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ABSOLUTE SUMMABILITY FACTORS I.

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In this paper* some general theorems are proved which give absolute summability factors for a certain class of matrix summability methods. The theorems were designed primarily to include the earlier work done for Cesàro methods (by Fekete for Cesàro means of integral order [3], by Anderson, Chow, and Peyerimhoff for Cesàro means of real (nonnegative) order [1], [2], [7], respectively); the theorems do contain methods other than Cesàro. Some of the methods used to develop the proofs are adaptations of the techniques used by Jurkat and Peyerimhoff to prove the corresponding theorems for ordinary summability factors ([4]). Let $\sum_{\nu=0}^{\infty} a_\nu$ be an infinite series. The following notation will be used when operating on the series with triangular matrix summability methods $A=(a_{\mu\nu})$:

$$s_n=\sum_{\nu=0}^{n} a_\nu,\ \sigma_n=\sum_{\nu=0}^{n} a_{n\nu}s_\nu \qquad \text{(sequence-sequence form)},$$

$$\sigma_n=\sum_{\nu=0}^{n} \bar{a}_{n\nu}a_\nu,\ \bar{a}_{n\nu}=\sum_{\mu=\nu}^{n} a_{n\mu},\ \bar{a}_{n\nu}=0 \text{ for } \nu>n \qquad \text{(series-sequence form)}$$

and

$$\beta_n=\sigma_n-\sigma_{n-1}=\sum_{\nu=0}^{n} \hat{a}_{n\nu}a_\nu,\ \hat{a}_{n\nu}=0 \text{ if } \nu>n \qquad \text{(series-series form)}$$

$$\sigma_{-1}=0,\ \hat{a}_{n\nu}=\bar{a}_{n,\nu}-\bar{a}_{n-1,\nu} \text{ for } \nu\leqq n.$$

The notation $A\geqq 0$ will mean $a_{n\nu}\leqq 0$ for $\nu<n$.
A method $A=(a_{\mu\nu})$ is said to be normal if $a_{nn}\neq 0$ (and $a_{\mu\nu}=0$ for $\nu>\mu$ since A is triangular). Hence, A normal implies A' exists (A' denotes the inverse

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of A).

The method A is said to be absolutely regular if $\sum_{\nu=0}^{\infty} |a_\nu| < \infty$ implies

$$\sum_{\nu=0}^{\infty} |\beta_\nu| < \infty, \text{ and } \sum_{\nu=0}^{\infty} a_\nu = \sum_{\nu=0}^{\infty} \beta_\nu .$$

$|A| \supseteqq |B|$ if AB' is absolutely regular.

Necessary and sufficient conditions for a method to be absolutely regular, as given by Knopp and Lorentz [6], are

(1) $\sum_{\nu=\mu}^{\infty} |\hat{a}_{\nu\mu}| \geqq M$ uniformly with respect to μ

(2) $\sum_{\nu=\mu}^{\infty} \hat{a}_{\nu\mu} = 1$ (this condition is used only to preserve the sums).

A series $\sum_{\nu=0}^{\infty} a_\nu$ is said to be absolutely summable by the method A if $\sum_{\nu=0}^{\infty} |\beta_\nu| < \infty$ (we denote this by $\sum_{\nu=0}^{\infty} a_\nu \in |A|$). The notation $\lambda_\nu \in |A|_r$ will mean $\sum_{\nu=0}^{\infty} a_\nu \lambda_\nu \in |A|$ whenever $\sum_{\nu=0}^{\infty} a_\nu \in |A|$ (sequences with this property are called absolute Hardy-Bohr factors).

A triangular method A is said to have an absolute mean value theorem if;

(i) $\hat{A} \geqq 0$

(ii) $a_{nn} > 0$

(iii) $\dfrac{\hat{a}_{n+1,\nu}}{\hat{a}_{n,\nu}} \searrow$ as $\nu \nearrow$, $\nu \leqq n$.

Conditions (i), (ii) and (iii) imply $\hat{A'} \leqq 0$.

THEOREM 1. *Let A be normal and absolutely regular. Also assume A has an absolute mean value theorem, and that $\hat{a}_{n\nu} \nearrow$ as $\nu \nearrow$, $\nu \leqq n$. Then $\varepsilon_\nu \in |A|_r$ if and only if $\varepsilon_\nu = \sum_{\mu=\nu}^{\infty} \hat{a}_{\mu\nu} c_\mu$ where $c_\mu = O(1)$.*

PROOF. Note the hypothesis that A has an absolute mean value is not used in the proof of the necessity.

$$\text{If } \sum_{\nu=0}^{\infty} a_\nu \in |A|, \text{ then } a_\nu = \sum_{\mu=0}^{\nu} \hat{a}'_{\nu\mu} \beta_\mu \quad \text{where } \sum_{\mu=0}^{\infty} |\beta_\mu| < \infty.$$

By hypothesis $\sum_{\nu=0}^{\infty} a_\nu \varepsilon_\nu \in |A|$, i.e. if $\widehat{\beta}_n=\sum_{\nu=0}^{n}\hat{a}_{n\nu}a_\nu\varepsilon_\nu$, then $\sum_{\nu=0}^{\infty}|\widehat{\beta}_\nu|<\infty$.
Introducing the inverse transformation we have

$$\widehat{\beta}_n=\sum_{\nu=0}^{n}\hat{a}_{n\nu}\varepsilon_\nu\sum_{\mu=0}^{\nu}\hat{a}_{\nu\mu}\beta_\mu=\sum_{\mu=0}^{n}\beta_\mu\sum_{\nu=\mu}^{n}\hat{a}_{n\nu}\hat{a}'_{\nu\mu}\varepsilon_\nu\,.$$

So
$$\widehat{\beta}_n=\sum_{\mu=0}^{n}c_{n\mu}\beta_\mu \text{ with } c_{n\mu}=\sum_{\nu=\mu}^{n}\hat{a}_{n\nu}\hat{a}_{\nu\mu}\varepsilon_\nu\,.$$

Thus the method $(c_{n\mu})$ transforms every absolutely convergent series into an absolutely convergent series, so by Knopp-Lorentz [6] we have $\sum_{n=\mu}^{\infty}|c_{n\mu}|\leqq M$, uniformly with respect to μ.

Thus
$$\sum_{n=0}^{\infty}\widehat{\beta}_n=\sum_{\mu=0}^{\infty}\beta_\mu c_\mu, \text{ with } c_\mu=\sum_{n=\mu}^{\infty}c_{n\mu}.$$

Choose $a_k=1$ and $a_\nu=0$ if $\nu\neq k$. With this choice for a_n we have $\beta_\mu=\hat{a}_{\mu k}$, $\widehat{\beta}_n=\hat{a}_{nk}\varepsilon_k$. So

$$\sum_{n=0}^{\infty}\widehat{\beta}_n=\sum_{n=0}^{\infty}\hat{a}_{nk}\varepsilon_k=c\quad \varepsilon_k=\sum_{\mu=k}^{\infty}\hat{a}_{\mu k}c_\mu,$$

since A is triangular and absolutely regular.
To prove the sufficiency we need to show

$$\sum_{n=0}^{\infty}|\beta_n|<\infty \text{ implies } \sum_{n=0}^{\infty}|\widehat{\beta}_n|<\infty.$$

By Knopp-Lorentz (1) we need only show $\sum_{n=\mu}^{\infty}|c_{n\mu}|\leqq M$, i.e. just $\sum_{n=\mu}^{\infty}\left|\sum_{\nu=\mu}^{n}\hat{a}_{n\nu}\hat{a}'_{\nu\mu}\varepsilon_\nu\right|\leqq M$ uniformly with respect to μ.
Introducing the representation for ε_ν, and using the fact that $|c_\mu|\leqq M$, we have, after interchanging the order of summation,

$$\sum_{n=\mu}^{\infty}\left|\sum_{\nu=\mu}^{n}\hat{a}_{n\mu}\hat{a}'_{\nu\mu}\varepsilon_\nu\right|\leqq M\sum_{\lambda=\mu}^{\infty}\sum_{n=\lambda}^{\infty}\left|\sum_{\nu=\mu}^{\lambda}\hat{a}_{n\nu}\hat{a}'_{\nu\mu}\hat{a}_{\lambda\nu}\right|+M\sum_{n=\mu}^{\infty}\sum_{\lambda=n+1}^{\infty}\left|\sum_{\nu=\mu}^{n}\hat{a}_{n\nu}\hat{a}'_{\nu\mu}\hat{a}_{\lambda\nu}\right|.$$

If $\lambda>\nu$, then $\sum_{\nu=\mu}^{\lambda}\hat{a}_{\lambda\nu}\hat{a}'_{\nu\mu}=0$. Since $\hat{a}_{\lambda\nu}\geqq 0$, $\hat{a}'_{\mu\mu}>0$, $\hat{a}'_{\nu\mu}\leqq 0$ $(\mu<\nu)$ and $\hat{a}_{n\mu}\nearrow$ as $\nu\nearrow$,

$(\nu \leqq n,)$ we have (observe $n \geqq \lambda$) $\sum_{\nu=\mu}^{\lambda} \hat{a}_{n\nu}\hat{a}_{\nu\mu}\hat{a}_{\lambda\nu} \leqq 0$ if $\lambda > \mu$. Write

$$\sum_{\lambda=\mu}^{\infty}\sum_{n=\lambda}^{\infty}\left|\sum_{\nu=\mu}^{\lambda}\hat{a}_{n\nu}\hat{a}'_{\nu\mu}\hat{a}_{\lambda\nu}\right| = 1-\sum_{\lambda=\mu+1}^{\infty}\sum_{n=\lambda}^{\infty}\sum_{\nu=\mu}^{\lambda}\hat{a}_{n\nu}\hat{a}'_{\nu\mu}\hat{a}_{\lambda\nu} = 2-\sum_{\lambda=\mu}^{\infty}\sum_{n=\lambda}^{\infty}\sum_{\nu=\mu}^{\lambda}\hat{a}_{n\nu}\hat{a}'_{\nu\mu}\hat{a}_{\lambda\nu}$$

$$= 2-\sum_{n=\mu}^{\infty}\sum_{\nu=\mu}^{n}\hat{a}_{n\nu}\hat{a}'_{\nu\mu}\sum_{\lambda=\nu}^{n}\hat{a}_{\lambda\nu}.$$

Applying a similar argument to

$$\sum_{n=\mu}^{\infty}\sum_{\lambda=n+1}^{\infty}\left|\sum_{\nu=\mu}^{n}\hat{a}_{n\nu}\hat{a}'_{\nu\mu}\hat{a}_{\lambda\nu}\right|$$

we obtain

$$\sum_{n=\mu}^{\infty}\sum_{\lambda=n+1}^{\infty}\left|\sum_{\nu=\mu}^{n}\hat{a}_{n\nu}\hat{a}'_{\nu\mu}\hat{a}_{\lambda\nu}\right| \leqq 2-2\hat{a}_{\mu\mu}-\sum_{n=\mu}^{\infty}\sum_{\nu=\mu}^{n}\hat{a}_{n\nu}\hat{a}_{\nu\mu}\sum_{\lambda=n+1}^{\infty}\hat{a}_{\lambda\nu}$$

and have

$$\sum_{n=\mu}^{\infty}\left|\sum_{\nu=\mu}^{n}\hat{a}_{n\nu}\hat{a}_{\nu\mu}\varepsilon_{\nu}\right| \leqq 4M-2Ma_{\mu\mu}-M\sum_{n=\mu}^{\infty}\sum_{\nu=\mu}^{n}\hat{a}_{n\nu}\hat{a}_{\nu\mu}$$

$$=4M-2Ma_{\mu\mu}-M\sum_{n=\mu}^{\infty}\delta_{n.\nu}\leqq 3M.$$

So the condition is sufficient. Using a technique similar to the one used in the proof of Theorem 1 we have the following corollary.

COROLLARY 1. *Let A be normal, absolutely regular, $\hat{A}' \leqq 0$ and $a_{nn}>0$ (the last two hypotheses imply $\hat{A} \geqq 0$). If $\varepsilon_n = O(\hat{a}_{nn})$, $\hat{a}_{nn}\searrow$, then $\varepsilon_n \in |A|_r$.*

Theorem 1 gives absolute Hardy-Bohr factors for all Cesàro methods C_α with $0 \leqq \alpha \leqq 1$, since C_α has an absolute mean value theorem only when $0 \leqq \alpha \leqq 1$.

From *B* to *BP*. The notation $\varepsilon_\nu = \varepsilon_\nu(\hat{A}, c)$ will mean $\varepsilon_\nu = \sum_{\mu=\nu}^{\infty} a_{\mu\nu}c_\mu$ where $c_\mu = O(1)$. We now extend Theorem 1 to methods of the form BP, where B has absolute Hardy-Bohr factor $\varepsilon_\nu = \varepsilon_\nu(\hat{B}, c)$. If $s_n \in |B|$ implies $s_{n-1} \in |A|$ (i. e. $\{0, s_0, s_1, \cdots\} \in |A|$) we write $|A| \dot{\supseteq} |B|$. We state the following lemmas without proofs, the proofs being similar to those given by Jurkat and Peyerimhoff in [4].

LEMMA 1. *If A and B are absolutely regular, triangular, normal and $|A| \dot{\supseteq} |B|$, then given a bounded sequence $c=\{c_\mu\}$ there exists a bounded sequence $c'=\{c'_\mu\}$ such that $\varepsilon_\nu(\hat{A}, c) = \varepsilon_\nu(\hat{B}, c')$.*

LEMMA 2. *Let $A=BP$, P a weighted mean. Then*

$$[\varepsilon_{\nu+1}(\hat{A}, c)-\varepsilon_\nu(\hat{A}, c)]\frac{P_{\nu-1}}{p_\nu}=\varepsilon_\nu(\hat{A}, c)-\varepsilon_\nu(\hat{B}, c),\ \nu\geqq 0\ (P_{-1}=0).$$

If $A=BP$ then the following identity holds ($\nu\geqq 1$)

$$\frac{\hat{a}_{n\nu}}{P_{\nu-1}}-\frac{\hat{a}_{n,\nu+1}}{P_\nu}=\hat{b}_{n\nu}\frac{p_\nu}{P_\nu P_{\nu-1}}\quad (\textit{for } \nu=0 \textit{ it reduces to } \hat{a}_{n0}=\hat{b}_{n0}).$$

THEOREM 2. *Let $A=BP$ where $|A|\supseteq|B|$, $|A|\dot{\supseteq}|B|$ and P is a weighted mean. if $\varepsilon_\nu(\hat{B}, c)\in|B|_r$ for every c, then $\varepsilon_\nu(\hat{A}, c)\in|A|_r$ for every c.*

PROOF. By partial summation Lemmas 1 and 2 and the above identity we have

$$\sum_{\nu=1}^{n}\hat{a}_{n\nu}a_\nu\varepsilon_\nu(\hat{A}, c)=\sum_{\nu=1}^{n}\hat{b}_{n\nu}\varepsilon_\nu(B, c')\frac{p_\nu}{P_\nu P_{\nu-1}}\sum_{\mu=1}^{\nu}P_{\mu-1}a_\mu$$
$$+\sum_{\nu=2}^{n}\hat{a}_{n,\nu}\varepsilon_{\nu-1}(\hat{B}, c')\frac{p_{\nu-1}}{P_{\nu-2}P_{\nu-1}}\sum_{\mu=1}^{\nu-1}P_{\mu-1}a_\mu.$$

If $\sum_{\nu=0}^{\infty}a_\nu\in|A|$, then $\sum_{\nu=0}^{\infty}\frac{p_\nu}{P_\nu P_{\nu-1}}\sum_{\mu=0}^{\nu}P_{\mu-1}a_\mu\in|B|$, and since $\varepsilon_\nu(\hat{B},c')\in|B|_r$ we have $\sum_{n=1}^{\infty}\left|\sum_{\nu=1}^{n}\hat{b}_{n\nu}\varepsilon_\nu(\hat{B}, c')\frac{p_\nu}{P_\nu P_{\nu-1}}\sum_{\mu=1}^{\nu}P_{\mu-1}a_\mu\right|<\infty.$

$\sum_{n=2}^{\infty}\left|\sum_{\nu=2}^{n}\hat{a}_{n,\nu}\varepsilon_{\nu-1}(B, c')\frac{p_\nu}{P_{\nu-2}P_{\nu-1}}\sum_{\mu=1}^{\nu-1}P_{\mu-1}a_\mu\right|<\infty$ since $\varepsilon_\nu(\hat{B},c)\in|B|_r$ and $|A|\dot{\supseteq}|B|$, consequently $\varepsilon_\nu(\hat{A}, c)\in|A|_r$.

The Induction from to $AP^{\kappa-1}$ to AP^{κ}. By repeated application of Theorem 2, $\varepsilon_\nu(B\hat{P}^\kappa,c)\in|BP^\kappa|_r$ (for every c) if $\varepsilon_\nu(\widehat{BP}^{\kappa-1},c)\in|BP^{\kappa-1}|_r$,* $|BP^\kappa|\supseteq|BP^{\kappa-1}|$ and $|BP^\kappa|\dot{\supseteq}|BP^{\kappa-1}|$, $K=1, 2, \cdots$. The conditions $|BP^\kappa|\supseteq|BP^{\kappa-1}|$ ($\kappa=1, 2, \cdots$) are satisfied if $|BP|\supseteq|B|$, which is no restriction at all in case of Cesàro means since they commute (i.e. $C_\alpha C_1 C_{-\alpha}=C_1$, which is absolutely regular). The conditions $|BP^\kappa|\dot{\supseteq}|BP^{\kappa-1}|$ ($\kappa=1,2,\cdots$) are of a more complicated nature and require a more detailed analysis. The following Lemma gives a sufficient condition for $|BP|\dot{\supseteq}|B|$.

LEMMA 3. *Let P be a weighted mean, $A=BP$ and assume $|A|\supseteq|B|$, B absolutely regular. If $\frac{p_\nu}{P_{\nu-1}}\in|B|_r$* (i. e. $\left\{0, \frac{p_1}{P_0}, \frac{p_2}{P_1}, \cdots\right\}\in|B|_r$) *then $|A|\dot{\supseteq}|B|$.*

*for every c

PROOF. Take $s_n \in |B|$. To establish the lemma we must show $s_{n-1} \in |A|$. Let $\alpha_n = \frac{p_n}{P_n P_{n-1}} \sum_{\nu=1}^{n} P_{\nu-1} a_{\nu-1}$, $n \geqq 1$ (the $\hat{P}$-transform of $a_{\nu-1}$). Shifting the index and writing a complete transformation we have (for $n \geqq 1$)

$$\alpha_n = \frac{p_n}{P_n P_{n-1}} \sum_{\nu=1}^{n} P_{\nu-1} a_\nu + \frac{p_n}{P_n P_{n-1}} \sum_{\nu=1}^{n} \frac{p_\nu}{P_{\nu-1}} (P_{\nu-1} a_\nu) - \frac{p_n}{P_{n-1}} a_n + + \frac{a_0 p_0 p_n}{P_n P_{n-1}}.$$

Since $|A| \supseteq |B|$ and $\frac{p_\nu}{P_{\nu-1}}$ $|B|_r$, $\sum_{n=1}^{\infty} \frac{p_n}{P_n P_{n-1}} \sum_{\nu=1}^{n} P_{\nu-1} a_\nu$ and

$\sum_{n=1}^{\infty} \frac{p_n}{P_n P_{n-1}} \sum_{\nu=1}^{n} P_{\nu-1} \left(\frac{p_\nu}{P_{\nu-1}} a_\nu \right) \in |B|$.

$\sum_{n=1}^{\infty} \frac{p_n}{P_{n-1}} a_n \in |B|$ since $\frac{p_n}{P_{n-1}} \in |B|_r$. $\sum_{n=1}^{\infty} \frac{a_0 p_0 p_n}{P_n P_{n-1}} \in |B|$ since the series is absolutely convergent. Thus $s_{n-1} \in |A|$.

The conditions $|BP^{\kappa-1}| \dot{\subseteq} |BP^{\kappa}|$ $(\kappa = 1, 2, \cdots)$ have now been replaced by the conditions $\frac{P}{P_{\nu-1}} \in |BP^{\kappa}|_r$ $(\kappa = 0, 1, \cdots)$. The following lemma will allow us to reduce all the conditions to the matrix B.

LEMMA 4. *Let P be a weighted mean, $A = BP$. Also assume* $\left\{ 0, \frac{p_1}{P_0}, \frac{p_2}{P_1}, \cdots \right\}$ $\in |B|_r$, $|BP| \supseteq |B|$, $\lambda_\nu \in |B|_r$ *and* $\frac{P_{\nu-1}}{P_\nu} (\lambda_{\nu+1} - \lambda_\nu) \in |B|_r$. *When these conditions are satisfied* $\lambda_\nu \in |A|_r$.

PROOF. Take $\sum_{\nu=0}^{\infty} a_\nu \in |A|$. By partial summation and, using the identity used in the proof of Theorem 2 we have

$$\sum_{\nu=1}^{n} \hat{a}_{n\nu} a_\nu \lambda_\nu = \sum_{\nu=1}^{n} \hat{b}_{n\nu} \lambda_\nu \frac{p_\nu}{P_\nu P_{\nu-1}} \sum_{\mu=0}^{\nu} P_{\mu-1} a_\mu$$
$$- \sum_{\nu=2}^{n} \hat{a}_{n\nu} (\lambda_\nu - \lambda_{\nu-1}) \frac{p_{\nu-2}}{P_{\nu-1}} \left(\frac{p_{\nu-1}}{P_{\nu-1} P_{\nu-2}} \sum_{\mu=1}^{\nu-2} P_{\mu-1} a_\mu \right).$$

Since $\sum_{\nu=0}^{\infty} a_\nu \in |A|$, $\sum_{\nu=0}^{\infty} \frac{p_\nu}{P_\nu P_{\nu-1}} \sum_{\mu=0}^{\nu} P_{\mu-1} a_\mu \in |B|$. By hypothesis $\lambda_\nu \in |B|_r$, so $\sum_{n=1}^{\infty} \left| \sum_{\nu=1}^{n} \hat{b}_{n\nu} \lambda_\nu \frac{p_\nu}{P_\nu P_{\nu-1}} \sum_{\mu=0}^{\nu} P_{\mu-1} a_\mu \right| < \infty$. The hypothesis $\frac{p_\nu}{P_{\nu-1}} \in |B|_r$ (i.e. $\left\{ 0, \frac{p_1}{P_0}, \frac{p_2}{P_1}, \cdots \right\}$ $\in |B|_r$) insures (by Lemma 3) us that $|A| \supseteq |B|$. Also by hypothesis

$\frac{P_{\nu-1}}{P_\nu}(\lambda_{\nu+1}-\lambda_\nu) \in |B|_r$, so $\sum_{n=2}^{\infty}\left|\sum_{\nu=2}^{n}\hat{a}_{n\nu}(\lambda_\nu-\lambda_{\nu-1})\frac{P_{\nu-2}}{p_{\nu-1}}\left(\frac{p_{\nu-1}}{P_{\nu-1}P_{\nu-2}}\sum_{\mu=1}^{\nu-1}P_{\mu-1}a_\mu\right)\right|$.
Hence $\lambda_\nu \in |A|_r$.

The following notation facilitates the use of Lemma 4 inductively. Let

$$\left(\frac{P_{n-1}}{p_n}\cdot\Delta\right)^0\lambda_n=\lambda_n \text{ and } \left(\frac{P_{n-1}}{p_n}\cdot\Delta\right)^s\lambda_n=\frac{P_{n-1}}{p_n}\left[\left(\frac{P_n}{p_{n+1}}\cdot\Delta\right)^{s-1}\lambda_{n+1}-\left(\frac{p_{n-1}}{p_n}\cdot\Delta\right)^{s-1}\lambda_n\right].$$

LEMMA 5. *If* $\left(\frac{P_{\nu-1}}{p_\nu}\cdot\Delta\right)^s\frac{p_\nu}{P_{\nu-1}} \in |B|_r$ (i. e. $\left\{0,\left(\frac{P_0}{p_1}\cdot\Delta\right)^s\frac{p_1}{P_0},\cdots\right\} \in |B|_r$), $s=0,1,\cdots,\kappa-1$, *and if* $|BP|\supseteq|B|$ *then* $|BP^m|\supseteq|BP^{m-1}|$ $(m=1,2,\cdots,\kappa)$.

PROOF. This follows immediately by repeated application of Lemma 4. The main result of the paper now follows immediately by Lemma 5 and Theorem 2.

THEOREM 3. *If* $\varepsilon_\nu(\hat{B},c)\in|B|_r$ *(for every c), P a weighted mean,* $|BP|\supseteq|B|$ *and* $\left(\frac{P_{\nu-1}}{p_\nu}\cdot\Delta\right)^s\frac{p_\nu}{P_{\nu-1}} \in |B|_r$ $(s=0,\ 1,\cdots,\ \kappa-1)$ *then* $\varepsilon_\nu(\widehat{BP^\kappa},\ c)\in$ $|BP^\kappa|_r$ *(for every c).*

To get absolute Hardy-Bohr factors for $C_\alpha, \alpha\geqq 0$ we apply Theorems 1 and 3, Corollary 1 and Lemma 1 in the following manner. Let $P=C_1$ and $\alpha=\theta+k$, where $0<\theta\leqq 1$ and $k\geqq 0$ (integral). Take $B=C_\theta$. By Theorem 1 $\varepsilon_\nu(\widehat{C_\theta},\ c)\in|C_\theta|_r$ for every c. Also $|C_\alpha|\approx|C_\theta C_k|\approx|C_\theta C_1^\kappa|$ (see [6] for the equivalence). For $P=C_1$ the expression $\left(\frac{P^{\nu-1}}{p_\nu}\cdot\Delta\right)^s\frac{p_\nu}{P_{\nu-1}}$ reduces to $(\nu,\Delta)^s\frac{1}{\nu}$. A short calculation shows $(\nu,\Delta)^s\frac{1}{\nu}=O\left(\frac{1}{\nu}\right)$ thus by Corollary 1 $(\nu,\Delta)^s\frac{1}{\nu}\in|C_\theta|_r$. Applying Theorem 3 we have $\varepsilon_\nu(\widehat{C_\theta C_1^\kappa},\ c)\in|C_\theta C_1^\kappa|_r$, or equivalently $\varepsilon_\nu(\widehat{C_\theta C_1^\kappa},c)\in|C_\alpha|_r$. By Lemma 1 $\varepsilon_\nu(\widehat{C_\theta C_1^\kappa},\ c)=\varepsilon_\nu(\widehat{C_\alpha},\ c')$. Thus $\varepsilon_\nu(\widehat{C_\alpha},\ c')\in|C_\alpha|_r$.

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