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A NOTE ON THE CLOSURE OF TRANSLATIONS IN L^p

MASAKITI KINUKAWA

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1. In this note, we shall consider a function $f(x)$ defined on the real axis $(-\infty, \infty)$. Suppose $p \geqq 1$ and $f \in L^1 \cap L^p$, then f is said to have the Wiener closure property (C_p), when the linear manifold spanned by the translates of f is dense in the space L^p. This property is equivalent to the following statement: if $\varphi(x) \in L^q \cap L^\infty$ and the convolution $f * \varphi(x) = 0$, then $\varphi(x) \equiv 0$, where $1/p + 1/q = 1$, (Cf. Herz [3]). Pollard [6] pointed out the close connection between the closure property (C_p) and a certain uniqueness problem for trigonometric integrals. Let us denote the set of zeros of the Fourier transform $\hat{f}(t)$ of $f(x)$ by $Z(f)$. We say that $f(x) \in L^p \cap L^1$ has the property (U_q) if the conditions

$$(a) \qquad \lim_{\sigma \to +0} \int_{-\infty}^{\infty} e^{-\sigma|x|} e^{ixt} \varphi(x)\, dx = 0 \quad \text{for} \quad t \notin Z(f)$$

and

$$(b) \qquad \varphi(x) \in L^q \cap L^\infty$$

can be satisfied simultaneously only by $\varphi(x) \equiv 0$.

Under the above terminology, Pollard's and Herz's result may be stated as follows:

I. For $1 \leqq p < \infty$, if $f \in L^p \cap L^1$ has the property (U_q), then f has the property (C_p).

II. For $2 \leqq p < \infty$, if $f \in L^p \cap L^1$ has the property (C_p), then f has the property (U_q).

It is essentially the same problems of spectral synthesis of bounded functions to ask whether the statement II for the case $1 < p < 2$ holds. Standing on this point of view, we shall show the following result:

THEOREM 1. *Suppose the following*:
i) $f \in L^p \cap L^1 \quad (1 < p < 2)$,

ii) *there exists a monotone decreasing function* $w(x) \in L^1(0, \infty)$ *such that* $|f(x)|^p \leqq w(|x|)$,
and

iii) f *has the property* (C_p).
Then f *has the property* (U_q), *where* $1/p + 1/q = 1$.

(That is, if f has a L^p-monotone majorant, then the property (C_p) is equivalent to the property (U_q).)

Considering the dual statement of Theorem 1, we see that for a proof of Theorem 1 it is sufficient to show that the following statement is true under the assumptions (i) and (ii) of Theorem 1: for $\varphi \in L^q \cap L^\infty$, if $\lim_{\sigma \to +0} U_\varphi(\sigma, t) = 0$ for $t \notin Z(f)$, then $f * \varphi = 0$, or $\int_{-\infty}^{\infty} \varphi(x) \overline{f}(x)\, dx = 0$, where $\overline{f}$ is the conjugate of f and

$$U_\varphi(\sigma, t) = \int_{-\infty}^{\infty} e^{-\sigma|x|} e^{+ixt} \varphi(x)\, dx\,.$$

2. We need some lemmas concerning the spectral analysis of bounded functions. For a function $\varphi(x) \in L^q \cap L^\infty$, we shall denote its spectral set by Sp.(φ), that is,

$$\text{Sp.}\,(\varphi) = \bigcap_k \{Z(k)\,;\, k * \varphi = 0,\, k \in L^1\}\,.$$

LEMMA 1. *Let F be a closed set on the real axis. If*

$$\lim_{\sigma \to +0} U_\varphi(\sigma, t) = 0 \quad \textit{for} \quad t \notin F,$$

then Sp.$(\varphi) \subset F$.

LEMMA 2. *Let I be any closed interval contained in the complement of* Sp.(φ), *then*

$$\lim_{\sigma \to +0} U_\varphi(\sigma, t) = 0\,, \quad \textit{uniformly on } t \in I.$$

Lemma 2 is due to Beurling ([1] and [2]).

Lemma 1 is essentially given by Pollard [6] and Herz [2]. Actually, Pollard proved the following:

LEMMA 3. *Suppose* $k(x) \in L^1 \cap L^p$ $(1 < p < 2)$, $|x|^{1/p}k(x) \in L^1$, *and* $\lim_{\sigma \to +0} U_\varphi(\sigma, t) = 0$ *for* $t \notin Z(k)$, *then* $k*\varphi = 0$.

On the other hand, $t_0 \notin \mathrm{Sp.}(\varphi)$ if and only if there exists a function $k(x) \in L^1$ such that $k*\varphi = 0$ but the Fourier transform $\widehat{k}(t)$ of $k(x)$ does not vanish on t_0. Take any $t_0 \notin F$. Since F is closed, there exists an open interval $I = (t_0 - \varepsilon, t_0 + \varepsilon)$ which is contained in the complement of F. Of course, we have $\lim_{\sigma \to +0} U_\varphi(\sigma, t) = 0$ for $t \in I$. We can find a function $k(x) \in L^1 \cap L^p$ $(1 < p < 2)$ such that $|x|^{1/p}k(x) \in L^1$ and I is the complement of $Z(k)$. (For example, take $k(x) = (1 - \cos \varepsilon t)e^{it_0 t}/(\varepsilon t^2)$, then $\widehat{k}(t) = 1 - |t - t_0|/\varepsilon$ for $t \in I$, and $=0$ for $t \notin I$.) Application of Lemma 3 shows that $k*\varphi = 0$. But $k(t_0) \neq 0$. Therefore, $t_0 \notin \mathrm{Sp.}(\varphi)$, that is, $\mathrm{Sp.}(\varphi) \subset F$. This completes the proof of Lemma 1.

From Lemmas 1 and 2, we have

LEMMA 4. *Let F be a closed set. If* $\lim_{\sigma \to +0} U_\varphi(\sigma, t) = 0$ *for* $t \notin F$, *then the above limit is convergent uniformly for t on any closed interval contained in the complement of F.*

3. Let A_p $(1 < p < 2)$ be the space of Fourier transforms of functions in L^p. Define a norm $\| \hat{f} \|_{A_p}$ in the space A_p by

$$\| \hat{f} \|_{A_p} = \| f \|_p = \left(\int_{-\infty}^{\infty} |f(x)|^p \, dx \right)^{1/p},$$

where $\hat{f}$ is the Fourier transform of f, that is,

$$\hat{f}(t) = \underset{\omega \to \infty}{\overset{(q)}{\mathrm{l.i.m.}}} \, (1/\sqrt{2\pi}) \int_{-\omega}^{\omega} f(x) e^{+ixt} \, dx.$$

We say that $g(t)$ is a normalized contraction of $\hat{f} \in A_p$, if $|g(t) - g(t')| \leqq |\hat{f}(t) - \hat{f}(t')|$ for any t and t', and if

$$\lim_{\lambda \to \infty} \int_{\lambda + a}^{\lambda + b} |g(t)|^q \, dt = 0 \quad \text{for each } a \text{ and } b.$$

Moreover, we say an element $\hat{f}$ of A_p is contractible in the space A_p, if every normalized contraction of $\hat{f}$ also belongs to the space A_p. And we say f is uniformly contractible in the space A_p if f is contractible in A_p and if $\lim_{n \to \infty} \| g_n \|_{A_p} = 0$ for any sequence $g_n(t)$ of normalized contractions of $\hat{f}(t)$ such that $\lim_{n\to\infty} g_n(t) = 0$. Using the above terms, we have the following theorem analogously to Beurling's result [2].

THEOREM 2. *Suppose that* (i) $f(x) \in L^1 \cap L^p$ $(1 < p < 2)$, (ii) $\hat{f}(t)$ *is uniformly contractible in the space* A_p, *and* (iii) *for* $\varphi \in L^q \cap L^\infty$, $\lim_{\sigma\to+0} U_\varphi(\sigma, t) = 0$ *on* $t \notin Z(f)$. *Then we have* $\int_{-\infty}^{\infty} \varphi(x)\bar{f}(x)\,dx = 0$.

We shall give a proof of Theorem 2 according to Beurling's argument. Take a sequence of circular projections $T_n(z)$, that is, $T_n(z) = z$ if $|z| \leqq 1/n$, and $= z/(n|z|)$ if $|z| > 1/n$. Since $\hat{f}(t)$ is the Fourier transform of $f \in L^1$, $\lim_{t\to\pm\infty} \hat{f}(t) = 0$. Hence we have $\lim_{\lambda\to\infty} \int_{\lambda+a}^{\lambda+b} |\hat{f}_n(t)|^q\,dt = 0$ for each n, a and b, where $\hat{f}_n(t) = T_n(\hat{f}(t))$. That is, $\hat{f}_n(t)$ is a normalized contraction of $f(t)$, and so the notation $\hat{f}_n(t)$ is justified by the assumption (ii). Since $\lim_{t\to\pm\infty} \hat{f}(t) = 0$, there exists a positive number R_n such that $\hat{f}(t) - \hat{f}_n(t) = 0$ for $|t| > R_n$. Put $E_n = Z(f) \cap [-R_n, R_n]$. Note $E_n \subset Z(f)$, and that $\hat{f}(t)$ is continuous. For each point $t_0 \in E_n$, there exists a neighborhood $N(t_0)$ of t_0 such that $|\hat{f}(t)| \leqq 1/n$ for $t \in N(t_0)$. Hence we have a finite number of open intervals $N(t_k)$ $(k = 1 \sim l)$ such that $\bigcup_{k=1}^{l} N(t_k) \supset E_n$ and $|f(t)| \leqq 1/n$ for $t \in \bigcup_{k=1}^{l} N(t_k)$. Put $[-R_n, R_n] - \bigcup_{k=1}^{l} N(t_k) = \bigcup_{j=1}^{m} I_j = I^{(n)}$, where each I_j is a closed interval contained in the complement of $Z(f)$ and

$$\hat{f}(t) - \hat{f}_n(t) = 0 \quad \text{for} \quad t \notin I^{(n)}. \tag{3.1}$$

Moreover, by Lemma 4,

$$\lim_{\sigma\to+0} U_\varphi(\sigma, t) = 0, \quad \text{uniformly for} \quad t \in I^{(n)}. \tag{3.2}$$

On the other hand, the function $\hat{f}(t) - \hat{f}_n(t)$ is bounded and its support is contained in the compact set $I^{(n)}$. Therefore $\hat{f}(t) - \hat{f}_n(t)$ is a Fourier transform of some function $G(x)$ in L^2, that is,

$$\hat{f}(t) - \hat{f}_n(t) = (1/\sqrt{2\pi})\, \underset{\omega\to\infty}{\overset{(2)}{\mathrm{l.i.m.}}} \int_{-\omega}^{\omega} G(x)\, e^{+ixt}\, dx.$$

This is also equlal to

$$(1/\sqrt{2\pi})\,\underset{\omega\to\infty}{\overset{(q)}{\text{l.i.m.}}}\int_{-\omega}^{\omega}\{f(x)-f_n(x)\}\,e^{+ixt}\,dx\,.$$

Now, applying the Fourier reciprocity, we have $f(x)-f_n(x)=G(x)\in L^2$. Hence we can apply the Parseval relation. That is, we have

$$(1/2\pi)\int_{-\infty}^{\infty}e^{-\sigma|x|}\,\varphi(x)\{\bar f(x)-\bar f_n(x)\}\,dx=\int_{-\infty}^{\infty}U_\varphi(\sigma,t)\{\overline{\hat f}(t)-\overline{\hat f}_n(t)\}dt\,.$$

Letting $\sigma\to+0$, we have

$$(1/2\pi)\int_{-\infty}^{\infty}\varphi(x)\{\bar f(x)-\bar f_n(x)\}dx=\lim_{\sigma\to+0}\int_{-\infty}^{\infty}U_\varphi(\sigma,t)\{\overline{\hat f}(t)-\overline{\hat f}_n(t)\}\,dt\,.$$

By (3. 1) and (3. 2), the right hand side is equal to

$$\lim_{\sigma\to+0}\int_{I(n)}U_\varphi(\sigma,t)\{\overline{\hat f}(t)-\overline{\hat f}_n(t)\}\,dt=0$$

Thus we can conclude

$$\int_{-\infty}^{\infty}\varphi(x)\{\bar f(x)-\bar f_n(x)\}\,dx=0\,,\tag{3. 3}$$

for a sufficiently large number n. By Hölder's inequality, we have

$$\left|\int_{-\infty}^{\infty}\varphi(x)\bar f_n(x)\,dx\right|\leqq\|\varphi\|_q\,\|f_n\|_p=\|\varphi\|_q\,\|\hat f_n\|_{A_p}\,.$$

Since $\hat f(x)$ is uniformly contractible, the right hand side of the above tends to zero when $n\to\infty$, that is,

$$\lim_{n\to\infty}\int_{\to\infty}^{\infty}\varphi(x)\bar f_n(x)\,dx=0\,.\tag{3. 4}$$

Summing the results (3. 3) and (3. 4) up, we can conclude that

$$\int_{-\infty}^{\infty}\varphi(x)\bar f(x)\,dx=0\,.$$

4. In order to finish the proof of Theorem 1, we need the following:

THEOREM 3. *Let $f(x) \in L^p$ $(1 < p < 2)$. Suppose that there exists a function $w(x)$ such that* (i) *$w(x)$ is even and positive,*
(ii) $x^2 w^{2/p}(x) \in L(0, \delta)$, $w^{2/p}(x) \in L(\delta, \infty)$ *for any* $\delta > 0$,

$$\text{(iii)} \quad \int^{\infty} x^{-3p/2} \left\{ \int_0^x u^2 w^{2/p}(u)\, du \right\}^{p/2} dx + \int_0^{\infty} x^{-p/2} \left\{ \int_x^{\infty} w^{2/p}(u)\, du \right\}^{p/2} dx < \infty$$

and (iv) $|f(x)|^p \leqq w(|x|)$. *Then the Fourier transform $\hat{f}(t)$ of $f(x)$ is uniformly contractible in the space A_p.*

A proof of Theorem 3 is a simple modification of the argument in the previous paper [5]. Let $\hat{f}_n(t)$ be a sequence of normalized contractions of $\hat{f}(t)$ with a property $\lim_{n\to\infty} \hat{f}_n(t) = 0$. Under the assumption of Theorem 3, $\hat{f}(t)$ is contractible in A_p, and so $\hat{f}_n(t)$ is a Fourier transform of $f_n(x) \in L_p$ (cf. [4]). Therefore, we need only to show that

$$\lim_{n\to\infty} \| \hat{f}_n \|_{A_p}^p = \lim_{n\to\infty} \int_{-\infty}^{\infty} | f_n(x)|^p\, dx = 0 .$$

Let $g(u) \in L^p$. By the Schwarz inequality, we have

$$S(x) = \int_0^x u^p | g(u)|^p\, dx \leqq x^{1-p/2} \left(\int_0^x u^2 | g(u)|^2\, du \right)^{p/2} .$$

Apply the partial integration to $\int_0^N | g(x)|^p\, dx = \int_0^N x^{-p} S'(x) dx$, then we have the following inequality:

$$\text{(4. 1)} \qquad \int_0^{\infty} |g(x)|^p\, dx \leqq p \int_0^{\infty} x^{-3p/2} \left(\int_0^x u^2 | g(u)|^2\, du \right)^{p/2} dx.$$

The Parseval relation and the assumptions assure the following inequalities:

$$\int_0^x u^2 | f_n(u) |^2\, du \leqq C x^2 \int_{-\infty}^{\infty} | f_n(u)|^2 \sin^2 u/x\, du$$

$$\text{(4. 2)} \qquad = C x^2 \int_{-\infty}^{\infty} | \hat{f}_n(u+1/x) - \hat{f}_n(u-1/x)|^2\, du$$

$$\leqq Cx^2 \int_{-\infty}^{\infty} | \hat{f}(u+1/x) - \hat{f}(u-1/x)|^2 \, du \tag{4.3}$$

$$= Cx^2 \int_{-\infty}^{\infty} | f(u) |^2 \sin^2 u/x \, du$$

$$\leqq C\left\{\int_0^x u^2 w^{2/p}(u) \, du + x^2 \int_x^{\infty} w^{2/p}(u) \, du\right\}. \tag{4.4}$$

From (4.1) ~ (4.4), we have

$$\| \hat{f}_n \|_{A_p}^p = \int_{-\infty}^{\infty} | f_n(x) |^p \, dx$$

$$\leqq C \int_{-\infty}^{\infty} dx \left\{ |x|^{-1} \int_{-\infty}^{\infty} | \hat{f}_n(u+1/x) - \hat{f}_n(u-1/x)|^2 \, du\right\}^{p/2}$$

$$\leqq C \int_{-\infty}^{\infty} dx \left\{ |x|^{-1} \int_{-\infty}^{\infty} | \hat{f}(u+1/x) - \hat{f}(u-1/x)|^2 \, du\right\}^{p/2}$$

$$\leqq C \int_0^{\infty} x^{-3p/2} \left\{\int_0^x u^2 w^{2/p}(u) \, du\right\}^{p/2} dx + C \int_0^{\infty} x^{-p/2} \left\{\int_x^{\infty} w^{2/p}(u) \, du\right\}^{p/2} dx$$

$$< \infty .$$

These inequalities assure the use of Lebesgue's theorem, and so we see that $\lim_{n\to\infty} \| \hat{f}_n \|_{A_p} = 0$. This completes the proof of Theorem 3.

Since a monotone decreasing functions $w(x) \in L^1(0, \infty)$ satisfies the conditions (ii) and (iii) of Theorem 3 (cf. [4]), we finish the proof of Theorem 1 through Theorems 2 and 3.

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INTERNATIONAL CHRISTIAN UNIVERSITY,
TOKYO, JAPAN.