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ON THE TENSOR PRODUCTS OF C^*-ALGEBERAS

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T. Turumaru [6] introduced a tensor product $A_1 \hat{\otimes}_\alpha A_2$ of two C^*-algebras A_1 and A_2, which is the C^*-algebra obtained as the completion of the *-algebraic tensor product $A_1 \odot A_2$ of A_1 and A_2 with respect to the α-norm $\| \ \|_\alpha$. As Wulfsohn [7] established, the α-norm has the property:

$$\left\| \sum_k x_{1,k} \otimes x_{2,k} \right\|_\alpha = \left\| \sum_k \pi(x_{1,k}) \otimes \pi_2(x_{2,k}) \right\|, \ x_{1,k} \in A_1, x_{2,k} \in A_2$$

for every faithful representations π_1 of A_1 and π_2 of A_2. It was observed in [5] that the α-norm is not necessarily the unique compatible norm in $A_1 \odot A_2$ and that it is the *least* one among the all compatible norms. On the other hand, A. Guichardet [4] gave, with the corresponding tensor product, the *greatest* compatible norm $\| \ \|_\nu$ in $A_1 \odot A_2$ the ν-norm. These arguments will bring forward many interesting problems on the relations between compatible norms in $A_1 \odot A_2$ and corresponding tensor products, and some of them will be considered in this paper. We shall discuss on B^*-norms in $A_1 \odot A_2$ in Theorems 1 and 2, and on the enveloping C^*-algebras of *-Banach algebras in Theorem 3. The auther wishes to express his hearty thanks to Prof. M. Fukamiya and Dr. M. Takesaki for their many valuable suggestions.

Let A_1, A_2 be *-Banach algebras,[1] $A_1 \odot A_2$ the *-algebraic tensor product of them. For norms $\| \ \|_{\beta'}$, $\| \ \|_\beta$, in $A_1 \odot A_2$, we say that $\| \ \|_\beta$ is smaller than $\| \ \|_{\beta'}$ in symbols $\| \ \|_{\beta'} \leqq \| \ \|_\beta$ if $\|u\|_\beta \leqq \|u\|_{\beta'}$ for all $u \in A_1 \odot A_2$. Of course the relation "$\leqq$" has the partial ordering property. A norm $\| \ \|_\beta$ in $A_1 \odot A_2$. is said to be compatible if it satisfies the condition

$$\|x_1 \otimes x_2\|_\beta \leqq \|x_1\| \|x_2\|, \ x_1 \in A_1, \ x_2 \in A_2. \text{ (cf. [5])}$$

Now let A_1, A_2 be C^*-algebras. The C^*-algebra $A_1 \hat{\otimes}_\beta A_2$ obtained as the completion of $A_1 \odot A_2$ with respect to a compatible B^*-norm $\| \ \|_\beta$ in $A_1 \odot A_2$[2] is

1) By a *-Banach algebra we mean any Banach algebra with an isometric involution.

called the tensor product of A_1 and A_2 with respect to $\| \|_\beta$. The α-norm in $A_1 \odot A_2$ is defined by the formula

$$\|u\|_\alpha = \|\pi_1 \otimes \pi_2(u)\|, \ u \in A_1 \odot A_2,$$

where π_1, π_2 are any fixed faithful representations of A_1, A_2, respectively.[3),4)] The value $\|u\|_\alpha$ of course does not depend on the choice of π_1 and π_2. The ν-norm is defined by the formula

$$\|u\|_\nu = \sup_\pi \|\pi(u)\|, \ u \in A_1 \odot A_2,$$

where π runs over the set of all representations of $A_1 \odot A_2$ such that

$$(*) \qquad \|\pi(x_1 \otimes x_2)\| \leqq \|x_1\| \|x_2\|, \ x_1 \in A_1, x_2 \in A_2.$$[5)]

For a *-Banach algebra A having at least one faithful representation, the enveloping C^*-algebra $C^*(A)$ of A means the C^*-algebra obtained as the completion of A with respect to the B^*-norm in A

$$\|x\|_* = \sup_\pi \|\pi(x)\|, \ x \in A,$$

where π denotes any representation of A. This notion is of course a generalization of that of the group C^*-algebra $C^*(G)$ of a locally compact group G, the enveloping C^*-algebra of $L^1(G)$.

THEOREM 1. *Let* A_1, A_2 be **-Banach algebras. Then each B^*-norm in* $A_1 \odot A_2$ is compatible.

To prove this we prepare

LEMMA 1. *Let $A_{1,1}$, $A_{2,1}$ be the *-Banach algebras obtained as the adjunctions of the identities to *-Banach algebras A_1, A_2, respectively. Then each B^*-norm $\| \|_\beta$ in $A_1 \odot A_2$ can be extented to a B^*-norm in $A_{1,1} \odot A_{2,1}$.*

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2) B^*-norm means any muliplicative norm $\| \|_\beta$ satisfying the condition $\|u^*u\|_\beta = \|u\|_\beta^2$ for all u.

3) We mean by a representation of a *-algebra any *-homomorphism into the algebra of all bounded linear operators on some Hilbert space.

4) In general, for representations π_1 of A_1 and π_2 of A_2, $\pi_1 \otimes \pi_1$ means the representation of $A_1 \odot A_2$ on the tensor product Hilbert space of representation spaces of π_1 and π_2 defined by the formula

$$\pi_1 \otimes \pi_2(u) = \textstyle\sum_k \pi_1(x_{1,k}) \otimes \pi_2(x_{2,k}), \ u = \sum_k x_{1,k} \otimes x_{2,k} \in A_1 \odot A_2.$$

5) This definitin of the ν-norm will be simplified in Corollary of Theorem 1 by omitting the condition (*) for π.

PROOF. For any $v \in A_{1,1} \odot A_{2,1}$, we put

$$||| v |||_\beta = \sup_u \|vu\|_\beta,$$

where u runs over $A_1 \odot A_2$ with $\|u\|_\beta \leqq 1$. This is a multiplicative norm in $A_{1,1} \odot A_{2,1}$ and an extension of $\| \ \|_\beta$. Moreover it is a B^*-norm. In fact, for any positive number $\varepsilon < 1$, there exists an element $u \in A_1 \odot A_2$ with $\|u\|_\beta \leqq 1$ such that $\varepsilon ||| v |||_\beta \leqq \|vu\|_\beta$. Then

$$\varepsilon^2 ||| v ||| \leqq \|uv\|_\beta^2 \leqq \|u^* v^* vu\|_\beta \leqq \|v^* vu\|_\beta \leqq ||| v^* v |||_\beta.$$

Since ε is arbitrary, we have

$$||| v |||_\beta^2 \leqq ||| v^* v |||_\beta;$$

and the opposite inequality is obvious. q.e.d.

PROOF OF THEOREM 1. We can assume that A_1 and A_2 have identities which are denoted by 1's. Let $\| \ \|_\beta$ be a B^*-norm in $A_1 \odot A_2$. The mapping $A_1 \ni x_1 \to x_1 \otimes 1 \in A_1 \hat{\otimes}_\beta A_2$ is a homomorphism (in fact an isomorphism) of A_1 into $A_1 \hat{\otimes}_\beta A_2$, hence

$$\|x_1 \otimes 1\|_\beta \leqq \|x_1\|, \ x_1 \in A_1.$$

Analogously we have

$$\|1 \otimes x_2\|_\beta \leqq \|x_2\|, \ x_2 \in A_2,$$

and therefore,

$$\|x_1 \otimes x_2\|_\beta = \|(x_1 \otimes 1)(1 \otimes x)_2\|_\beta \leqq \|x_1\| \|x_2\|, \ x_1 \in A_1, \ x_2 \in A_2,$$

which completes the proof.

COROLLARY. *Let A_1, A_2 be C^*-algebras, then*

$$\|u\|_\nu = \sup_\pi \|\pi(u)\|, \ u \in A_1 \odot A_2,$$

where π runs over the set of all representations of $A_1 \odot A_2$.

PROOF. For any $u \in A_1 \odot A_2, \|u\|_\nu \geqq \sup_\rho \|\rho(u)\|$, where ρ runs over the set of all faithful representations of $A_1 \odot A_2$ which satisfiy (*). Moreover the right-hand side is equal to $\sup_\sigma \|\sigma(u)\|$, where σ runs over the set of all faithful representations, because by Theorem 1 we know that any faithful representation of $A_1 \odot A_2$ necessarily satisfies (*). Then, τ denoting the restriction on $A_1 \odot A_2$ of a faithful representation of $A_1 \hat{\otimes}_\nu A_2$, we have

$$\|u\|_\nu \geqq \sup_\rho \|\rho(u)\| = \sup_\sigma \|\sigma(u)\| \geqq \|\tau(u)\| = \|u\|_\nu,$$

and also the desired formula. q.e.d.

THEOREM 2. *Let A_1, A_2 be C*-algebras, then the set of all B*-norms in $A_1 \odot A_2$ becomes a complete lattice under the ordering "$\leqq$" with the least ellement $\| \; \|_\alpha$ and the greatest element $\| \; \|_\nu$.*

PROOF. For a given set N of B^*-norms in $A_1 \odot A_2$, we put

$$\|u\|_{\beta_0} = \sup_\pi \|\pi(u)\|, u \in A_1 \odot A_2,$$

where π runs over the set of all representations of $A_1 \odot A_2$ which are continuous with respect to every $\| \; \|_\beta$ in N. Here, remark that this set contains every representations of the product type $\pi_1 \otimes \pi_2$. $\| \; \|_{\beta_0}$ is a B^*-norm and is smaller than each $\| \; \|_\beta \in N$. And for every B^*-norm $\| \; \|_\beta'$ in $A_1 \odot A_2$ which is smaller than each $\| \; \|_\beta \in N$, $\| \; \|_{\beta'} \leqq \| \; \|_{\beta_0}$. Hence $\| \; \|_{\beta_0}$ is the infimum of N. Also we put

$$\|u\|_{\beta_1} = \sup_{\| \; \|_\beta \in N} \|u\|_\beta, u \in A_1 \odot A_1,$$

then this is not only a B^*-norm in $A_1 \odot A_2$ but also the supremum of N. q.e.d.

Theorem 2 has an interpretation. For each $u \in A_1 \hat{\otimes}_\nu A_2$, choosing a sequence $\{u_n\}$ in $A_1 \odot A_2$ converging to u with respect to $\| \; \|_\nu$, we can define well a homomorphism π_β of $A_1 \hat{\otimes}_\nu A_2$ onto $A_1 \hat{\otimes}_\beta A_2$ by

$$\pi_\beta(u) = \| \; \|_\beta - \lim_n u_n, u \in A_1 \hat{\otimes}_\nu A_2.$$

The kernal $\pi_\beta^{-1}(0) = I_\beta$ of π_β is a closed two-sided ideal in $A_1 \hat{\otimes}_\nu A_2$ with $I_\beta \cap A_1 \odot A_2 = \{0\}$. We consider the correspondence $\| \; \|_\beta \to I_\beta$ of the set of all B^*-norms in $A_1 \odot A_2$ onto the set of all closed two-sided ideals in $A_1 \hat{\otimes}_\nu A_2$ intersecting $A_1 \odot A_2$ only at 0. This becomes one-to-one because $A_1 \hat{\otimes}_\nu A_2 / I_\beta$ is isomorphic to $A_1 \hat{\otimes}_\beta A_2$, and moreover order-preversing. Now Theorem 2 makes us state

COROLLARY. *The set of all closed two-sided ideals in $A_1 \hat{\otimes}_\nu A_2$ intersecting*

$A_1 \odot A_2$ *only at* 0 *becomes a complete lattice under the inclusion ordering with the least element* $\{0\} = I_\nu$ *and the greatest element* I_α.

The following lemma is essentially due to Guichardet [3].

LEMMA 2. *Let* A_1, A_2 *be *-Banach algebras with approximating identities. For any representation* π *of* $A_1 \odot A_2$ *which is continuous with respect to the* γ*-norm* $\| \ \|_\gamma$ *in* $A_1 \odot A_2$, *there exist representations* π^1 *of* A_1 *and* π^2 *of* A_2 *such that*

$$\pi(x_1 \otimes x_2) = \pi^1(x_1)\pi^2(x_2) = \pi^2(x_2)\pi^1(x_1),\ x_1 \in A_1,\ x_2 \in A_2.$$

PROOF. Just as in the proof of Proposition 1 of [3], we put

$$\pi^1(x_1) = \text{strong-lim}_\eta \pi(x_1 \otimes e_{2,\eta}), x_1 \in A_1,$$

$$\pi^2(x_2) = \text{strong-lim}_\xi \pi(e_{1,\xi} \otimes x_2),\ x_2 \in A_2,$$

$\{e_{1,\xi}\}$, $\{e_{2,\eta}\}$ being approximating identities of A_1, A_2, respectively. Then π^1, π^2 are required representations. q.e.d.

THEOREM 3. *Let* A_1, A_2 *be *-Banach algebras with approximating identities each of which has at least one faithful representation. Then the enveloping* C^**-algebra* $C^*(A_1 \hat{\otimes}_\gamma A_2)$, $A_1 \hat{\otimes}_\gamma A_2$ *being the projective tensor product of* A_1 *and* A_2, *is isomorphic to the tensor product* $C^*(A_1) \hat{\otimes}_\nu C^*(A_2)$ *of the enveloping* C^**-algebras* $C^*(A_1)$ *and* $C^*(A_2)$.

PROOF. Under the natural identifications, we may consider that the *-algebra $A_1 \odot A_2$ is contained both in $C^*(A_1 \hat{\otimes}_\gamma A_2)$ and in $C^*(A_1) \hat{\otimes}_\nu C^*(A_2)$. We shall prove that $\| \ \|_\nu = \| \ \|_*$[6] in $A_1 \odot A_2$. Since $\| \ \|_\nu$ in $A_1 \odot A_2$ is compatible by Theorem 1, $\| \ \|_\nu \leqq \| \ \|_\gamma$, and the restriction π on $A_1 \odot A_2$ of a faithful repesentation of $C^*(A_1) \hat{\otimes}_\gamma C^*(A_2)$ can be extended to a representation of $A_1 \hat{\otimes}_\gamma D_2$. Then,

$$\|u\|_\nu = \|\pi(u)\| \leqq \sup_\rho \|\rho(u)\| = \|u\|_*,\ u \in A_1 \odot A_2,$$

where ρ denotes any representation of $A_1 \otimes_\gamma A_2$. Next we see the opposite inequality. Since the restriction σ on $A_1 \odot A_2$ of a faithful representation of $C^*(A_1 \hat{\otimes}_\gamma A_2)$ is continuous with respect to $\| \ \|_\gamma$, there exist representations σ^1 of A_1 and σ^2 of A_2 such that $\sigma\ (x_1 \otimes x_2) = \sigma^1(x_1)\sigma^2(x_2)$, $x_1 \in A_1, x_2 \in A_2$. We can extend σ^1, σ^2 to representations σ_1, σ_2 of $C^*(A_1), C^*(A_2)$, respectively. Then

$$\tau(u) = \sum_k \sigma_1(x_{1,\,k})\sigma_2(x_{2,\,k}) \text{ for } u = \sum_k x_{1,\,k} \otimes x_{2,k} \in C^*(A_1) \odot C^*(A_2)$$

is a representation of $C^*(A_1) \odot C^*(A_2)$ and an extension of σ. Thus

6) We sall denote the norm in the enveloping C*-algelna by $\| \ \|_*$.

$$\|u\|_*=\|\sigma(u)\|=\|\tau(u)\|\leqq\|u\|_\nu,\ u\in A_1\odot A_2.$$

To complete the proof, it is sufficient to see that $A_1\odot A_2$ is dense in $C^*(A_1\hat{\otimes}_\gamma A_2)$ with respect to $\|\ \|_*$ and also that it is dense in $C^*(A_1)\hat{\otimes}_\nu C^*(A_2)$ with respect to $\|\ \|_\nu$. Since $A_1\hat{\otimes}_\gamma A_2$ is dense in $C^*(A_1\hat{\otimes}_\gamma A_2)$ with respect to $\|\ \|_*$ and $A_1\odot A_2$ is dense in $A_1\hat{\otimes}_\gamma A_2$ with respect to $\|\ \|_\gamma$ which is greater than $\|\ \|_*$, $A_1\odot A_2$ is dense in $C^*(A_1\hat{\otimes}_\gamma A_2)$ with respect to $\|\ \|_*$. Let $u\in C^*(A_1)\hat{\otimes}_\nu C^*(A_2)$, then there is a is a sequence $\{x_{1,k}^n\}$ in $C^*(A_1)$ and $\{x_{2,k}^n\}$ in $C^*(A_2)$ with $x_{2,k}^n\neq 0$ such as

$$\|u-\sum_{k=1}^{k_n}x_{1,k}^n\otimes x_{2,k}^n\|_\nu\to 0,\ n\to\infty.$$

And there exist sequences $\{y_{1,k}^n\}$ in A_1 with $y_{1,k}^n\neq 0$ and $\{y_{2,k}^n\}$ in A_2 such that

$$\|x_{1,k}^n-y_{1,k}^n\|_*\leqq\frac{1}{nk_n\max_k\|x_{2,k}^n\|_*},$$

$$\|x_{2,k}^n-y_{2,k}^n\|_*\leqq\frac{1}{nk_n\max_k\|y_{1,k}^n\|_*},\ \mathrm{n}=1,2,\cdots.$$

Then,

$$\|u-\sum_k y_{1,k}^n\otimes y_{2,k}^n\|_\nu$$

$$\leqq\|u-\sum_k x_{1,k}^n\otimes x_{2,k}^n\|_\nu+\|\sum_k x_{1,k}^n\otimes x_{2,k}^n-\sum_k y_{1,k}^n\otimes y_{2,k}^n\|_\nu$$

$$\leqq\|\cdots\|_\nu+\sum_k\|(x_{1,k}^n-y_{1,k}^n)\otimes x_{2,k}^n\|_\nu+\sum_k\|y_{1,k}^n\otimes(x_{2,k}^n-y_{2,k}^n)\|_\nu$$

$$\leqq\|\cdots\|_\nu+\sum_k\|x_{1,k}^n-y_{1,k}^n\|_*\|x_{2,k}^n\|_*+\sum_k\|y_{1,k}^n\|_*\|x_{2,k}^n-y_{2,k}^n\|_*$$

$$\leqq\|\cdots\|_\nu+\frac{1}{\mathrm{n}}+\frac{1}{\mathrm{n}}\to 0,\ \mathrm{n}\to\infty.\qquad \text{q.e.d.}$$

COROLLARY ([4]). *Let G_1, G_2 be locally compact groups, then $C^*(G_1\times G_2)$, $G_1\times G_2$ being the direct product of G_1 and G_2, is isomorphic to $C^*(G_1)\hat{\otimes}_\nu C^*(G_2)$.*

The proof is obtained immediately via the Grothendieck theorem ([2], Theorem 1 and etc.) which asserts that the projective tensor product $L^1(G_1)\hat{\otimes}_\gamma L(G_2)$ is isomorphic to $L^1(G_1\times G_2)$.

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