Tôhoku Math. Journ.
Vol. 18, No. 3, 1966

ON THE STRONG LAW OF LARGE NUMBERS

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(Received March 8, 1966)

1. In the present note $f(x)$, $-\infty < x < \infty$, will denote a real function satisfying the following conditions

$$(1.1) \qquad \int_0^1 f(x)dx=0, \quad \int_0^1 f(x)^2dx=1, \; f(x+1)=f(x),$$

and $R(N)$ will dtenote

$$R(N)=\left(\int_0^1 [f(x)-S_N(x)]^2dx\right)^{\frac{1}{2}},$$

where $S_N(x)$ is the N-th order partial sum of the Fourier series of $f(x)$. By $n_1 < n_2 < \cdots\cdot$ (for simplicity we shall occasionally denote n_s by $n(s)$) we shall denote an arbitrary sequence of positive integers satisfying

$$(1.2) \qquad \frac{n_{s+1}}{n_s} \geqq \theta > 1 \qquad s=1,2,\cdots$$

P.Erdös [1] proved the following

THEOREM A. *If $f(x)$ and $\{n_s\}$ satisfies* (1.1) *and* (1.2) *respectively, and for $\alpha > 1$,*

$$(1.3) \qquad R(N)=O\left(\frac{1}{\log\log N}\right)^{\alpha}$$

then for almost every x,

$$(1.4) \qquad \lim_{N\to\infty}\frac{1}{N}\sum_{s=1}^{N} f(n_sx)=0.$$

In his paper, Erdös conjectured that the condition (1. 3) would be replaced by the following condition

$$R(N)=O\left(\frac{1}{(\log\log\log N)^{c}}\right) \tag{1. 5}$$

Thus, there are both cases $0<\alpha\leqq 1$ in (1. 3) or (1. 5) in which we are interested.

Our object of the present note will prove the following theorem, which contains Theorem A in the two points, i.e. α and the rapidity of tending to 0 in (1. 4).

THEOREM. *Let* $f(x)$ *satisfy* (1. 1) *and* (1. 3), *and let* $\{n_k\}$ *satisfy* (1. 2). *If a real number* γ *satisfies*

$$\gamma>\frac{1}{2}-\alpha \quad \textit{for } 0<\alpha\leqq\frac{5}{2} \quad \textit{or} \quad \gamma=-2 \quad \textit{for } \frac{5}{2}<\alpha, \tag{1. 6}$$

then for almost every x,

$$\lim_{N\to\infty}\frac{1}{N(\log N)^{\gamma}}\sum_{s=1}^{N}f(n_s x)=0. \tag{1. 7}$$

Without any loss of generality we may suppose $\gamma<0$ for $\frac{1}{2}<\alpha$, so we obtain the following corollary.

COROLLARY. *If* (1. 3) *is satisfied for* $\alpha>\frac{1}{2}$, *then for almost every* x, (1. 4) *holds.*

If we use the same method as the proof of Theorem A, we can prove the following theorem, which is weaker than our theorem in the case $0<\alpha\leqq\frac{5}{2}$.

THEOREM B. *If* (1. 3) *is satisfied, then for almost every* x,

$$\lim_{N\to\infty}\frac{1}{N(\log N)^{\delta}}\sum_{s=1}^{N}f(n_s x)=0.$$

where $0\leqq\alpha<5$ *and* $2\delta>1-\alpha$.

REMARK. The proof of the case $\alpha=0$ of Theorem B proceeds the same as that of the case $\alpha>0$ of the theorem, but in our Theorem these do not hold. Therefore the case $\alpha=0$ of our Theorem holds by Theorem B.

2. LEMMA. *Let $\tau(s)$ and $\mu(s)$ be strictly increasing sequences of integers satisfying for $s=1,2,\cdots$*

$$\tau(s)>\mu(s)>1. \tag{2. 1}$$

If $f(x)$ and $\{n_s\}$ satisfy (1. 3) *and* (1. 2), *respectively, then for any positive integers M and N such that $0\leqq M<N$, we have*

$$\int_0^1 \sum_{s=M}^{N} c_s[S_{\tau(s)}(n_s x)-S_{\mu(s)}(n_s x)]^2\,dx \leqq A\frac{N-M}{(\log(N-M))^\alpha}\sum_{s=M}^{N}\frac{c_s^2}{(\log\log\mu(s))^\alpha} \tag{2. 2}$$

$$\int_0^1 \sum_{s=M}^{N} c_s[f(n_s x)-S_{\mu(s)}(n_s x)]^2\,dx \leqq A\frac{N-M}{(\log(N-M))^\alpha}\sum_{s=M}^{N}\frac{c_s^2}{(\log\log\mu(s))^\alpha} \tag{2. 3}$$

$$\int_0^1 \sum_{s=M}^{N} [c_s S_{\tau(s)}(n_s x)]^2 dx \leqq A\frac{N-M}{(\log(N-M))^\alpha}\sum_{s=M}^{N} c_s^2 \tag{2. 4}$$

where A's denote absolute constants respectively.

This lemma was proved by author in [2]. Though the proof of Theorem is in need of (2. 3) and (2. 4) only, we prove the most complicated case (2. 2), because (2. 3) and (2. 4) are analogous to (2. 2).

PROOF. Putting

$$I(s,t)=\int_0^1 [S_{\tau(s)}(n_s x)-S_{\mu(s)}(n_s x)][S_{\tau(t)}(n_t x)-S_{\mu(t)}(n_t x)]\,dx,$$

and let the Fourier series of $f(x)$ be

$$f(x)\sim\sum_{p=1}^{\infty} a_p\cos(2\pi p x+\theta_p),$$

then for any pair of integers (p, q) such that

$$n_s p = n_t q,\ \mu(s) \leqq p \leqq \tau(s),\ \mu(t) \leqq q \leqq \tau(t)$$

$$|I(s, t)| = \frac{1}{2}\left|\sum_{(p,q)} a_p a_q\right| = \frac{1}{2}\left|\sum_{(p,q)} a_q a_{n_t q/n_s}\right|$$

$$\leqq \frac{1}{2}\left(\sum_{(p,q)} a_q^2 \cdot \sum_{(p,q)} a_{n_t q/n_s}^2\right)^{\frac{1}{2}}$$

$$\leqq \frac{1}{2}\left(\sum_q a_q^2 \sum_p 1\right)^{\frac{1}{2}}\left(\sum_q a_{n_t q/n_s}^2\ \sum_p 1\right)^{\frac{1}{2}}$$

$$\leqq \frac{1}{2}\left(\sum_{\mu(t)\leqq q} a_q^2\right)^{\frac{1}{2}}\left(\sum_{\mu(t)\leqq q} a_{n_t q/n_s}^2\right)^{\frac{1}{2}} \leqq \frac{1}{2} R(\mu(t)) R\left(\frac{n_t}{n_s}\mu(t)\right)$$

$$\leqq A \frac{1}{(\log\log\mu(t))^\alpha (\log(t-s))^\alpha},$$

where $t > s$. If $s=t$, then by the same way we obtain

$$|I(s,s)| \leqq \frac{A}{(\log\log\mu(s))^{2\alpha}}.$$

Therefore we have

$$\int^1 \sum_{s=M}^{N} c_s [S_{\tau(s)}(n_s x) - S_{\mu(s)}(n_s x)]^2\, dx$$

$$= \sum_{s=M}^{N} c_s^2 I(s, s) + 2 \sum_{M\leqq s<t\leqq N} c_s c_t I(s, t)$$

$$\leqq A \sum_{s=M}^{N} \frac{c_s^2}{(\log\log\mu(s))^{2\alpha}} + 2A \sum_{s=M}^{N-1}\sum_{t=s+1}^{N} \frac{c_s c_t}{(\log\log\mu(t))^\alpha (\log(t-s))^\alpha}$$

$$\leqq A \sum_{s=M}^{N} \frac{c_s^2}{(\log\log\mu(s))^\alpha}$$

$$+ A \sum_{r=1}^{N-M} \frac{1}{(\log r)^\alpha} \sum_{s=M}^{N-r} \frac{c_s}{(\log\log\mu(s+r))^{\alpha/}} \cdot \frac{c_{s+r}}{(\log\log\mu(s+r))^{\alpha/2}}$$

$$\leqq A \sum_{s=M}^{N} \frac{c_s^2}{(\log\log\mu(s))^\alpha} + A \sum_{r=1}^{N-M} \frac{1}{(\log r)^\alpha} \sum_{s=M}^{N} \frac{c_s^2}{(\log\log\mu(s))^\alpha}$$

$$\leqq A\sum_{s=M}^{N}\frac{c_s^2}{(\log\log\mu(s))^\alpha}\left(1+\sum_{r=1}^{N-M}\frac{1}{(\log r)^\alpha}\right)$$

$$\leqq A\frac{N-M}{(\log(N-M))^\alpha}\sum_{s=M}^{N}\frac{c_s^2}{(\log\log\mu(s))^\alpha}\,.$$

This is (2. 2).

3. We shall prove Theorem by two steps separately.

(I) *the first step.* We consider an equation

$$\frac{x}{(\log x)^{\alpha-1}}=\sqrt{s}$$

If $\alpha>1$ then it has only one root bigger than $e^{\alpha-1}$, and if $0<\alpha\leqq 1$ then it has only one root bigger than 1. If we denote this root by $\lambda(s)$, where $s>s_0$*), then it is easily seen that for $s\geqq s_0$

$$\lambda(s)\uparrow+\infty$$

$$s^{1/4}<\lambda(s)<s^{3/4}. \tag{3. 1}$$

Let us put

$$\tau(s)=\begin{cases}\exp[\lambda(s)] & s=s_0,\,s_0+1,\cdots\\ \tau(s_0) & s=1,\,2,\cdots,\,s_0-1.\end{cases} \tag{3. 2}$$

Then we have for any real γ and α such that $\alpha>0$,

$$\sum_{s=2}^{\infty}\frac{\log\tau(s)}{s^2(\log s)^{2\gamma}(\log\log\tau(s))^{\alpha-1}}$$

$$=A\sum_{s=2}^{\infty}\frac{\log\tau(s)}{s^2(\log s)^{2\gamma}(\log\lambda(s))^{\alpha-1}}$$

$$=A\sum_{s=2}^{\infty}\frac{s^{3/4}}{s^2(\log s)^{2\gamma+\alpha-1}}\leqq A\sum_{s}\frac{1}{s^{5/4}}<\infty \tag{3. 3}$$

Now we consider a trigonometric series

*) If $0<\alpha\leqq 1$ then $s_0=1$ and if $1<\alpha$ then $s_0=\left(\frac{e}{\alpha-1}\right)^{2(\alpha-1)}$

(3. 4)
$$\sum_{s=2}^{\infty} \frac{1}{s(\log s)^\gamma} S_{\tau(s)}(n_s x),$$

and our object of this step (I), is to prove the almost everywhere convergence of (3. 4) without any restriction concerning $\alpha > 0$ and γ.

We define by induction, a sequence $\{\nu_k\}$ of positive integers, i.e.

(3. 5)
$$\nu_{k+1} = \nu_k + \left[\frac{2\log\tau(\nu_k)}{\log\theta}\right] \qquad k=0, 1, 2, \cdots$$

(3. 6)
$$\log\tau(\nu_0) \geqq \max(1+\log\theta, \log\tau(s_0)).$$

Thus defined sequence $\{\nu_k\}$ evidently satisfies,

(3. 7)
$$0 < \nu_0 < \nu_1 < \cdots \rightarrow \infty.$$

Let us put for $k=0, 1, 2, \cdots$

(3. 8)
$$X_k(x) = \sum_{s=\nu_k+1}^{\nu_{k+1}} \frac{1}{s(\log s)^\gamma} S_{\tau(s)}(n_s x).$$

Then frequencies of each term of this trigonometric series lie in an interval

(3. 9)
$$J_k = [n(\nu_k), \tau(\nu_{k+1})n(\nu_{k+1})],$$

so that J_{k-1} and J_{k+1} are mutually disjoint, and in addition there exists a gap $[\tau(\nu_k)n(\nu_k), n(\nu_{k+1})]$ in these intervals. From (3. 5) and (3. 6), this gap intervals satisfy the following formula.

(3.10)
$$\frac{n(\nu_{k+1})}{n(\nu_k)\tau(\nu_k)} \geqq \theta^{\nu_{k+1}-\nu_k} \frac{1}{\tau(\nu_k)} \geqq e \qquad k=0, 1, 2, \cdots$$

Now from (3. 10), (2. 4), (3. 5) and (3. 3) we have

$$\int_0^1 \left[\sum_{k=M}^{N} X_{2k}(x)\right]^2 dx = \sum_{k=M}^{N} \int_0^1 X_{2k}(x)^2 dx$$

$$= A \sum_{k=M}^{N} \frac{\nu_{2k+1} - \nu_{2k}}{(\log(\nu_{2k+1} - \nu_{2k}))^\alpha} \sum_{s=\nu_{2k}+1}^{\nu_{2k+1}} \frac{1}{s^2(\log s)^{2\gamma}}$$

$$\leqq A\sum_{k=M}^{N}\frac{\log\tau(\nu_{2k})}{(\log\log\tau(\nu_{2k}))^{\alpha}}\sum_{s=\nu_{2k}+1}^{\nu_{2k+1}}\frac{1}{s^2(\log s)^{2\gamma}}$$

$$\leqq A\sum_{s=\nu_{2M}+1}^{\nu_{2N+1}}\frac{\log\tau(s)}{s^2(\log s)^{2\gamma}(\log\log\tau(s))^{\alpha}}<\infty.$$

This shows that the series $\sum X_{2k}(x)$ and $\sum X_{2k+1}(x)$ converge in the L_2-mean, respectively, and there exist L_2 integrable functions $h_1(x)$ and $h_2(x)$ such that

$$h_1(x)\sim\sum_0^{\infty}X_{2k}(x),\quad h_2(x)\sim\sum_0^{\infty}X_{2k+1}(x).$$

Combining these and the well known Kolmogoroff theorem [3], then the series $\sum_0^{\infty}X_k(x)$ converges almost everywhere.

Now for the proof of the convergence of (3. 4), it is sufficient that,

$$\sum_{k=0}^{\infty}\int_0^1\max_{\nu_k<m<\nu_{k+1}}\left[\sum_{s=\nu_k+1}^{m}\frac{1}{s(\log s)^{\gamma}}S_{\tau(s)}(n_s x)\right]^2 dx<\infty. \tag{3.11}$$

Making use of the well known Menchoff's devices, we have the left hand side of (3.11)

$$\leqq\sum_{k=0}^{\infty}\log(\nu_{k+1}-\nu_k)\sum_{v=0}^{\log(\nu_{k+1}-\nu_k)}\sum_{u=0}^{(\nu_{k+1}-\nu_k)2^{-v}-1}\int_0^1\left[\sum_{s=\nu_k+u2^{v}+1}^{\nu_k+(u+1)2^{v}}\frac{1}{s(\log s)^{\gamma}}S_{\tau(s)}(n_s x)\right]^2 dx$$

$$\leqq A\sum_{k=0}^{\infty}\log(\nu_{k+1}-\nu_k)\sum_{v=0}^{\log(\nu_{k+1}-\nu_k)}\frac{2^v}{v^{\alpha}}\sum_{s=\nu_k+1}^{\nu_{k+1}}\frac{1}{s^2(\log s)^{2\gamma}}$$

$$\leqq A\sum_{k=0}^{\infty}\log(\nu_{k+1}-\nu_k)\frac{\nu_{k+1}-\nu_k}{(\log(\nu_{k+1}-\nu_k))^{\alpha}}\sum_{s=\nu_k+1}^{\nu_{k+1}}\frac{1}{s^2(\log s)^{2\gamma}}$$

$$=A\sum_{k=0}^{\infty}\frac{\log\tau(\nu_k)}{(\log\log\tau(\nu_k))^{\alpha-1}}\sum_{s=\nu_k+1}^{\nu_{k+1}}\frac{1}{s^2(\log s)^{2\gamma}}$$

$$=A\sum_{s}\frac{\log\tau(s)}{s^2(\log s)^{2\gamma}(\log\log\tau(s))^{\alpha-1}}.$$

By (3. 3) this last series is convergent, so that we obtain (3. 11),

Thus we complete our object of this step, i.e. for any $\alpha > 0$ and γ.

$$\sum_{s=\nu_0}^{\infty} \frac{1}{s(\log s)^{\gamma}} S_{\tau(s)}(n_s x)$$

converges almost everywhere.

(II) *the second step.* In this step we shall prove

$$\lim_{N\to\infty} \frac{1}{N(\log N)^{\gamma}} \sum_{s=1}^{N} [f(n_s x) - S_{\tau(s)}(n_s x)] = 0 \quad \text{a.e.,} \tag{3.12}$$

provided that (i) $0 < \alpha \leqq \frac{5}{2}$ and $\gamma > \frac{1}{2} - \alpha$, or (ii) $\alpha > \frac{5}{2}$ and $\gamma = -2$. Its proof is analogous with Erdös' one and only different point is to make use of (2. 3).

Let $0 < \alpha \leqq \frac{5}{2}$ and $\gamma > \frac{1}{2} - \alpha$. For the proof of (3. 12) we may suppose that γ is sufficiently near to $\frac{1}{2} - \alpha$, so we assume that

$$0 < \alpha \leqq \frac{1}{2} \quad \text{and} \quad 0 \leqq \frac{1}{2} - \alpha < \gamma < 2 \tag{3.13}$$

$$\frac{1}{2} < \alpha \leqq \frac{5}{2} \quad \text{and} \quad -2 \leqq \frac{1}{2} - \alpha < \gamma < 0. \tag{3.13)}$$

Now we proceed on the same line as Erdös. For any positive integers z and N such that $0 \leqq z < z + N$,

$$\int_0^1 \left| \sum_{s=z}^{z+N} [f(n_s x) - S_{\tau(s)}(n_s x)]^2 \right| dx \tag{3.14}$$

$$\leqq A \frac{N}{(\log N)^{\alpha}} \sum_{s=z}^{z+N} \frac{1}{(\log\log \tau(s))^{\alpha}}.$$

Therefore putting

$$M(z, N, \lambda) = \left(x; \left| \sum_{z}^{z+N} [f(n_s x) - S_{\tau(s)}(n_s x)] \right| \geqq \lambda N (\log N)^{\gamma} \right),$$

then

$$mM(z, N, \lambda) \leqq A \frac{1}{\lambda^2 N(\log N)^{2\gamma+\alpha}} \sum_{s=z}^{z+N} \frac{1}{(\log\log\tau(s))^{\alpha}}.$$

Hence we consider special cases of the above formula

$$mM(1, 2^n-1, \lambda) \leqq \frac{A}{\lambda^2 2^n n^{2\gamma+\alpha}} \sum_{s=1}^{2^n} \frac{1}{(\log\log\tau(s))^{\alpha}}, \tag{3.15}$$

$$mM\left(2^n+2u\cdot 2^{n-k}, 2^{n-k}, \frac{\lambda 2^k}{(k+1)^2}\right) \tag{3.16}$$

$$\leqq \frac{A(k+1)^4}{\lambda^2 2^{n+k}(n-k)^{2\gamma+\alpha}} \sum_{s=2^n+2u\cdot 2^{n-k}+1}^{2^n+(2u+1)\cdot 2^{n-k}} \frac{1}{(\log\log\tau(s))^{\alpha}}$$

where $0 \leqq u < 2^{k-1}$, $0 \leqq k \leqq n$, $0<n<\infty$ and λ is a positive number.

Writing

$$M_n = M(1, 2^n-1, \lambda) \cup \left\{ \bigcup_{k=0}^{n} \bigcup_{u=0}^{2^{k-1}-1} M(2^n+2u\cdot 2^{n-k}, 2^{n-k}, \frac{\lambda 2^k}{(k+1)^2}) \right\},$$

then

$$\sum_{n=1}^{\infty} mM_n$$

$$\leqq \sum_{n=1}^{\infty} mM(1, 2^n-1, \lambda) + \sum_{n=1}^{\infty} \sum_{k=0}^{n} \sum_{u=0}^{2^{k-1}-1} mM\left(2^n+2u\cdot 2^{n-k}, 2^{n-k}, \frac{\lambda 2^k}{(k+1)^2}\right)$$

$$\equiv P+Q.$$

Making use of (3. 1), (3. 13) and (3. 13′), then whether or not $2\gamma+\alpha \geqq 0$, we have

$$P = \sum_{n=1}^{\infty} mM(1, 2^n-1, \lambda) = \sum_{n=1}^{\infty} \frac{A}{\lambda^2 2^n \cdot n^{2\gamma+\alpha}} \sum_{s=1}^{2^n} \frac{1}{(\log\log\tau(s))^{\alpha}}$$

$$\leqq \frac{A}{\lambda^2} \sum_{s=1}^{\infty} \frac{1}{(\log\log\tau(s))^{\alpha}} \sum_{2^n \geqq s} \frac{1}{2^n \cdot n^{2\gamma+\alpha}}$$

$$\leqq \frac{A}{\lambda^2} \sum_{s=1}^{\infty} \frac{1}{s(\log s)^{2\gamma+\alpha}(\log\log\tau(s))^{\alpha}}$$

$$=\frac{A}{\lambda^2}\sum_{s=1}^{\infty}\frac{1}{s(\log s)^{2\gamma+2\alpha}}<\infty,$$

and

$$Q=A\sum_{n=1}^{\infty}\sum_{k=0}^{n}\sum_{u=0}^{2^{k-1}-1}\frac{(k+1)^4}{\lambda^2 2^{n+k}(n-k)^{2\gamma+\alpha}}\sum_{s=2^n+2u\cdot 2^{n-k}+1}^{2^n+(2u+1)\cdot 2^{n-k}}\frac{1}{(\log\log\tau(s))^{\alpha}}$$

$$\leqq\frac{A}{\lambda^2}\sum_{n=1}^{\infty}\frac{1}{2^n}\sum_{k=0}^{n}\frac{(k+1)^4}{2^k(n-k)^{2\gamma+\alpha}}\sum_{s=2^n+1}^{2^{n+1}}\frac{1}{(\log\log\tau(s))^{\alpha}}$$

$$\leqq\frac{A}{\lambda^2}\sum_{n=1}^{\infty}\frac{1}{2^n}\left(\sum_{s=2^n+1}^{2^{n+1}}\frac{1}{\log\log\tau(s))^{\alpha}}\right)\cdot\frac{1}{n^{2\gamma+\alpha}}$$

$$\leqq\frac{A}{\lambda^2}\sum_{s=1}^{\infty}\frac{1}{s(\log s)^{2\gamma+\alpha}(\log\log\tau(s))^{\alpha}}$$

$$=\frac{A}{\lambda^2}\sum_{s=1}^{\infty}\frac{1}{s(\log s)^{2\gamma+2\alpha}}<\infty.$$

That is to say

$$\sum_{n=1}^{\infty} mM_n<\infty. \tag{3.17}$$

Now we suppose that $2^n < m < 2^{n+1}$ and $|\gamma|\leqq 2$, and let us apply the Borel-Cantelli lemma,

$$\left|\sum_{s=1}^{m}[f(n_s x)-S_{\tau(s)}(n_s x)]\right|\leqq\lambda 2^n(\log 2^n)^{\gamma}\sum_{k=0}^{n-1}\frac{1}{(k+1)^2}\left(\frac{n-k}{n}\right)^{\gamma}, \tag{3.18}$$

provided that

$$x\notin\bigcup_{n=1}^{\infty}M_n,$$

where $m\left(\limsup_{n\to\infty} M_n\right)=0$.

If (3.13′) holds, then $-2<\gamma<0$ and from (3.18)

$$\left|\sum_{s=1}^{m}[f(n_s x)-S_{\tau(s)}(n_s x)]\right| \leqq \lambda 2^n(\log 2^n)^\gamma \sum_{k=0}^{n-1}\frac{1}{(k+1)^2}\left(\frac{n}{n-k}\right)^{|\gamma|}$$

$$=\lambda 2^n(\log 2^n)^\gamma\left(\sum_{k=0}^{[n/2]}+\sum_{k=[n/2]+1}^{n-1}\right)\frac{1}{(k+1)^2}\left(\frac{n}{n-k}\right)^{|\gamma|}$$

$$\leqq \lambda 2^n(\log 2^n)^\gamma\left\{\frac{n^{|\gamma|}}{(n/2)^{|\gamma|}}\sum_{k=0}^{\infty}\frac{1}{(k+1)^2}+\frac{n^{|\gamma|}}{\left(\frac{n}{2}+1\right)^2}\sum_{j=1}^{[n/2]}\frac{1}{j^{|\gamma|}}\right\}$$

$$\leqq A\lambda 2^n(\log 2^n)^\gamma < A\lambda m(\log m)^\gamma.$$

On the other hand if (3.13) holds, then $0<\gamma<2$ and

$$\left|\sum_{s=1}^{m}[f(n_s x)-S_{\tau(s)}(n_s x)]\right| \leqq \lambda 2^n(\log 2^n)^\gamma \sum_{k=0}^{n-1}\frac{1}{(k+1)^2}$$

$$\leqq A\lambda m(\log m)^\gamma.$$

Since $\lambda>0$ is an arbitrary positive number, in both cases (3.13) and (3.13), we obtain, for almost every x,

$$\lim_{m\to\infty}\frac{1}{m(\log m)^\gamma}\sum_{s=1}^{m}[f(n_s x)-S_{\tau(s)}(n_s x)]=0. \tag{3.19}$$

Thus from (3.19) and the convergence of (3.4), we conclude that (3.13) and (3.13′) imply (1.7).

Lastly we must consider the case (ii) $\alpha>\frac{5}{2}$ and $\gamma=-2$. But in this case we can apply the same methods as the above calculus, so that we omit it.

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