed and of same length. Then the integral cohomology ring of* M *is a truncated polynomial ring. Hence* M *has the same integral cohomology ring as the symmetric space of compact type of rank* 1.

THEOREM 6. (Klingenberg [8]) *Let* M *be a compact Kaehlerian manifold of positive holomorphic sectional curvature* $\mathrm{hol}\, K$, $0 < \mathrm{hol}\, K \leqq 1$ *and let the equality* $d(M) = \pi$ *be satisfied. Then* M *is isometric to the complex projective space with usual metric.*

THEOREM 7. (Myers [11]) *Let* M *be a complete Riemannian manifold and the sectional curvature* K *of* M *satisfy the inequalities* $0 < k \leqq K$

everywhere, where k is a constant. Then M is compact and the diameter $d(M)$ *of M satisfies the inequality* $d(M) \leqq \pi/\sqrt{k}$.

THEOREM 8. (Berger [3]) *Let M be a compact simply connected Riemannian manifold and the sectional curvature K of M satisfy the inequalities* $0 < k \leqq K$ *everywhere, where k is a constant. Furthermore if the inequality* $d(M) > \pi/2\sqrt{k}$ *is satisfied, then M is a homological sphere. In particular M is homeomorphic to a sphere, if* $\dim M \neq 3, 4$.

THEOREM 9. (Toponogov's comparison theorem, cf. [9], [13]) *Let M be a compact Riamannian manifold and the sectional curvature K of M satisfy the inequalities* $0 < k \leqq K$ *everywhere, where k is a constant. Let* p, q, r *be three points on M and* $\Gamma = \{\gamma(s)\}$, $\Lambda = \{\lambda(s)\}$ *be two geodesics on M such that* $\Gamma \in G(p, q)$, $\Lambda \in G(p, r)$, $\lambda(0) = \gamma(0) = p$. *We denote by* $S_2(k)$ *the 2 dimensional sphere with constant curvature k and denote by* $\Delta(\hat{p}\hat{q}\hat{r})$ *the triangle on* $S_2(k)$ *such that* $d(\hat{p}, \hat{q}) = d(p, q)$, $d(\hat{p}, \hat{r}) = d(p, r)$ *and that the angle* α *at* $\hat{p}$ *verifies* $\cos\alpha = \langle\gamma'(0), \lambda'(0)\rangle$. *If* $d(\hat{q}, \hat{r})$ *denote the length of the third side of the triangle* $\Delta(\hat{p}\hat{q}\hat{r})$ *of* $S_2(k)$, *then we have the inequality* $d(q, r) \leqq d(\hat{q}, \hat{r})$.

THEOREM 10. (Toponogov, cf. [9], [13]) *Let M be a complete Riemannian manifold and the sectional curvature K of M satisfy the inequalities* $0 < k \leqq K$ *everywhere, where k is a constant. And let* $d(M) = \pi/\sqrt{k}$ *be satisfied. Then M is isometric to the sphere with constant curvature k.*

4. Proof of theorems.

PROOF OF THEOREM B. We assume that M is not hemeomorphic to sphere. Let p, q be two points of M such that $d(p, q) = d(M)$. If we have $d(r, s) < \pi$, $(r, s \in M)$, then the two points r and s can be joined by exactly one shortest geodesic. By using Theorem 3 we can find a point r of M such that $d(p, r) \geqq \pi$ and $d(q, r) \geqq \pi$. By our assumption we have $d(p, q) = d(M) \geqq \pi$ and $L(M) = 3\pi$. Hence we have $d(M) = \pi$. By using Theorems 4 and 5, we obtain Theorem B. Q.E.D.

PROOF OF THEOREM C. Since M is kaehlerian manifold of $\dim_C M \geqq 2$, M is not homeomorphric to a sphere. By the same argument as the Theorem B, we have $d(M) = \pi$. And by using Theorem 6, we obtain Theorem C. Q.E.D.

PROOF OF THEOREM D. We have $3d(M) \geqq L(M)$ by the definitions of $L(M)$ and $d(M)$. Hence we have $d(M) > \pi/2\sqrt{k}$. So Theorem D is reduced to Theorem 8. Q.E.D.

PROOF OF THEOREM E. We prove $d(M)=\pi/\sqrt{k}$. Then, by using Theorem 10, M is isometric to the sphere with constant curvature k.

Since we have $L(M)=2\pi/\sqrt{k}$, we have a geodesic triangle $\Delta(pqr)$ on M with circumference $2\pi/\sqrt{k}$. Then, by using Theorem 9 we can easily see that the following two cases can only occur:

(i) One of the three numbers $d(p,q)$, $d(q,r)$, $d(r,p)$ is equal to $\pi/\sqrt{k}$.

(ii) Three geodesic arcs pq, qr, rp compose the closed geodesic of length $2\pi/\sqrt{k}$.

Under the assumption of Theorem E we have $d(M)\leqq\pi/\sqrt{k}$.

In the case (i) we have $d(M)=\pi/\sqrt{k}$.

In the case (ii) we assume $d(M)<\pi/\sqrt{k}$ and are led to a contradiction. By this assumption we have at least two geodesic arcs of length $\geqq\pi/2\sqrt{k}$ among three geodesic arcs pq, qr, rp which compose the closed geodesic Γ. Let them be pq and rp. And let Γ be divided into two parts of the same length by the two points p and p' on Γ. Then we can find that the point p' lies on Γ between q and r. Since we have $d(M)<\pi/\sqrt{k}$, we have a shortest geodesic $\Theta=\{\theta(v)\}\in G(p,p')$ $(0\leqq v\leqq m,\ m=d(p,p'),\ \theta(0)=p',\ \theta(m)=p,\ \Theta\neq\Gamma)$. Let the geodesic subarc $p'qp$ of Γ be $\Gamma_1=\{\gamma_1(v)\}$ $(0\leqq v\leqq l,\ l=\pi/\sqrt{k},\ \gamma_1(0)=p',\ \gamma_1(l)=p)$. And let the geodesic subarc $p'rp$ of Γ be $\Gamma_2=\{\gamma_2(v)\}$ $(0\leqq v\leqq l,\ l=\pi/\sqrt{k},\ \gamma_2(0)=p',\ \gamma_2(l)=p)$. Then we have either

$$<\gamma'_1(0),\theta'(0)>\ \geqq 0 \quad\text{or}\quad <\gamma'_2(0),\theta'(0)>\ \geqq 0\,.$$

First we assume $<\gamma'_1(0),\theta'(0)>\ \geqq 0$. We divide it into two cases: (a) $<\gamma'_1(0),\theta'(0)>\ >0$, (b) $<\gamma'_1(0),\theta'(0)>\ =0$. In the case (a), we use the cosine rule of spherical trigonometry and Theorem 9. We construct a geodesic triangle $\Delta(\hat{p}'\hat{p}\hat{q})$ on $S_2(k)$ such that $d(\hat{p}',\hat{p})=d(p',p)$, $d(\hat{p}',\hat{q})=d(p',q)$ and the angles $\sphericalangle(\hat{p}\hat{p}'\hat{q})=\sphericalangle(pp'q)$. Then we have by using Theorem 9

$$d(\hat{p},\hat{q})\geqq d(p,\hat{q})=\pi/\sqrt{k}-d(p',q)\,. \tag{1}$$

On the other hand we have by using the the cosine rule of spherical trigonometry

$$d(\hat{p},\hat{q})<\pi/\sqrt{k}-d(\hat{p}',\hat{q})=\pi/\sqrt{k}-d(p',q)\,. \tag{2}$$

From (1) and (2) we are led to a contradiction.

In the case (b) we also have $<\gamma'_1(0),\theta'(0)>\ =0$. By the assumption we have either $d(p,q)>\pi/2\sqrt{k}$ or $d(p,r)>\pi/2\sqrt{k}$. If we have $d(p,q)>\pi/2\sqrt{k}$, we can use for the geodesic triangle $\Delta(qp'p)$ the same argument as (a) and we are led to a contradiction. If we have $d(p,r)>\pi/2\sqrt{k}$, we are also led to a contradiction by using the same argumet as (a) for the geodesic triangle $\Delta(rp'p)$.

In the case $<\gamma_2'(0), \theta'(0)> \geqq 0$, we are led to a contradiction by using the same argument. Hence, in the case (ii) we also have $d(M)=\pi/\sqrt{k}$. Q.E.D.

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