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ON PERTURBATION THEORY FOR SEMI-GROUPS OF OPERATORS

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1. In this paper we shall deal with perturbation theory for semi-groups of operators of class (C_0) defined on a Banach space.

Let $\{T(\xi; A); \xi \geqq 0\}$ be a semi-group of operators of class (C_0) and let A be the infinitesimal generator of $\{T(\xi; A); \xi \geqq 0\}$. R.S.Phillips [2] first proved that if B is a bounded linear operator, then $A+B$ generates a semi-group of operators of class (C_0). Later he has introduced the following class $\mathfrak{P}(A)$ of perturbing operators [1].

DEFINITION 1. *A linear operator B is said to belong to the class $\mathfrak{P}(A)$ if*

(a) $D(B)=D(A)$ *and* $BR(\lambda;A)$[1] *is a bounded linear operator for some* λ *in the resolvent set of* A,

(b) $BT(\xi; A)$ *defined on* $D(A)$ *is bounded for all* $\xi > 0$,

(c) $\int_0^1 \|BT(\xi;A)\|_A d\xi < \infty$, *where the subscript A means that the norm is taken relative to the subspace* $D(A)$.

We shall now consider the class $\{B\}$ of operators satisfying the following conditions:

(a′) $D(B) \supset D(A)$ and $BR(\lambda; A)$ is a bounded linear operator for some λ in the resolvent set of A.

(b') There exists a constant $K > 0$ such that

$$\int_0^1 \|BT(\xi; A)x\| d\xi \leqq K\|x\| \qquad \text{for all } x \in D(A).$$

Then we can show that *if A is the infinitesimal generator of a semi-group $\{T(\xi; A); \xi \geqq 0\}$ (or group $\{T(\xi; A); -\infty < \xi < \infty\}$) of class (C_0) and $B \in \{B\}$, then for sufficiently small $|\varepsilon|$, $A+\varepsilon B$ generates a semi-group (or group) of class (C_0).*

It is obvious that if $\{T(\xi; A); -\infty < \xi < \infty\}$ is a group of operators of class (C_0), then each $B \in \mathfrak{P}(A)$ is bounded on $D(A)$.

1) The notations $D(A)$ and $R(\lambda;A)$ denote the domain of operator A and the resolvent of operator A, respectively.

On the other hand, in this group case, it is shown by an example (see section 4) that our class contains unbounded operators. Therefore our class properly includes the class $\mathfrak{P}(A)$. Main results are given in section 3.

2. Let X be Banach space and $\mathfrak{B}(X)$ be the Banach algebra of all bounded linear operators from X into itself.

DEFINITION 2. *$\{T(\xi); \xi \geqq 0\}$ is said to be a semi-group of operators of class (C_0) if*

(i) *$T(\xi) \in \mathfrak{B}(X)$ for each $\xi \geqq 0$,*

(ii) *$T(0)=I$ (the identity) and $T(\xi+\eta)=T(\xi)T(\eta)$ for each $\xi, \eta \geqq 0$,*

(iii) *$\lim_{\xi\to 0+} T(\xi)x=x$ for each $x \in X$.*

The *infinitesimal generator* A of $\{T(\xi); \xi \geqq 0\}$ is defined as the limit in norm

$$\lim_{h\to 0+} \frac{1}{h} [T(h)-I]x=Ax \tag{2.1}$$

whenever this limit exists. And if A is the infinitesimal generator of a semi-group of operators, we denote the corresponding semi-group by $\{T(\xi; A); \xi \geqq 0\}$.

The following theorem is due to R.S.Phillips (see [1] or [2]).

THEOREM I. *Let $\{T(\xi; A); \xi \geqq 0\}$ be a semi-group of class (C_0) and suppose that $B \in \mathfrak{B}(X)$. Then $A+B$ defined on $D(A)$ is again the infinitesimal generator of a semi-group of class (C_0) and*

$$T(\xi; A+B)=\sum_{k=0}^{\infty} S_k(\xi) \qquad \text{for } \xi \geqq 0, \tag{2.2}$$

where $S_0(\xi)=T(\xi; A)$ and $S_k(\xi)x=\int_0^{\xi} T(\xi-\sigma; A)BS_{k-1}(\sigma)x d\sigma$ for $x \in X$ and $k \geqq 1$; the series (2.2) converges absolutely, uniformly with respect to ξ in any compact interval. For each k, $S_k(\xi)$ is strongly continuous for $\xi \geqq 0$.

We also use the following theorem which was given by H.F.Trotter [3].

THEOREM II. *Let $\{T(\xi; A_n); \xi \geqq 0\}_{n=1,2,\dots}$ be a sequence of semi-groups of class (C_0) satisfying the condition*

$$\|T(\xi; A_n)\| \leqq C \exp(\gamma\xi) \qquad \text{for } \xi \geqq 0,$$

where C and γ are constants independent of n and ξ. Suppose that

(i) *$Ax=\lim_{n\to\infty} A_n x$ exists on a dense subset of X,*

(ii) *for some $\lambda > \gamma$, the range $\mathfrak{R}(\lambda-A)$ of $\lambda-A$ is dense in X.*

Then the closure of A is the infinitesimal generator of a semi-group $\{T(\xi; A); \xi \geqq 0\}$ of class (C_0), and $T(\xi; A)x=\lim_{n\to\infty} T(\xi; A_n)x$ for $x \in X$ and $\xi \geqq 0$.

3. We first remark that if $\{T(\xi; A); \xi \geqq 0\}$ is a semi-group of class (C_0), then there exist real constants $M > 0$ and $\omega \geqq 0$ such that

(3.1) $$\|T(\xi; A)\| \leqq M \exp(\omega\xi) \qquad \text{for } \xi \geqq 0.$$

LEMMA 1. *Let $\{T(\xi; A); \xi \geqq 0\}$ be a semi-group of class (C_0) and suppose that $B_n \in \mathfrak{B}(X)$ for each positive integer n.*

If

(3.2) $$\sup_n \int_0^1 \|B_n T(\xi; A)x\| d\xi < \infty \qquad \textit{for each } x \in X,$$

then $r_\lambda = \sup_{\|x\| \leqq 1} r_\lambda(x)$ is finite for each $\lambda > \omega$, where

$$r_\lambda(x) = \sup_n \int_0^\infty e^{-\lambda\xi} \|B_n T(\xi; A)\, x\| d\xi \textit{ for } x \in X.$$

PROOF. Let us put

$$p(x) = \sup_n \int_0^1 \|B_n T(\xi; A)x\| d\xi \qquad \text{for } x \in X.$$

It is easy to see that $p(x)$ has the following properties:

(a) $0 \leqq p(x) < \infty$ for each $x \in X$.

(b) $p(x+y) \leqq p(x) + p(y)$ and $p(\alpha x) = |\alpha| p(x)$ for $x, y \in X$ and any complex number α.

(c) $p(x)$ is a lower semi-continuous function defined on the whole space X.

Therefore by the Gelfand lemma there exists a constant $K > 0$ such that

$$p(x) \leqq K\|x\| \qquad \text{for } x \in X.$$

Then for each $\lambda > \omega$ and each $x \in X$ we have

$$\int_0^\infty e^{-\lambda\xi} \|B_n T(\xi; A)x\| d\xi = \sum_{k=0}^\infty \int_k^{k+1} e^{-\lambda\xi} \|B_n T(\xi; A)x\| d\xi$$

$$\leqq \sum_{k=0}^\infty e^{-\lambda k} \int_0^1 \|B_n T(\xi; A)[T(k; A)x]\| d\xi$$

$$\leqq \sum_{k=0}^\infty e^{-\lambda k} p(T(k; A)x) \leqq K \sum_{k=0}^\infty e^{-\lambda k} \|T(k; A)x\|$$

$$\leqq MK \sum_{k=0}^\infty e^{-(\lambda-\omega)k} \|x\| = MK[1 - e^{-(\lambda-\omega)}]^{-1} \|x\|.$$

Thus $0 \leqq r_\lambda = \sup_{\|x\| \leqq 1} r_\lambda(x) < \infty$ for each $\lambda > \omega$. This concludes the proof.

Since r_λ defined on the open interval (ω, ∞) is a non-negative, monotone non-increasing function of λ, the limit

(3.3) $$r=\lim_{\lambda\to\infty} r_\lambda$$

exists and $0 \leqq r < \infty$.

LEMMA 2. *Let* $\{T(\xi; A); \xi \geqq 0\}$ *be a semi-group of class* (C_0) *and suppose that* $B_n \in \mathfrak{B}(X)$ *for each positive integer n. By Theorem* I, *for each n,* $A+B_n$ *defined on* $D(A)$ *generates a semi-group* $\{T(\xi, A+B_n); \xi \geqq 0\}$ *of class* (C_0) *and*

$$T(\xi; A+B_n)=\sum_{k=0}^{\infty} S_k^{(n)}(\xi) \qquad \textit{for } \xi \geqq 0,$$

where $S_0^{(n)}(\xi)=T(\xi; A)$ *and* $S_k^{(n)}(\xi)x=\int_0^\xi T(\xi-\sigma; A)B_n S_{k-1}^{(n)}(\sigma)x d\sigma$ *for* $x \in X$ *and* $k \geqq 1$.

If

(3.2) $$\sup_n \int_0^1 \|B_n T(\xi; A)x\| d\xi < \infty \qquad \textit{for each } x \in X,$$

then

(3.4) $$\sup_n \|S_k^{(n)}(\xi)\| \leqq M(r_\lambda)^k \exp(\lambda\xi)$$

for $\lambda > \omega$, $\xi \geqq 0$ *and* $k \geqq 0$.

PROOF. From the Fubini theorem we have

(3.5) $$S_k^{(n)}(\xi)x=\int_0^\xi S_{k-1}^{(n)}(\xi-\eta)B_n T(\eta; A)x d\eta$$

for $n \geqq 1$, $k \geqq 1$, $\xi \geqq 0$ and $x \in X$. It follows from Lemma 1 that $0 \leqq r_\lambda < \infty$ and $r_\lambda(x) \leqq r_\lambda \|x\|$ for $\lambda > \omega$ and $x \in X$.

We shall now prove (3.4) by induction. For $k=0$, this is obvious from (3.1). Suppose that it is true for $k=m$. Then, by (3.5),

$$\|S_{m+1}^{(n)}(\xi)x\| = \left\|\int_0^\xi S_m^{(n)}(\xi-\eta)B_n T(\eta;A)x d\eta\right\|$$

$$\leqq \int_0^\xi \|S_m^{(n)}(\xi-\eta)\| \|B_n T(\eta;A)x\| d\eta$$

$$\leqq M(r_\lambda)^m \int_0^\xi e^{\lambda(\xi-\eta)} \|B_n T(\eta; A)x\| d\eta$$

$$\leqq M(r_\lambda)^m e^{\lambda\xi}\int_0^\infty e^{-\lambda\eta}\|B_nT(\eta;A)x\|d\eta$$

$$\leqq M(r_\lambda)^m e^{\lambda\xi}r_\lambda(x)\leqq M(r_\lambda)^{m+1}e^{\lambda\xi}\|x\|$$

for $\lambda>\omega$, $\xi\geqq 0$ and $x\in X$. Hence we have

$$\sup_n \|S_{m+1}^{(n)}(\xi)\|\leqq M(r_\lambda)^{m+1}e^{\lambda\xi}$$

for $\lambda>\omega$ and $\xi\geqq 0$. This concludes the proof.

From these lemmas and Trotter's theorem (Theorem II) we have the following

THEOREM 1. *Let* $\{T(\xi;A);\xi\geqq 0\}$ *be a semi-group of class* (C_0) *and suppose that* $B_n\in\mathfrak{B}(X)$ *for each positive integer* n.

If $\lim_{n\to\infty} B_nx=Bx$ *for* $x\in D(A)$ *and if*

$$\sup_n\int_0^1\|B_nT(\xi;A)x\|d\xi<\infty \qquad \text{for each } x\in X, \tag{3.2}$$

then there exists an $\varepsilon_0>0$ *which is finite or* ∞ *such that for each* ε *with* $|\varepsilon|<\varepsilon_0$, $A+\varepsilon B$ *defined on* $D(A)$ *generates a semi-group of class* (C_0).

PROOF. We put $\varepsilon_0=1/r$ if $r\neq 0$, and we put $\varepsilon_0=\infty$ if $r=0$. The theorem is trivial for $\varepsilon=0$. For given ε with $0<|\varepsilon|<\varepsilon_0$, there exists a positive number $\lambda_\varepsilon>\omega$ such that

$$|\varepsilon|r_{\lambda_\varepsilon}<1.$$

Since $\tilde{B}_n=\varepsilon B_n\in\mathfrak{B}(X)$, it follows from Theorem I that $A+\tilde{B}_n$ generates a semi-group $\{T(\xi;A+\tilde{B}_n);\xi\geqq 0\}$ of class (C_0) and

$$T(\xi;A+\varepsilon B_n)=T(\xi;A+\tilde{B}_n)=\sum_{k=0}^\infty \tilde{S}_k^{(n)}(\xi) \qquad \text{for } \xi\geqq 0, \tag{3.6}$$

where $\tilde{S}_0^{(n)}(\xi)=T(\xi;A)$ and $\tilde{S}_k^{(n)}(\xi)x=\int_0^\xi T(\xi-\sigma;A)\tilde{B}_n\tilde{S}_{k-1}^{(n)}(\sigma)xd\sigma$ for $x\in X$ and $k\geqq 1$.

Further $\tilde{S}_k^{(n)}(\xi)=\varepsilon^kS_k^{(n)}(\xi)$ and hence by (3.4) we get

$$\sup_n \|\tilde{S}_k^{(n)}(\xi)\|\leqq M(|\varepsilon|r_{\lambda_\varepsilon})^k e^{\lambda_\varepsilon\xi} \tag{3.7}$$

for $k\geqq 0$ and $\xi\geqq 0$. Hence by (3.6) we have

$$\|T(\xi;A+\varepsilon B_n)\|\leqq\sum_{k=0}^\infty\|\tilde{S}_k^{(n)}(\xi)\|\leqq M(1-|\varepsilon|r_{\lambda_\varepsilon})^{-1}e^{\lambda_\varepsilon\xi} \tag{3.8}$$

for $\xi \geqq 0$.

Since $B_n \in \mathfrak{B}(x)$ and $R(\lambda; A)x = \int^{\infty} e^{-\lambda\xi} T(\xi; A)x d\xi$ for $\lambda > \omega$ and $x \in X$, we have

$$\varepsilon B_n R(\lambda; A)x = \varepsilon \int^{\infty} e^{-\lambda\xi} B_n T(\xi; A)x d\xi$$

for $x \in X$. Let λ be a real number with $\lambda \geqq \lambda_\varepsilon$. Hence

$$\|\varepsilon B_n R(\lambda; A)x\| \leqq |\varepsilon| r_\lambda(x) \leqq |\varepsilon| r_\lambda \|x\| \qquad \text{for } x \in X.$$

Passing to the limit as $n \to \infty$ we have $\|\varepsilon BR(\lambda; A)x\| \leqq |\varepsilon| r_\lambda \|x\|$ for $x \in X$, so that

$$\|\varepsilon BR(\lambda; A)\| \leqq |\varepsilon| r_\lambda \leqq |\varepsilon| r_{\lambda_\varepsilon} < 1.$$

Then $[\lambda - (A + \varepsilon B)]^{-1}$ exists and $[\lambda - (A + \varepsilon B)]^{-1} = R(\lambda; A) \sum_{k=0}^{\infty} [\varepsilon BR(\lambda; A)]^k \in \mathfrak{B}(X)$. This shows that $A + \varepsilon B$ defined on $D(A)$ is a closed linear operator and $\Re(\lambda - (A + \varepsilon B)) = X$.

Thus it follows from Theorem II that $A + \varepsilon B$ defined on $D(A)$ is the infinitesimal generator of a semi-group $\{T(\xi; A + \varepsilon B); \xi \geqq 0\}$ of class (C_0), and $T(\xi; A + \varepsilon B)x = \lim_{n\to\infty} T(\xi; A + \varepsilon B_n)x$ for $x \in X$ and $\xi \geqq 0$. This concludes the proof.

THEOREM 2. *Let $\{T(\xi; A); \xi \geqq 0\}$ be a semi-group of class (C_0). Suppose that*

(i) B is a linear operator with $D(B) \supset D(A)$ and $BR(\lambda; A) \in \mathfrak{B}(X)$ for some $\lambda > \omega$,

(ii) there exists a constant $K > 0$ such that

$$\int^{1} \|BT(\xi; A)x\| d\xi \leqq K\|x\|$$

for all $x \in D(A)$.
Then there exists an $\varepsilon_0 > 0$ which is finite or ∞ such that for each ε with $|\varepsilon| < \varepsilon_0$, $A + \varepsilon B$ defined on $D(A)$ generates a semi-group of class (C_0).

PROOF. Let us put

$$B_n = nBR(n; A)$$

for each positive integer n with $n > \omega$, and let us put $B_n = 0$ for each positive integer n with $n \leqq \omega$. From the resolvent equation we have

$$BR(n; A) = BR(\lambda; A) - (n - \lambda) BR(\lambda; A) R(n; A)$$

for $n > \omega$. It follows from this formula and the assumption (i) that $B_n \in \mathfrak{B}(X)$ for all $n \geqq 1$. Then we have

$$\lim_{n \to \infty} B_n x = Bx$$

for $x \in D(A)$. In fact, for each $x \in D(A)$ there exists an element $y \in X$ such that $x = R(\lambda; A)y$. Hence $B_n x = nBR(n; A)R(\lambda; A)y = BR(\lambda; A)[nR(n; A)y] \to BR(\lambda; A)y = Bx$ as $n \to \infty$.

Furthermore from the assumption (ii) we have

$$\int_0^1 \|B_n T(\xi; A)x\| d\xi = \int_0^1 \|nBR(n; A)T(\xi; A)x\| d\xi$$

$$= \int_0^1 \|BT(\xi; A)[nR(n; A)x]\| d\xi$$

$$\leqq K\|nR(n; A)x\| \leqq \frac{n}{n-\omega} MK\|x\|$$

for $n > \omega$ and $x \in X$. Hence

$$\sup_n \int_0^1 \|B_n T(\xi; A)x\| d\xi < \infty$$

for each $x \in X$. The result now follows from Theorem 1. This concludes the proof.

COROLLARY 1. *Let* $\{T(\xi; A); \xi \geqq 0\}$ *be a semi-group of class* (C_0). *Suppose that*

(i′) B *is a closed linear operator with* $D(B) \supset D(A)$,

(ii′) *there exists a constant* $K > 0$ *such that*

$$\int^1 \|BT(\xi; A)x\| d\xi \leqq K\|x\|$$

for all $x \in D(A)$.

Then there exists an $\varepsilon_0 > 0$ *which is finite or* ∞ *such that for each* ε *with*

$|\varepsilon|<\varepsilon_0$, *$A+\varepsilon B$ defined on $D(A)$ generates a semi-group of class (C_0).*

PROOF. For any $\lambda>\omega$, $BR(\lambda;A)$ is a closed linear operator defined on the whole space X and therefore it is bounded by the closed graph theorem. Thus (i′) implies the assumption (i) of Theorem 2. This completes the proof.

THEOREM 3. *Let $\{T(\xi;A);\xi\geqq 0\}$ be a semi-group of class (C_0). Suppose that*

(i″) B is a linear operator with $D(B)\supset D(A)$ and $BR(\lambda;A)\in\mathfrak{B}(X)$ for some $\lambda>\omega$ (or B is a closed linear operator with $D(B)\supset D(A)$),

(ii″) $BT(\xi;A)$ defined on $D(A)$ is bounded for all $\xi>0$ and

$$\int_0^1 \|U(\xi)x\|d\xi<\infty^{2)} \qquad \textit{for each } x\in X,$$

where for each $\xi>0$, $U(\xi)$ denotes the linear bounded extension of $BT(\xi;A)$ to the whole space X.

Then there exists an $\varepsilon_0>0$ which is finite or ∞ such that for each ε with $|\varepsilon|<\varepsilon_0$, $A+\varepsilon B$ defined on $D(A)$ generates a semi-group of class (C_0).

PROOF. Let us put $p(x)=\int_0^1\|U(\xi)x\|d\xi$ for $x\in X$. If $\lim_{n\to\infty}x_n=x$, then it follows from the Fatou lemma that

$$p(x)=\int_0^1 \lim_{n\to\infty}\|U(\xi)x_n\|d\xi$$

$$\leqq \varliminf_{n\to\infty}\int_0^1\|U(\xi)x_n\|d\xi=\varliminf_{n\to\infty}p(x_n).$$

Thus $p(x)$ is a lower semi-continuous function defined on the whole space X. Furthermore $0\leqq p(x)<\infty$, $p(x+y)\leqq p(x)+p(y)$ and $p(\alpha x)=|\alpha|p(x)$ for x, $y\in X$ and any complex number α. Therefore by the Gelfand lemma there exists a constant $K>0$ such that

$$\int_0^1\|U(\xi)x\|d\xi=p(x)\leqq K\|x\| \qquad \text{for all } x\in X.$$

This implies the condition (ii) of Theorem 2, and hence the theorem follows from Theorem 2 and Corollary 1. This completes the proof.

2) We note that for each $x\in X$, $U(\xi)x$ is Bochner measurable on $(0,\infty)$.

The following result is due to R.S.Phillips [1].

COROLLARY 2. *Let* $\{T(\xi; A); \xi \geqq 0\}$ *be a semi-group of class* (C_0). *If* $B \in \mathfrak{P}(A)$, *then for each complex number* ε, $A+\varepsilon B$ *defined on* $D(A)$ *generates a semi-group of class* (C_0).

PROOF. $B \in \mathfrak{P}(A)$ implies the assumptions of Theorem 3. We now prove that $\varepsilon_0=\infty$. We define B_n similarly as in the proof of Theorem 2. For each $x \in X$, $n > \omega$ and $\lambda > \omega$

$$\int_0^\infty e^{-\lambda\xi}\|B_n T(\xi; A)x\|d\xi = \int_0^\infty e^{-\lambda\xi}\|BT(\xi; A)[nR(n; A)x]\|d\xi$$

$$= \int_0^1 e^{-\lambda\xi}\|BT(\xi; A)[nR(n; A)x]\|d\xi + \int_1^\infty e^{-\lambda\xi}\|BT(\xi; A)[nR(n; A)x]\|d\xi$$

$$\leqq \frac{Mn}{n-\omega}\left[\int_0^1 e^{-\lambda\xi}\|BT(\xi; A)\|_A d\xi + e^{-\lambda}\|BT(1)\|_A \int_0^\infty e^{-\lambda\xi}\|T(\xi; A)\|d\xi\right]\|x\|.$$

Hence if we define r_λ similarly as in Lemma 1, then for each $\lambda > \omega$ we have

$$r_\lambda \leqq M'\left[\int_0^1 e^{-\lambda\xi}\|BT(\xi; A)\|_A d\xi + e^{-\lambda}\frac{M\|BT(1)\|_A}{\lambda-\omega}\right],$$

where M' is a positive constant independent of λ. The above right hand side tends to zero when $\lambda \to \infty$, and hence $r = \lim_{\lambda\to\infty} r_\lambda = 0$. Thus it follows from the definition of ε_0 in Theorem 1 that $\varepsilon_0 = \infty$. This completes the proof.

4. In this sectin we deal with groups of operators of class (C_0). We note that if $\{T(\xi; A); -\infty < \xi < \infty\}$ is a group of operators of class (C_0), then there exist real constants $M > 0$ and $\omega \geqq 0$ such that

$$\|T(\xi; A)\| \leqq M \exp(\omega|\xi|) \quad \text{for all } \xi, \tag{4.1}$$

and

$$\{\lambda; |\lambda| > \omega\} \subset \rho(A), \tag{4.2}$$

where $\rho(A)$ is the resolvent set of A.

THEOREM 4. *Let* $\{T(\xi; A); -\infty < \xi < \infty\}$ *be a group of class* (C_0).
Suppose that

(i) *B is a linear operator with* $D(B) \supset D(A)$ *and* $BR(\lambda; A) \in \mathfrak{B}(X)$ *for some real* λ *with* $|\lambda| > \omega$, *(or B is a closed linear operator with* $D(B) \supset D(A)$,*)*

(ii) *there exists a constant* $K>0$ *such that*

$$\int_0^1 \|BT(\xi; A)x\| d\xi \leqq K\|x\|$$

for all $x \in D(A)$.
Then there exists an $\varepsilon_0>0$ *which is finite or* ∞ *such that for each* ε *with* $|\varepsilon|<\varepsilon_0$, $A+\varepsilon B$ *defined on* $D(A)$ *generates a group of class* (C_0).

PROOF. From the resolvent equation we get

$$BR(\mu; A)=BR(\lambda; A)-(\mu-\lambda)BR(\lambda; A)R(\mu; A)$$

for all real μ with $|\mu|>\omega$. Then by (i) we have

$$BR(\mu; A) \in \mathfrak{B}(X) \tag{4.3}$$

for all real μ with $|\mu|>\omega$. (If B is a closed linear operator with $D(B)\supset D(A)$, then we have the same result from the closed graph theorem.)

Setting $T_+(\xi)=T(\xi; A)$ for $\xi \geqq 0$, $\{T_+(\xi); \xi \geqq 0\}$ is a semi-group of class (C_0) and A is its infinitesimal generator. Thus by Theorem 2 there exists an $\varepsilon_1>0$ which is finite or ∞ such that for each ε with $|\varepsilon|<\varepsilon_1$, $A+\varepsilon B$ defined on $D(A)$ generates a semi-group $\{T_+(\xi; A+\varepsilon B); \xi \geqq 0\}$ of class (C_0).

We next put $T_-(\xi)=T(-\xi; A)$ for $\xi \geqq 0$. Then $\{T_-(\xi); \xi \geqq 0\}$ is a semi-group of class (C_0) and $-A$ is its infinitesimal generator. By (4.2), $\mu \in \rho(-A)$ for $\mu>\omega$, and by (4.3) and $R(\mu; -A)=-R(-\mu; A)$ we have

$$BR(\mu; -A) \in \mathfrak{B}(X) \qquad \text{for each } \mu>\omega.$$

Moreover by the assumption (ii)

$$\begin{aligned}\int_0^1 \|BT_-(\xi)x\| d\xi &= \int_0^1 \|BT(-\xi; A)x\| d\xi \\ &= \int_0^1 \|BT(1-\xi; A)T(-1)x\| d\xi \\ &= \int_0^1 \|BT(\xi; A)T(-1)x\| d\xi \\ &\leqq K\|T(-1)x\| \leqq K\|T(-1)\| \|x\|\end{aligned}$$

for all $x \in D(-A)=D(A)$. These imply the assumptions of Theorem 2. Then there exists an $\varepsilon_2>0$ which is finite or ∞ such that for each ε with $|\varepsilon|<\varepsilon_2$,

$-A-\varepsilon B$ defined on $D(A)$ generates a semi-group $\{T_-(\xi; -A-\varepsilon B); \xi \geqq 0\}$ of class (C_0).

We now put $\varepsilon_0=\min(\varepsilon_1, \varepsilon_2)$. Setting $S_\varepsilon(\xi)=T_+(\xi; A+\varepsilon B)T_-(\xi; -A-\varepsilon B)$ for each ε with $|\varepsilon|<\varepsilon_0$, $S_\varepsilon(\xi)x (x \in D(A))$ is (strongly) continuously differentiable for $\xi \geqq 0$ and

$$\frac{d}{d\xi}S_\varepsilon(\xi)x=[T_+(\xi; A+\varepsilon B)(A+\varepsilon B)]T_-(\xi; -A-\varepsilon B)x$$
$$+T_+(\xi; A+\varepsilon B)[(-A-\varepsilon B)T_-(\xi; -A-\varepsilon B)]x=0,$$

and further $S_\varepsilon(\xi)x \to x$ as $\xi \to 0$. It follows that $S_\varepsilon(\xi)x\equiv x$ for each $x \in D(A)$ and since $D(A)$ is dense in X the same is true for each $x \in X$. Thus $T_+(\xi; A+\varepsilon B)T_-(\xi; -A-\varepsilon B)=I$; a similar argument shows that $T_-(\xi; -A-\varepsilon B)T_+(\xi; A+\varepsilon B)=I$ and hence $T_-(\xi; -A-\varepsilon B)=[T_+(\xi; A+\varepsilon B)]^{-1}$. We define

$$T(\xi; A+\varepsilon B)=\begin{cases} T_+(\xi; A+\varepsilon B) & \text{for } \xi \geqq 0, \\ T_-(-\xi; -A-\varepsilon B) & \text{for } \xi < 0, \end{cases}$$

where $|\varepsilon| < \varepsilon_0$. Then it is easy to see that $\{T(\xi; A+\varepsilon B); -\infty<\xi<\infty\}$ is a group of class (C_0) and $A+\varepsilon B$ is its infinitesimal generator. This concludes the proof.

If $\{T(\xi; A); -\infty<\xi<\infty\}$ is a group of class (C_0), then each $B \in \mathfrak{P}(A)$ is bounded on $D(A)$. In fact, the assumption $\|BT(\xi; A)\|_A<\infty$ for $\xi > 0$ implies that

$$\|Bx\| = \|[BT(\xi; A)]T(-\xi; A)x\| \leqq \|BT(\xi; A)\|_A \|T(-\xi; A)\| \, \|x\|$$

for $x \in D(A)$.

Finally we show that there exists an unbounded operator satisfying the assumptions of Theorem 4.

EXAMPLE. We consider the Lebesgue space $L_1(-\infty, \infty)$ and we define

$$[T(\xi)x](t)=x(t+\xi) \tag{4.4}$$

for all real ξ and $x \in L_1(-\infty, \infty)$. It is easy to see that $\{T(\xi); -\infty<\xi<\infty\}$ is a group of operators of class (C_0) and $D(A)$ is the class of all absolutely continuous functions such that $x(t)$ and $x'(t)$ belong to $L_1(-\infty.\infty)$.

Let $b(t)$ be a function such that $b(t) \in L_1(-\infty, \infty)$ and $b(t) \notin L_\infty(-\infty, \infty)$, and we define an operator B by

$$[Bx](t)=b(t)x(t) \tag{4.5}$$

for $x(t) \in L_1(-\infty, \infty)$ such that $b(t)x(t) \in L_1(-\infty, \infty)$. If $x \in D(A)$, then $x \in L_\infty(-\infty, \infty)$. This shows that $D(B) \supset D(A)$.

We shall next prove that B is a closed linear operator. Suppose $x_n \in D(B)$, $\|x_n - x\| = \int_{-\infty}^{\infty} |x_n(t) - x(t)|\, dt \to 0$ and $\|Bx_n - y\| = \int_{-\infty}^{\infty} |b(t)x_n(t) - y(t)|\, dt \to 0$. Then there exists a subsequence $\{n_i\}$ such that $\lim x_{n_i}(t) = x(t)$ a.e. t, and for arbitrary $\varepsilon > 0$ there exists a positive integer n_ε such that

$$\int_{-\infty}^{\infty} |b(t)x_{n_i}t(t) - y(t)|\, dt < \varepsilon$$

for $n \geqq n_\varepsilon$. Hence

$$\int_{-\infty}^{\infty} |b(t)x_{n_i}(t) - y(t)|\, dt < \varepsilon$$

for $n_i \geqq n_\varepsilon$. Passing to the limit as $n_i \to \infty$, by the Fatou lemma, we have

$$\int_{-\infty}^{\infty} |b(t)x(t) - y(t)|\, dt \leqq \varepsilon;$$

so that $b(t)x(t) = y(t)$ a.e. t. Consequently $x \in D(B)$ and $Bx = y$. Thus B is a closed linear operator.

Further

$$\int_{-\infty}^{\infty} \|BT(\xi)x\|\, d\xi = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} |b(t)x(t+\xi)|\, dt\right] d\xi = \|b\|\,\|x\|$$

for all $x \in D(A)$.

Therefore the operator B satisfies the assumptions of Theorem 4. But the operator B isn't bounded since $b \notin L_\infty(-\infty, \infty)$.

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