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A NOTE ON THE CHOQUET BOUNDARY OF A RESTRICTED FUNCTION ALGEBRA

NOZOMU MOCHIZUKI

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Let A be a function algebra on a compact Hausdorff space Y, i.e., a uniformly closed subalgebra of the algebra of continuous functions $C(Y)$ which separates the points and contains the constants. We denote by X, $M(A)$ and $\partial(A)$ the maximal ideal space, the Choquet boundary and the Silov boundary, respectively. We assume that A is represented on X. The Choquet boundary is characterized by the extreme points of the positive linear functionals of norm 1, so $M(A)$ is independent of the representing space X. A subset E of X is said to be a peak set in X if there exists $f \in A$ such that $|f(p)|=1$ on E, $|f(p)|<1$ on $X-E$. In the relationship between the algebra A and the representing space, a subset which is an intersection of peak sets plays a distinguished role. In connection with this, the following is known ([3]).

THEOREM. *If E is an intersection of peak sets in X, then the restricted algebra $A|E$ is closed in $C(E)$ and $\partial(A|E) \subset \partial(A) \cap E$.*

The object of this note is to prove the following

THEOREM. *Let E be an intersection of peak sets in X, then $M(A|E) = M(A) \cap E$.*

The following is proved in [2].

LEMMA. *$p_0 \in M(A)$ if and only if, for every neighborhood U of p_0 in X, there exists a sequence $\{f_n\}$ in A such that $\|f_n\| \leqq 1$, $|f_n(p_0)| \rightarrow 1$ and $f_n(p) \rightarrow 0$ uniformly for $X-U$.*

PROOF OF THEOREM. First, suppose that $p_0 \in M(A) \cap E$. Let $U \cap E$ be a neighborhood of p_0 in E. There is a sequence $\{f_n\}$ in A satisfying the condition for U. Let $g_n = f_n|E$. $\{g_n\}$ satisfies the condition for $U \cap E$. Thus, $p_0 \in M(A|E)$. Conversely, let $p_0 \in M(A|E)$. We prove that $p_0 \in M(A)$. Let U be an open neighborhood of p_0. Take $\{g_n\}$ in $A|E$ such that $|g_n(p_0)| \rightarrow 1$, $\|g_n\|_E \leqq 1$ and $g_n(E-U) \rightarrow 0$. Replacing g_n by $(1-1/n)g_n$ if necessary, we may assume that

$\|g_n\|_E < 1$. Since $A|E$ is isometric with $A/k(E)$. *) We have $\|g_n\|_E = \|\dot{f}_n\|$ for some $f_n \in A$ such that $\|f_n\| \leqq 1$. Since $g_n = f_n|E$, $|f_n(p_0)| \to 1$ and $\sup_{E-U} |f_n(p)| \to 0$, so

$$|f_{n_k}(p_0)| > 1 - 1/k \quad \text{and} \quad \sup_{E-U} |f_{n_k}(p)| < 1/2k, \quad \text{for some } n_k.$$

There exists an open set G containing $E-U$ such that

$$\sup_G |f_{n_k}(p)| < \sup_{E-U} |f_{n_k}(p)| + 1/2k < 1/k,$$

and by compactness of $E-U$ we can select a finite intersection E_0 of peak sets such that $E_0 - U \subset G$. E_0 is itself a peak set, so a function $g \in A$ peaks on E_0 and, since $G \cup U$ is an open set containing E_0, we have

$$\sup_{(G\cup U)^c} |g^{m_k}(p) f_{n_k}(p)| \leqq \sup_{(G\cup U)^c} |g^{m_k}(p)| < 1/k$$

for a large m_k. Let $h_k = g^{m_k} f_{n_k}$. Since $U^c \subset (G \cup U)^c \cup G$ and $\sup_G |h_k(p)| \leqq \sup_G |f_{n_k}(p)| < 1/k$, we have

$$\sup_{Uc} |h_k(p)| < 1/k.$$

Clearly, $\|h_k\| \leqq 1$ and $|h_k(p_0)| = |f_{n_k}(p_0)| > 1 - 1/k$. Thus, we have $p_0 \in M(A)$.

References

[1] E. BISHOP AND K. DE LEEUW, The representations of linear functionals by measures on sets of extreme points, Ann. Inst. Fourier, 9(1959).

[2] N. MOCHIZUKI, The tensor product of function algebras, Tôhoku Math. Journ., 17(1965).

[3] J. TOMIYAMA, Some remarks on antisymmetric decomposition of function algebras, ibid., 16(1964).

TÔHOKU UNIVERSITY.

*) $k(E)$ denotes the kernel of E and $\dot{f}$ means the equivalence class in $A/k(E)$ which contains f.