Tôhoku Math. Journ.
Vol. 18, No. 2, 1966

A NOTE ON THE LITTLEWOOD-PALEY FUNCTION $g^*(f)$

SATORU IGARI

(Received February 28, 1966)

In a previous paper [1], we studied the functions of Littlewood-Paley, Lusin and Marcinkiewicz. And now we wish to consider a proof for the function g^* independently of the decomposition theorem on Fourier series. The function $g^*(x,f)$ is defined for f of $L(0,1)$ as follows;

$$g^*(x,f) = g^*(f) = \left(|\hat{f}_0|^2 + \sum_{n=1}^{\infty} \frac{|s_n(x,f)-\sigma_n(x,f)|^2}{n} \right)^{1/2},$$

where $\hat{f}_n$ is the n-th Fourier coefficient of f and s_n, σ_n denote the n-th partial sum of the Fourier series of f and its n-th Cesàro mean respectively.

THEOREM. *Let* $1 < p < \infty$, *then*

$$A_p\| f \|_p \leqq \| g^*(f)\|_p \leqq A_p'\| f \|_p$$

for all f *of* $L^p(0,1)$, A_p, A_p' *being positive constants independent of* f.

Let F_n be n-th Fejér kernel, that is,

$$F_n(x) = \sum_{\nu=-n}^{n} \left(1 - \frac{|\nu|}{n+1}\right) e^{2\nu\pi ix} = \frac{1}{n+1} \left\{ \frac{\sin(2n+1)\pi x}{\sin \pi x} \right\}^2$$

and let us denote $k_0(x)=1$ and

$$k_n(x) = \frac{F_{2n}(x)-F_n(x)}{c_n\sqrt{n}} \quad n = 1, 2, \cdots,$$

c_n being constants bounded away both from 0 and from infinity which will be defined later. Our proof composes, as that of [1] on the decomposition theorem, of estimation of the vector valued kernel $K(x) = (k_0(x), k_1(x), \cdots)$. Let us put $\mathfrak{K}f=K*f$ for f of $L(0,1)$ and $\mathfrak{L}g=\int <g(y), K(x-y)> dy$ for g of

$L(l^2)$, where $<\cdot,\cdot>$ means the inner product of the sequence space l^2. If $f\in L^2(0,1)$, then

$$\|\mathfrak{K}f\|_2^2 = |\hat{f}_0|^2 + \sum_{\nu=1}^{\infty} |\hat{f}_\nu|^2 \left(\sum_{n=|\nu|}^{\infty} \frac{n\nu^2}{c_n^2(2n+1)^2(n+1)^2} + \sum_{n=[|\nu|+1/2]}^{|\nu|-1} \frac{(2n+1-|\nu|)^2}{c_n^2 n(2n+1)^2}\right)$$
$$\leqq A\|f\|_2^2$$

for a constant A. Therfore $\mathfrak{K}$ is of strong type (2, 2). On the other hand we have $\int <\mathfrak{K}f, g>dx = \int f\mathfrak{L}g dx$ for f of L^2 and g of $L^2(l^2)$, so that $\mathfrak{L}$ is also of strong type (2, 2).

LEMMA. *We have*

$$\int_{2^{-1}>|x|>2^{-M}} |K(x+y)-K(x)|dx \leqq B$$

for all $|y|\leqq 2^{-M-1}$, $M=1,2,\cdots$, *where* B *is a constant.*

PROOF. By the definition of Fejér kernel, we have

$$|k_n(x+y)-k_n(x)| \leqq |k_n(x+y)|+|k_n(x)| \leqq \frac{A}{n\sqrt{n}\,x^2} \quad {}^{1)}$$

for all $2^{-1}>|x|>2^{-M}$, $|y|<2^{-M-1}$ and $M=1,2,\cdots$. On the other hand

$$|F_n(x+y)-F_n(x)| \leqq \frac{1}{n}\left|\frac{\sin(n+1)\pi(x+y)}{\sin\pi(x+y)} + \frac{\sin\pi(n+1)x}{\sin\pi x}\right|$$
$$\cdot\left|\frac{\sin(n+1)\pi(x+y)}{\sin\pi(x+y)} - \frac{\sin\pi(n+1)x}{\sin\pi x}\right|$$
$$\leqq \frac{An|y|}{|x|}.$$

Therefore

$$|k_n(x+y)-k_n(x)| \leqq A\sqrt{n}\,|y|/|x|.$$

Hence for arbitrary positive integer N, we have

1) Constants A may be different in each occasion.

$$\sum_{n=0}^{\infty} |k_n(x+y) - k_n(x)|^2 \leqq A \sum_{n=1}^{2^N} nx^{-2}y^2 + A \sum_{n=2^N+1}^{\infty} n^{-3}x^{-4}$$
$$\leqq A(2^{2N}y^2x^{-2} + 2^{-2N}x^{-4}),$$

for all $2^{-1} > |x| > 2^{-M}$ and $|y| < 2^{-M-1}$. Consequently

$$\int_{2^{-1}>|x|>2^{-M}} |K(x+y) - K(x)|\, dx$$

$$\leqq A \int_{2^{-1}>|x|>2^{-M}} (2^{-M+N}|x|^{-1} + 2^{-N}x^{-2})\, dx$$

$$\leqq A \sum_{\nu=1}^{M-1} \int_{2^{-\nu-1}}^{2^{-\nu}} (2^{-M+N}x^{-1} + 2^{-N}x^2)\, dx$$

$$\leqq A \sum_{\nu=1}^{M-1} (2^{-M+N} + 2^{-N+\nu}).$$

If we choose $N=[(\nu+M)/2]$ in the ν-th term, the last sum is not greater than

$$A \sum_{\nu=1}^{M-1} (2^{-M/2}2^{\nu/2} + 2^{-M/2}2^{\nu/2}) \leqq A,$$

which proves our lemma.

PROOF OF THEOREM. $\mathfrak{K}$ and $\mathfrak{L}$ are singular integral operators by the above lemma and therefore of strong type (p, p), $1 < p \leqq 2$ (cf. for example, the arguments in [1]), from which it results that $\mathfrak{K}$ and $\mathfrak{L}$ are of strong type (p, p) for $1 < p < \infty$ by conjugacy method. We remark that if $|j| \leqq n$, then j-th coefficient of n-th component of $\mathfrak{K}f$ is equal to that of $[s_n(x,f)-\sigma_n(x,f)]/c_n\sqrt{n}$. Therofore applying generalized M. Riesz theorem to $\|\mathfrak{K}f\|_p \leqq A_p\|f\|_p$, we get the first half inequality of the theorem. If $g = (g_0, g_1, \cdots) = (\hat{f}_0, \cdots, [s_n(x,f) - \sigma_n(x,f)]/\sqrt{n}, \cdots)$ belongs to $L^p(l^2)$, $1 < p < \infty$ and if we put $\mathfrak{L}_N g = k_0 * g_0 + \cdots + k_N * g_N$, then $\|\mathfrak{L}_N g\|_p$ is bounded. We denote one of its weak limit point by f', then we have $\|f'\|_p \leqq \limsup_{N\to\infty} \|\mathfrak{L}_N g\|_p \leqq A_p\|g\|_p = A_p\|g^*(f)\|_p$. It remains only to show that $f' = f$. For a suitable sequence $\{N_j\}$, we have

$$\hat{f}'_n = \lim_{j\to\infty} \int \mathfrak{L}_{N_j} g(x)\, e^{-2\pi i n x}\, dx$$
$$= \lim_{j\to\infty} \sum_{m=n}^{N_j} \frac{n^2}{c_m(2m+1)(m+1)^2} \hat{f}_n .$$

The last sum is equal to $\hat{f}_n$ if $c_m = m^2/(2m+1)^2, m \geqq 1$. Therefore our proof is completed.

REFERENCE

[1] S. IGARI, On the decomposition theorems of Fourier transforms with weighted norms, Tôhoku Math. Journ., 15(1963), 6–36.

TÔHOKU UNIVERSITY.