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SOME NOTES ON THE GROUP OF AUTOMORPHISMS OF CONTACT AND SYMPLECTIC STRUCTURES

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Introduction. In this paper, we shall prove that the group of automorphisms of a contact or a symplectic structure defined on a compact manifold M^n acts transitively on it. For this purpose, we first notice that we can find a linear mapping h from the linear space of differentiable functions $\mathfrak{F}$ defined on M^n to that of infinitesimal automorphisms $\boldsymbol{L}$ of these structures; these mappings were introduced by J.W.Gray and P.Libermann in the case of a contact structure and a symplectic structure respectively, and many properties of h were studied by them, (cf. [3][2)] and [4]), but in this paper, we only need the fact that for any differentiable function ρ over M^n, $h(\rho)$ gives an infinitesimal automorphism of these structures. Next, we consider n functions defined over M^n which give a canonical coordinate system around a point $P \in M^n$ associated with such structures, and making use of these functions and the mapping h, we shall prove our theorems.

1. The transitivity of the group of automorphisms of a contact structure. Let $M^n (n=2m+1)$ be a differentiable manifold with a contact structure defined by a 1-form η, i.e., let M^n admit a 1-form η satisfying the relation

$$\eta \wedge \overbrace{d\eta \wedge \cdots \wedge d\eta}^{m} \neq 0, \tag{1.1}$$

where $d\eta$ and $\wedge$ mean the exterior derivative of η and exterior product respectively. Then, we can find a uniquely determined vector field ξ defined over M^n which satisfies the relations

$$i(\xi)\eta = 1 \text{ and } i(\xi)d\eta = 0, \tag{1.2}$$

1) The material of this work is a section of thesis for the degree of Doctor of Science in Tôhoku University, 1965.
2) Numbers in brackets refer to the bibliography at the end of the paper.

where $i(X)\omega$ means an interior product of a form ω by a vector X.

Now, a diffeomorphism $f: M^n \rightarrow M^n$ is said to be a contact transformation if it satisfies

$$f^*\eta=\rho\eta,$$

where f^* is the endomorphism of the ring of differential forms over M^n induced by f and ρ is a function over M^n which does not vanish at any point of M^n. Especially, if $\rho\equiv 1$, i.e., if f satisfies

$$f^*\eta=\eta,$$

then f is said to be a strict contact transformation. It is clear that the set of all (strict) contact transformations over M^n constitutes a group under the natural rule of composition. In order to study such a group of (strict) contact transformations, we start with infinitesimal (strict) contact transformations.

A vector field X over M^n is said to be an infinitesimal contact transformation if it satisfies

$$\mathfrak{L}(X)\eta=\sigma\eta,$$

where $\mathfrak{L}(X)$ denotes the Lie differentiation with respect to X and σ is a function defined over M^n. Especially, if σ vanishes identically, i.e., if X satisfies

$$\mathfrak{L}(X)\eta\equiv 0,$$

X is said to be an infinitesimal strict contact transformation.

Now, we shall give the definition of the mapping h: $\mathfrak{F}\rightarrow \boldsymbol{L}$.

Let D be a $2m$-dimensional distribution $D: P\rightarrow D_P$ defined by

$$D_P=\{X\in T_P(M^n);\ i(X)\eta=0\},$$

and let D^* be a $2m$-dimensional codistribution $D^*: P\rightarrow D_P^*$ defined by

$$D_P^*=\{\theta\in T_P^*(M^n);\ i(\xi)\theta=0\},$$

where $T_P(M^n)$ and $T_P^*(M^n)$ denote the tangent and cotangent vector space of M^n at $P\in M^n$ respectively. Next, if we consider a linear mapping α from the linear space of vector fields over M^n to that of 1-forms over M^n defined by

$$\alpha(X)=i(X)d\eta,$$

then, making use of (1.1) and (1.2), we can easily see that α gives a one-to-one isomorphism from the linear space of vector fields which belong to the distribution D to that of 1-forms which belong to the codistribution D^*. Let β be the inverse mapping of $\alpha \mid D$ and let ρ be a differentiable function over M^n. Then, we have

$$i(\xi)(d\rho-(\xi\rho)\eta)=\xi\rho-\xi\rho=0,$$

which shows that the 1-form $d\rho-(\xi\rho)\eta$ belongs to the codistribution D^*. Hence, we can define a vector field $\gamma(\rho)$ by the relation

$$\gamma(\rho)=\beta(d\rho-(\xi\rho)\eta).$$

From this definition, we get the fact that the relations

$$i(\gamma(\rho))\eta=0 \text{ and } i(\gamma(\rho))d\eta=d\rho-(\xi\rho)\eta$$

hold good. So we obtain

$$\begin{aligned}\pounds(\gamma(\rho))\eta&=i(\gamma(\rho))d\eta+d(i(\gamma(\rho))\eta)\\&=d\rho-(\xi\rho)\eta.\end{aligned}$$

Making use of this and the relation

$$\pounds(\rho\xi)\eta=i(\rho\xi)d\eta+d(i(\rho\xi)\eta)=d\rho,$$

we see that if we define a mapping h from the linear space of differentiable functions over M^n to that of vector fields over M^n by

$$h(\rho)=\rho\xi-\gamma(\rho), \tag{1.3}$$

we have

$$\pounds(h(\rho))\eta=(\xi\rho)\eta.$$

Therefore, we get the following

THEOREM 1.1. *Let ρ be a differentiable function over M^n. Then the vector field $h(\rho)$ defined by* (1.3) *gives an infinitesimal contact transformation over M^n. Moreover, if ρ satisfies the relation*

$$\xi\rho=0, \tag{1.4}$$

$h(\rho)$ gives an infinitesimal strict contact transformation.

Next, we shall prove that if M^n is a compact manifold with a (regular) contact structure, then for any two points P and Q of M^n, we can always find a (strict) contact transformation of M^n which sends P to Q. By virtue of E. Cartan's result (Cf. [2]), for any point P of M^n, we can always find a local coordinate system $(x^\lambda, y^\lambda, z)(\lambda=1,\cdots,m)$ around P with respect to which the contact form η can be expressed as follows:

$$\eta=dz-\sum_\lambda y^\lambda dx^\lambda.$$

On the other hand, for any differentiable function u defined on an open set containing P, we can always find a differentiable function defined over M^n which coincides with u in a certain neighborhood of P. Therefore, we have the following

LEMMA 1. *Let M^n be a manifold with a contact structure defined by η. Then, for any point P of M^n, we can always find an open neighborhood U of P and $(n=2m+1)$ functions $x^1,\cdots,x^m, y^1,\cdots,y^m, z$ which satisfy the following conditions*

(1) *In U, the contact form η is expressed as*

$$\eta=dz-\sum_\lambda y^\lambda dx^\lambda.$$

(2) *The set of functions $(x^\lambda, y^\lambda, z)$ defines a diffeomorphism from U onto an open set V in $(2m+1)$-dimensional Euclidean space E^{2m+1} defined by*

$$V=\{(u^1,\cdots,u^{2m+1})\in E^{2m+1};\ |u^i|<\varepsilon \text{ for all } i\},$$

where ε is a positive real number smaller than $\frac{1}{m}$, and

$$x^\lambda(P)=0,\quad y^\lambda(P)=0,\quad z(P)=0$$

hold good.

We call such a coordinate neighborhood U a *canonical coordinate neighborhood* and such a local coordinate system a *canonical coordinate system* around P. Now, we shall prove the following

THEOREM 1.2. *Let M^n be a compact manifold with a contact structure defined by η, and U be a canonical coordinate neighborhood around a point P of M^n. Then, for any two points Q and Q' of U, we can always find a contact transformation of M^n which sends Q to Q'.*

PROOF. It is evident that we only need to show the existence of a contact transformation of M^n which sends P to Q. First, we notice the fact that since M^n is compact, any infinitesimal (strict) contact transformation generates a global 1-parameter group of (strict) contact transformations. We denote the 1-parameter group of strict contact transformations generated by $\xi=h(1)$ by f_t.

Next, we consider the 1-parameter group of contact transformations generated by an infinitesimal contact transformation $h(\rho)$ for the function ρ defined by

$$\rho=\sum \alpha_\lambda x^\lambda+\sum\beta_\lambda y^\lambda+\gamma, \tag{1. 3}$$

where α_λ, β_λ and γ are constant. Then, since the relations

$$\xi=\frac{\partial}{\partial z} \quad \text{and} \quad d\eta=\sum_\lambda dx^\lambda\wedge dy^\lambda$$

hold with respect to this coordinate system in U, we have

$$i\left(\frac{\partial}{\partial x^\lambda}\right)d\eta=dy^\lambda,\ i\left(\frac{\partial}{\partial y^\lambda}\right)d\eta=-dx^\lambda,\ i\left(\frac{\partial}{\partial z}\right)d\eta=0,$$

and

$$i\left(\frac{\partial}{\partial x^\lambda}\right)\eta=-y^\lambda,\ i\left(\frac{\partial}{\partial y^\lambda}\right)\eta=0,\ i\left(\frac{\partial}{\partial z}\right)\eta=1.$$

So, we get

$$\gamma(x^\lambda)=\beta(dx^\lambda-\frac{\partial x^\lambda}{\partial z}\eta)=\beta(dx^\lambda)=-\frac{\partial}{\partial y^\lambda},$$

$$\gamma(y^\lambda)=\beta(dy^\lambda-\frac{\partial y^\lambda}{\partial z}\eta)=\beta(dy^\lambda)=\frac{\partial}{\partial x^\lambda}+y^\lambda\frac{\partial}{\partial z},$$

and

$$\gamma(1)=0.$$

Therefore, in the canonical coordinate neighborhood U, the vector field $h(\rho)$ is given as follows:

$$\begin{aligned} h(\rho)&=\rho\xi-\gamma(\rho)\\ &=\left(\sum\alpha_\lambda x^\lambda+\sum\beta_\lambda y^\lambda+\gamma\right)\frac{\partial}{\partial z}-\left(-\sum\alpha_\lambda\frac{\partial}{\partial y^\lambda}+\sum\beta_\lambda\frac{\partial}{\partial x^\lambda}+(\sum\beta_\lambda y^\lambda)\frac{\partial}{\partial z}\right)\\ &=-\sum\beta_\lambda\frac{\partial}{\partial x^\lambda}+\sum\alpha_\lambda\frac{\partial}{\partial y^\lambda}+(\sum\alpha_\lambda x^\lambda+\gamma)\frac{\partial}{\partial z}. \end{aligned}$$

So, the trajectory of $h(\rho)$ in U starting from a point of U of coordinates $(x_0^\lambda, y_0^\lambda, z_0)$ is given by

$$
(1.4)\qquad
\begin{aligned}
x^\lambda &= x_0^\lambda - \beta_\lambda t,\\
y^\lambda &= y_0^\lambda + \alpha_\lambda t,\\
z &= z_0 + (\alpha_\lambda x_0^\lambda + \gamma)t - \frac{\sum_\lambda \alpha_\lambda \beta_\lambda}{2} t^2.
\end{aligned}
$$

Now, suppose the coordinates of Q with respect to the canonical coordinate system to be $(a^\lambda, b^\lambda, c)$. If we put

$$a = -\frac{1}{2}\sum_\lambda a^\lambda b^\lambda,$$

then we have

$$|a| = \frac{1}{2}\sum_\lambda |a^\lambda|\,|b^\lambda| < \frac{1}{2} m\varepsilon^2 < \frac{\varepsilon}{2}.$$

Next, we take points R and S in U whose coordinates are $(0, 0, a)$ and $(a^\lambda, b^\lambda, 0)$ respectively, and we take $b^\lambda, -a^\lambda$ and 0 as $\alpha_\lambda, \beta_\lambda$ and γ in (1.3) respectively. Then, by virtue of (1.4), the trajectory of $h(\rho)$ starting from R is given by

$$
(1.5)\qquad
\begin{aligned}
x^\lambda(t) &= a^\lambda t,\\
y^\lambda(t) &= b^\lambda t,\\
z(t) &= a(1-t^2),
\end{aligned}
$$

for any t such that

$$|x^\lambda(t')| < \varepsilon,\ |y^\lambda(t')| < \varepsilon,\ |z(t')| < \varepsilon$$

hold for every t' not larger than t. So, for $0 \leqq t \leqq 1$, the trajectory of $h(\rho)$ is expressed by (1.5). Therefore, if we denote the 1-parameter group of contact transformations generated by $h(\rho)$ by g_t, g_1 sends R to S. Hence, the contact transformation $f_c \cdot g_1 \cdot f_a$ sends P to Q which proves our assertion. Q.E.D.

Next, let P and Q be arbitrary two points of M^n. And let C be a curve in M^n from P to Q. Then C can be covered by a finite number of canonical coordinate neighborhoods $U_1, U_2, \cdots, U_k$. If we take a sequence of points P_0,

$P_1, \cdots, P_k$ such that $P_0=P, P_k=Q$ and $P_\alpha \in U_\alpha \cap U_{\alpha+1}$ $(\alpha=1, \cdots, k-1)$, then by virtue of Theorem 1.2, we can find a contact transformation f_α which sends $P_{\alpha-1}$ to P_α for each α. So, if we set

$$f=f_k \cdot f_{k-1} \cdots f_1,$$

then f is a contact transformation which sends P to Q.

Therefore, we get the following

THEOREM 1.3. *Let M^n be a compact manifold with a contact structure defined by η. Then, for any two points P and Q of M^n, we can always find a contact transformation of M^n which sends P to Q.*

Next, we consider a compact manifold with a regular contact structure. To begin with, we shall prove the following

LEMMA 2. *Let M^n be a compact manifold, and let ξ be a regular vector field over M^n. If ρ is a differentiable function defined in a certain neighborhood U of a point P of M^n, and ρ satisfies the condition*

$$\xi\rho=0.$$

Then, we can find a differentiable function ρ' defined all over M^n which satisfies the following conditions:

(1) $\xi\rho'=0$.

(2) *On a certain neighborhood V of P, ρ' coincides with ρ.*

PROOF. Since M^n is compact and ξ is regular, the quotient space $M^n/\{\xi\} =B$ is a differentiable manifold, and M^n is a bundle space of a differentiable fiber bundle over B whose fibers are the trajectories of ξ (Cf. [5]). We denote the projection from M^n to B by π. Since $\xi\rho=0$, the function ρ is constant along the fibres in a certain neighborhood U' of P. So, we can find a function σ defined on an open set V' containing $\pi(P)$ such that

$$\rho=\sigma\circ\pi$$

in U'. Now, we can find a function σ' globally defined over B which coincides with σ on a certain neighborhood V' of $\pi(P)$. Then, if we set

$$\rho'=\sigma'\circ\pi,$$

the function ρ' satisfies the conditions (1) and (2). Q.E.D.

By virtue of Lemma 2, on a compact manifold with a regular contact structure, we may suppose the functions x^λ and y^λ in Lemma 1 satisfy the condition

$$\xi x^\lambda=0,\ \xi y^\lambda=0 \qquad \text{over } M^n.$$

So, the function ρ defined by (1.3) satisfies the condition

$$\xi\rho=0,$$

which shows that $h(\rho)$ is an infinitesimal strict contact transformation. Therefore, the contact transformations f_t and g_t which we used in the proof of Theorem 1.3 are strict contact transformations. Hence, we get the following

THEOREM 1.4. *Let M^n be a compact manifold with a regular contact structure. Then, for any two points P and Q of M^n, we can always find a strict contact transformation of M^n which sends P to Q.*

2. The transitivity of the group of automorphisms of a sympletic structure. Let $M^n(n=2m)$ be a differentiable manifold with a symplectic structure defined by a 2-form Ω, i.e., M^n admits a 2-form Ω satisfying the relations

$$\overbrace{\Omega\wedge\cdots\wedge\Omega}^{m}\neq 0 \qquad \text{and} \qquad d\Omega=0. \tag{2.1}$$

And, as in the previous section, a diffeomorphism f of M^n is said to be a symplectic transformation if it satisfies

$$f^*\Omega=\Omega,$$

and a vector field X is said to be an infinitesimal symplectic transformation if it satisfies

$$\pounds(X)\Omega=0.$$

If we define a mapping α from the linear space of vector fields over M^n to that of 1-forms over M^n by the relation

$$\alpha(X)=i(X)\Omega,$$

then, by virtue of (2.1), α gives a one-to-one isomorphism of these two linear spaces. Now, let β be the inverse mapping of α and let ρ be a differentiable function over M^n. If we define a mapping h by

$$h(\rho)=\beta(d\rho), \tag{2.2}$$

then we have

$$i(h(\rho))\Omega = d\rho.$$

Therefore, we get

$$\pounds(h(\rho))\Omega = i(h(\rho))d\Omega + d(i(h(\rho))\Omega) = 0.$$

Hence, we get the following

THEOREM 2.1. *Let ρ be a differentiable function over M^n. Then the vector field $h(\rho)$ defined by* (2.2) *gives an infinitesimal symplectic transformation over M^n.*

Now, in the same way as Lemma 1, we can verify the following

LEMMA 3. *Let M^n be a manifold with a symplectic structure defined by Ω. Then, for any point P of M^n, we can always find an open neighborhood U of P and $2m$ functions $x^1, \cdots, x^m, y^1, \cdots, y^m$ which satisfy the following conditions*:

(1) *In U, the symplectic form Ω is expressed as*

$$\Omega = \sum_{\lambda} dx^{\lambda} \wedge dy^{\lambda}.$$

(2) *The set of functions $(x^{\lambda}, y^{\lambda})$ defines a diffeomorphism from U onto an open set V in $2m$-dimensional Euclidean space E^{2m} defined by*

$$V = \{(u^1, \cdots, u^{2m}) \in E^{2m};\ |u^i| < \varepsilon \text{ for all } i\},$$

and

$$x^{\lambda}(P) = 0,\ y^{\lambda}(P) = 0$$

hold good.

We call such a coordinate neighborhood U a *canonical coordinate neighborhood*, and such a local coorinate system a *canonical coordinate system* of the symplectic structure.

Next, suppose M^n to be compact and let Q be a point in U whose canonical coordinate is $(a^{\lambda}, b^{\lambda})$. If we consider a differentiable function ρ defined by

$$\rho = -\sum b^{\lambda}x^{\lambda} + \sum a^{\lambda}y^{\lambda},$$

then the 1-parameter group of symplectic transformations generated by the infinitesimal symplectic transformation $h(\rho)$ sends P to Q for $t=1$.

Therefore we get the following

THEOREM 2.2. *Let M^n be a compact manifold with a symplectic structure defined by Ω, and let U be a canonical coordinate neighborhood around a point P of M^n. Then, for any two points Q and Q' of U, we can always find a symplectic transformation of M^n which sends Q to Q'.*

Making use of this theorem in the same way as in §1, we get the following

THEOREM 2.3. *Let M^n be a compact manifold with a symplectic structure. Then, for any two points P and Q of M^n, we can always find a symplectic transformation of M^n which sends P to Q.*

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