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SOME TRANSFORMATIONS ON K-CONTACT AND NORMAL CONTACT RIEMANNIAN MANIFOLDS

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1. Introduction. In [3] and [5], infinitesimal transformations on K-contact and normal contact Riemannian manifolds were studied, and global transformations on almost contact and contact Riemannian manifolds were discussed in [4]. In this note, we shall add some results concerning global transformations on K-contact and normal contact Riemannian manifolds. In §2, some preliminary notions and identities are given for later use. In §3, it will be shown that homothetic and affine transformations on K-contact Riemannian manifolds must be isometries. In §4, transformations on η-Einstein manifolds will be concerned. The author wishes to thank Professor Sasaki and Mr. Tanno for their suggestions and kind advices.

2. Preliminaries ([1], [2]). Let M be an $n(=2m+1, m \geqq 1)$ dimensional C^∞-manifold with a contact structure η. We take an arbitrary point x and a local coordinate system (x^k, U) around x. If we put $2\phi_{ji}=\partial_j\eta_i-\partial_i\eta_j$, there exists a Riemannian metric g_{ji} in M such that $\phi_i^h=g^{hr}\phi_{ir}$ and $\xi^i=g^{ir}\eta_r$ define a (ϕ,ξ,η,g)-structure with η_i and g_{ji}. That is,

$$\begin{aligned} &\xi^i\eta_i = 1, \quad \operatorname{rank}(\phi_j^i) = n-1\,, \\ (1.1) \qquad &\phi_j^i\xi^j = 0, \quad \phi_j^i\phi_k^j = -\delta_k^i + \xi^i\eta_k\,, \\ &g_{ji}\xi^i = \eta_j, \quad g_{ji}\phi_k^j\phi_h^i = g_{kh} - \eta_k\eta_h \end{aligned}$$

hold good. This structure is called a contact metric structure, and the manifold with this structure is called a contact Riemannian manifold. If we define,

$$(1.2) \qquad \phi^{kh} = g^{kr}\phi_r^h = g^{ki}g^{hr}\phi_{ir}\,,$$

this is a skew-symmetric contravariant tensor.

For a contact Riemannian manifold, torsion tensor fields $N_{ji}{}^h$ and N_j^i can be defined. The condition $N_j^i=0$ is equivalent to the fact that ξ^i is a Killing vector, and the contact Riemannian manifold which satisfies this condition is called a K-contact Riemannian manifold. In a K-contact Riemannian manifold,

$$\nabla_j \eta_i = \phi_{ji}, \tag{1.3}$$

$$\nabla_k \phi_{ji} + R_{ijk}{}^r \eta_r = 0, \tag{1.4}$$

$$R_{kji}{}^h \xi^k \xi^i = \eta_j \xi^h - \delta_j^h, \quad R_{kjih} \xi^k \xi^h = g_{ji} - \eta_j \eta_i \tag{1.5}$$

hold good, where $R_{kji}{}^h$ is the curvature tensor.

On the other hand, the contact Riemannian manifold which satisfies $N_{ji}{}^h=0$ is called a normal contact Riemannian manifold. It is known that a normal contact Riemannian manifold is K-contact. In this case, we have

$$\nabla_k \phi_{ji} = \eta_j g_{ki} - \eta_i g_{kj}, \tag{1.6}$$

$$\xi^k R_{kji}{}^r = \xi^r g_{ji} - \eta_i \delta_j^r, \quad R_{kji}{}^r \eta_r = \eta_k g_{ji} - \eta_j g_{ki}, \tag{1.7}$$

$$\phi_j^r R_r^l + (1/2)\phi^{rk} R_{r\,kj}{}^l = (n-2)\phi_j^l, \tag{1.8}$$

where we have put

$$R_{ji} = R_{rji}{}^r \quad \text{and} \quad R_l^r = g^{rs} R_{sl}.$$

Now, K-contact Riemannian manifold $(m>1)$ in which Ricci's tensor takes the form

$$R_{ji} = a g_{ji} + b \eta_j \eta_i \tag{1.9}$$

is called a K-contact η-Einstein manifold, where a and b become constants. Then,

$$a + b = n - 1 \tag{1.10}$$

holds good and in a normal contact η-Einstein manifold, we get

$$\frac{1}{2}\phi^{rk} R_{rkj}{}^i = (b-1)\phi_j^i. \tag{1.11}$$

In the sequel we will be concerned with differentiable transformations on M. These transformations induce algebra automorphisms of algebra over real numbers of all tensor fields defined on M, and they preserve types and commute with contractions. The notation 'bar' will be used to denote the geometric objects which are transformed by the induced transformation.

By the automorphism of a contact Riemannian manifold M, we mean the transformation of M which leaves invariant ϕ_j^i, ξ^i, η_i, and g_{ji} of (ϕ, ξ, η, g)-structure.

3. Transformations on K-contact Riemannian manifolds.

PROPOSITION 1. ([4]) *In a contact Riemannian manifold M, any conformal transformation μ which is also a contact transformation is an isometry, and if $\bar{\eta}(\xi)>0$ holds, μ is an automorphism.*

PROOF. By definition we can write $\bar{g}_{ji} = \rho g_{ji}$, $\bar{g}^{ji} = (1/\rho) g^{ji}$, and $\bar{\eta}_j = \sigma\eta_j$ for some positive scalar ρ and scalar σ. By $(1.1)_5$ $\bar{\xi}^i = (\sigma/\rho)\xi^i$ holds good. Contracting both sides of $\bar{\phi}_{ji} = \sigma\phi_{ji} + (1/2)(\partial_j\sigma\cdot\eta_i - \partial_i\sigma\cdot\eta_j)$ with ξ^i, we know that $\bar{\phi}_{ji} = \sigma\phi_{ji}$ and consequently $\bar{\phi}_j^i = (\sigma/\rho)\phi_j^i$ is true. Now we have $\sigma^2 = \rho^2 = \rho = 1$ by $(1.1)_4$. Q.E.D.

THEOREM 1. *In a K-contact Riemannian manifold M, any homothetic transformation μ is an isometry.*

PROOF. We can write $\bar{g}_{ji} = \rho g_{ji}$, and $\bar{g}^{ji} = (1/\rho) g^{ji}$ where ρ is a positive constant. Because a homothetic transformation is an affine transformation, we have

$$\rho R_{kjih}\bar{\xi}^k\bar{\xi}^h = \rho g_{ji} - \bar{\eta}_j\bar{\eta}_i \,. \tag{2.1}$$

by (1.5). Transvecting (2.1) with $\xi^j\xi^i$, we get

$$\rho(g_{kh} - \eta_k\eta_h)\bar{\xi}^k\bar{\xi}^h = \rho - \bar{\eta}_j\bar{\eta}_i\xi^j\xi^i \,. \tag{2.2}$$

But since $g_{kh}\bar{\xi}^k\bar{\xi}^h = 1/\rho$ and $\eta_k\bar{\xi}^k = (1/\rho)\xi^k\bar{\eta}_k$ hold good, we have

$$\rho(1-\rho) = (1-\rho)(\xi^r\bar{\eta}_r)^2 \,. \tag{2.3}$$

So we get $\rho = 1$ or $(\xi^r\bar{\eta}_r)^2 = \rho$. In the second case,

$$\bar{\eta}_k = (\xi^r\bar{\eta}_r)\eta_k - \phi_k^j\phi_j^h\bar{\eta}_h$$

is true by $(1.1)_4$. Since μ is a homothetic transformation, we have $\|\bar{\eta}_k\|^2 = g^{kj}\bar{\eta}_k\bar{\eta}_j = \rho$. On the other hand, $\|(\xi^r\bar{\eta}_r)\eta_k\|^2 = (\xi^r\bar{\eta}_r)^2 = \rho$ is true and, $(\xi^r\bar{\eta}_r)\eta_k$ and $\phi_k^j\phi_j^h\bar{\eta}_h$ are mutually orthogonal, so $\|\phi_k^j\phi_j^h\bar{\eta}_h\| = 0$ holds good. That is, μ is a contact transformation. Then by Proposition 1 we know that μ is isometric. Q.E.D.

THEOREM 2. *In a K-contact Riemannian manifold M, any affine transformation μ is an isometry.*

PROOF. Since μ is an affine transformation, by Ricci's identity,

$$0 = \nabla_l \nabla_m \bar{g}_{ik} - \nabla_m \nabla_l \bar{g}_{ik} = -R_{lmi}{}^a \bar{g}_{ak} - R_{lmk}{}^a \bar{g}_{ai},$$

that is,

$$R_{lmi}{}^a \bar{g}_{ak} + R_{lmk}{}^a \bar{g}_{ai} = 0 \tag{2.4}$$

holds good. Transvecting (2. 4) with $\xi^i \xi^k \xi^l$, we have by (1. 5)

$$\bar{g}_{kr}\xi^r = (\xi^a \xi^b \bar{g}_{ab})\eta_k . \tag{2.5}$$

Now, transvecting (2. 4) with $\xi^i \xi^l$, we get

$$(\eta_m \xi^a - \delta^a_m)\bar{g}_{ak} + \xi^l R_{lmk}{}^a \xi^i \bar{g}_{ia} = 0 . \tag{2.6}$$

Substituting (2. 5) into (2. 6), we obtain

$$\bar{g}_{mk} = (\xi^a \xi^b \bar{g}_{ab})\, g_{mk} . \tag{2.7}$$

Thus μ is a conformal transformation. But any affine transformtion which is also a conformal transformation must be homothetic. So Theorem 2 reduces to Theorem 1. Q.E.D.

PROPOSITION 2.*) *In a K-contact Riemannian manifold M, any projective transformation μ which is at the same time a contact transformation with constant associated function σ is an isometry. Moreover, if σ is positive, μ is an automorphism.*

PROOF. Since μ is a projective transformation, we have

$$\bar{\Gamma}^k_{ji} = \Gamma^k_{ji} + p_j \delta^k_i + p_i \delta^k_j$$

for some covariant vector field p_i, where Γ^k_{ji} is Christoffel's symbols. Then,

$$\bar{\phi}_{ij} = \partial_i \bar{\eta}_j - \bar{\Gamma}^k_{ij}\bar{\eta}_k = \sigma\phi_{ij} - \sigma(p_i\eta_j + p_j\eta_i) \tag{2.8}$$

*) This result is also obtained by Mr. Tanno, and in infinitesimal case in a normal contact Riemannian manifold by Mr. Mizusawa. ([5])

holds good. Adding to (2.8) the identity which is obtained from (2.8) by permuting i and j, we get

$$p_i \eta_j + p_j \eta_i = 0. \tag{2.9}$$

Now transvecting (2.9) with g^{ji} and ξ^i respectively, we have $p_i = 0$. Thus projective transformation which is also a contact transformation with constant associsted function must be an affine transformation. Then Proposition 2 reduces to Theorem 2 and Proposition 1. Q.E.D.

4. Transformations on η-Einstein manifolds.

PROPOSITION 3. *In a K-contact η-Einstein manifold ($b \neq 0$, $m > 1$), an isometry μ which satisfies $\bar{\eta}(\xi) > 0$ is an automorphism.*

PROOF. Since $\bar{R}_{ji} = R_{ji}$ holds good, we have $\bar{\eta}_i \bar{\eta}_j = \eta_i \eta_j$. Transvecting this identity with ξ^i, we know that μ is a contact transformation. Thus Proposition 3 reduces to Proposition 1. Q.E.D.

In the sequel we will be concerned with two theorems which are studied in [3] in infinitesimal case.

THEOREM 3. *In a normal contact η-Einstein manifold M ($b \neq 0$, $m > 1$), any conformal transformation μ is an isometry, and if it satisfies $\bar{\eta}(\xi) > 0$, μ is an automorphism.*

PROOF. If we put $\bar{g}_{ji} = \rho g_{ji}$, where ρ is a positive scalar, and $\tau = (1/2)\log \rho$, then we have

$$\bar{\Gamma}^i_{jk} = \Gamma^i_{jk} + (\tau_k \delta^i_j + \tau_j \delta^i_k - \tau^i g_{jk}), \tag{3.1}$$

where $\tau_k = \partial_k \tau$ and $\tau^k = g^{kj} \tau_j$. Next,

$$\begin{aligned} \bar{R}_{kji}{}^h = R_{kji}{}^h + \delta^h_k A_{ji} - \delta^h_j A_{ki} + A^h_k g_{ji} - A^h_j g_{ki} \\ - (\delta^h_k g_{ji} - \delta^h_j g_{ki}) \tau_r \tau^r \end{aligned} \tag{3.2}$$

holds good, where $A_{ji} = \tau_j \tau_i - \nabla_j \tau_i$ is symmetric and $A^h_k = g^{hr} A_{rk}$. In particular, we have

$$\bar{R}_{jk} = R_{jk} + (n-2) A_{jk} + A^r_r g_{jk} - (n-1) \tau_r \tau^r g_{jk}. \tag{3.3}$$

Contracting (3.2) with $\bar{\eta}_k$ and using (1.7) we get .

(3.4) $(\rho+\tau_r\tau^r)(\overline{\eta}_j g_{ki}-\overline{\eta}_k g_{ji}) = -R_{kji}{}^r\overline{\eta}_r+(A_{ki}\overline{\eta}_j-A_{ji}\overline{\eta}_k)+(A_j^r g_{ki}-A_k^r g_{ji})\overline{\eta}_r\,.$

And next (3.5) comes from (1.9) and (3.3).

(3.5) $$(n-2)A_{ki} = \{a(\rho-1)+(n-1)\tau_r\tau^r - A_r^r\}\, g_{ki} + b(\overline{\eta}_k\overline{\eta}_i-\eta_k\eta_i)\,.$$

Now transvecting (3.4) with $(1/2)\phi^{kj}$, we get by (1.11)

(3.6) $$(\rho-1+b+\tau_r\tau^r)\phi_i^j\overline{\eta}_j = \phi^{kj}A_{ki}\overline{\eta}_j + \phi_i^j A_j^r\overline{\eta}_r\,.$$

Next, contracting (3.5) with $\phi^{kj}\overline{\eta}_j$ and $\phi_i^k g^{ir}\overline{\eta}_r$ respectivily, we have

(3.7) $$(n-2)\phi^{kj}A_{ki}\overline{\eta}_j = \alpha\phi_i^j\overline{\eta}_j\,,$$

(3.8) $$(n-2)\phi_i^k A_k^r\overline{\eta}_r = (\alpha+b\rho)\phi_i^k\overline{\eta}_k\,,$$

by virtue of (1.1) and (1.2) where we have put

$$\alpha = a(\rho-1)+(n-1)\tau_r\tau^r - A_r^r\,.$$

Substituting (3.7) and (3.8) into (3.6), we obtain

(3.9) $$\{(n-2)(\rho-1+b+\tau_r\tau^r) - b\rho - 2\alpha\}\,\phi_i^j\overline{\eta}_j = 0\,.$$

On the other hand, transvecting (3.5) with g^{ji},

(3.10) $$2A_r^r = n\tau_r\tau^r + (n-b)(\rho-1)$$

holds good. Then using (3.10) we have

$$(n-2)(\rho-1+b+\tau_r\tau^r) - b\rho - 2\alpha = b(n-3) \neq 0\,.$$

Thus, from (3.9) we know that $\phi_i^j\overline{\eta}_j=0$, that is, μ is a contact transformation. Then Theorem 3 reduces to Proposition 1. Q.E.D.

THEOREM 4. *In a normal contact η-Einstein manifold M ($b\neq 0$, $m>1$), any projective transformation μ is an isometry. Moreover if $\overline{\eta}(\xi)>0$ holds good, μ is an automorphism.*

PROOF. By definition of projective transformation, we get

(3.11) $$\overline{\Gamma}_{ji}^k = \Gamma_{ji}^k + p_j\delta_i^k + p_i\delta_j^k$$

for some covariant vector p_i. If we put $A_{ji}=p_j p_i-\nabla_j p_i$,

$$\bar{R}_{kji}{}^h = R_{kji}{}^h + \delta_k^h A_{ji} - \delta_j^h A_{ki} \tag{3.12}$$

and especially

$$\bar{R}_{kj} = R_{kj} + (n-1) A_{kj} \tag{3.13}$$

hold good.

LEMMA 1. *In a normal contact η-Einstein manifold, we have for some scalar α*

$$\bar{g}_{kj} = \alpha(g_{kj} + A_{kj}). \tag{3.14}$$

PROOF OF LEMMA 1. By Ricci's identity and (3.12) we get

$$R_{lki}{}^a \bar{g}_{aj}+R_{lkj}{}^a \bar{g}_{ai} = A_{il}\bar{g}_{kj}+A_{jl}\bar{g}_{ki}-A_{ki}\bar{g}_{jl}-A_{kj}\bar{g}_{il}. \tag{3.15}$$

Transvecting (3.15) with $\xi^l\xi^i$, we have

$$\xi^a\xi^b\bar{g}_{ab}(g_{kj}+A_{kj}) = (1+\xi^a\xi^b A_{ab})\bar{g}_{kj} + \xi^r(\bar{g}_{kr}\eta_j-\bar{g}_{jr}\eta_k) + \xi^a\xi^b(A_{aj}\bar{g}_{bk}-A_{ak}\bar{g}_{bj}). \tag{3.16}$$

Adding to (3.16) the identity which is obtained from (3.16) by permuting k and j, we get (3.14) unless $g_{kj}+A_{kj} = 0$. Next, if $g_{kj}+A_{kj} = 0$, operating ∇_l to the both sides of $\nabla_k p_j = p_k p_j + g_{kj}$, and using Ricci's identity,

$$R_{lkj}{}^r p_r = g_{kj} p_l - g_{lj} p_k \tag{3.17}$$

holds good. Transvecting (3.17) with $(1/2)\phi^{lk}$, we get $\phi_j^r p_r=0$ by virtue of $b\neq 0$. So we can write $p_l = \sigma\eta_l$. By differentiating this identity covariantly and taking notice of the shew-symmetric property of ϕ_{kl}, we obtain

$$2\sigma\phi_{kl} = \nabla_l\sigma\cdot\eta_k - \nabla_k\sigma\cdot\eta_l. \tag{3.18}$$

Contracting (3.18) with ξ^k, we have $\nabla_l\sigma = \beta\eta_l$. Substituting this into (3.18) we get $\sigma = 0$. Thus in this case μ is an affine transformation and consequently an isometry by virtue of Theorem 2.

LEMMA 2. *In a normal contact η-Einstein manifold, if α of Lemma*

1 *is a constant,* μ *is an isometry.*

PROOF OF LEMMA 2. Substituting $\bar{g}_{ij} = \alpha(g_{ij}+A_{ij})$ into

$$\nabla_k \bar{g}_{ij} = 2p_k \bar{g}_{ij} + p_i \bar{g}_{kj} + p_j \bar{g}_{ik},$$

we have

$$\nabla_k \nabla_i p_j = 2(p_k \nabla_i p_j + p_i \nabla_k p_j + p_j \nabla_k p_i) - 4p_k p_i p_j - 2p_k g_{ij} - p_i g_{kj} - p_j g_{ik}.$$

Then, using Ricci's identity we know that $p_i = \sigma\eta_i$ holds and that μ is an affine transformation by the same method as Lemma 1.

On the other hand, next (3. 19) comes from (3. 12) and (1. 7).

$$R_{kji}{}^r \bar{\eta}_r + \bar{\eta}_k A_{ji} - \bar{\eta}_j A_{ki} = \bar{\eta}_k \bar{g}_{ji} - \bar{\eta}_j \bar{g}_{ki}. \tag{3. 19}$$

Now, transvecting (3. 19) with $(1/2)\phi^{kj}$, we get

$$\bar{\eta}_j\{\phi^{kj}\bar{g}_{ki} + (b-1)\phi_i^j - \phi^{kj}A_{ki}\} = 0$$

by virtue of (1. 11). Using (3. 14) this identity can be turned into

$$(1-\alpha)(\phi_i^j + \phi^{kj}A_{ki})\bar{\eta}_j = b\phi_i^j\bar{\eta}_j. \tag{3. 20}$$

On the other hand, from (3. 13) and (3. 14) we have

$$a(1-\alpha)(g_{ki}+A_{ki}) = b(\bar{\eta}_k\bar{\eta}_i - \eta_k\eta_i - A_{ki}). \tag{3. 21}$$

Transvecting (3. 21) with ϕ^{kj},

$$a(1-\alpha)(\phi_i^j + \phi^{kj}A_{ki})\bar{\eta}_j = -b\phi^{kj}A_{ki}\bar{\eta}_j \tag{3. 22}$$

holds good. By these identities we get

$$\{b-(1-\alpha)(1-a)\}\phi_i^j\bar{\eta}_j = 0. \tag{3. 23}$$

If $b=(1-\alpha)(1-a)$, our theorem reduces to Lemma 2 and Proposition 3. If $\phi_i^j\bar{\eta}_j=0$, μ is a contact transformation. We can put $\bar{\eta}_j=\sigma\eta_j$. Then by (3.13) and (3.14),

$$\bar{g}_{ji} = c_1 g_{ji} + c_2\eta_j\eta_i$$

holds good, where $c_1=b/\{a-(n-1)/\alpha\}$ and $c_2=b(1-\sigma^2)/\{a-(n-1)/\alpha\}$. That is,

$$\bar{g}^{ji} = (1/c_1)g^{ji} - (c_2/c_1(c_1+c_2))\,\xi^j\xi^i$$

is true. If $a-(n-1)/\alpha=0$, our theorem reduces to Lemma 2. Thus, we get $\bar{\xi}^j=\sigma'\xi^j$ and hence σ is a constant. Then by Proposition 2 we know that Theorem 4 is true. Q.E.D.

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