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HARDY-BOHR THEOREMS[1]

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The following results are a generalization of results obtained recently by Jurkat and Peyerimhoff [8]. In this paper we will be concerned exclusively with triangular summability methods. Let $A = (a_{n\nu})$ be the triangular matrix associated with the method A. Given a series $\sum a_n$ we use the notation

$$s_n = \sum_{\nu=0}^{n} a_\nu, \ \sigma_n = \sum_{\nu=0}^{n} a_{n\nu} s_\nu = \sum_{\nu=0}^{n} \bar{a}_{n\nu} a_\nu$$

$$\beta_n = \sigma_n - \sigma_{n-1} = \sum_{\nu=0}^{n} \hat{a}_{n\nu} a_\nu \ \ (n \geqq 0, \ \sigma_{-1}=0)$$

where the relations between the matrices A, $\bar{A}$ and $\hat{A}$ are

$$(1) \qquad \bar{a}_{n\nu} = \sum_{\mu=\nu}^{n} a_{n\mu}, \ \ \hat{a}_{n\nu} = \bar{a}_{n\nu} - \bar{a}_{n-1,\nu} = \sum_{\mu=\nu}^{n} (a_{n\mu} - a_{n-1,\mu})$$

$$\bar{a}_{n\nu} = \hat{a}_{n\nu} = 0 \text{ if } \nu > n, \ \ \bar{a}_{-1,\nu} = 0.$$

DEFINITION 1. A summability method $A = (a_{n\nu})$ is said to be normal if $a_{nn} \neq 0$. In this case the inverse matrix exists and is denoted by $A' = (a'_{n\nu})$.

DEFINITION 2. If σ_n converges, then we say that $\sum a_n \in A$.

DEFINITION 3. If $\sum_{n=0}^{\infty} |\beta_n| < \infty$, then we use the notation $\sum a_n \in |A|$.

DEFINITION 4. If a sequence $\{\epsilon_n\}$ is such that $\sum a_n \epsilon_n \in A$ whenever $\sum a_n \in A$, we say that $\epsilon_n \in A_r$.

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1) This paper constitutes a significant part of a thesis for the Doctor of Philisophy degree in Mathematics, presented to the faculty, Department of Mathematics, University of Utah.

DEFINITION 5. If a sequence $\{\epsilon_n\}$ is such that $\sum a_n\epsilon_n \in |A|$ whenever $\sum a_n \in A$, then we use the notation $\epsilon_n \in (A, |A|)_r$.

Part 1 contains a theorem which gives, for a certain class of matrices, necessary and sufficient conditions for a sequence $\{\epsilon_n\}$ to be such that $\epsilon_n \in (A, |A|)_r$. In part 2 we show that under certain restrictions on the methods B and P, P being a weighted arithmetical mean, factors can be obtained for the method BP if factors are known for B. In part 3 we outline an induction argument which gives factors for BP^k when factors are known for B. We thus obtain as a special case the results for the Cesàro means which were obtained by Bosanquet and Chow [4], and Peyerimhoff [12], in the special case $\epsilon_n \in (C_k, |C_k|)_r$, $k > 0$. Here we use the fact that $C_k \approx H_k$ and $|C_k| \approx |H_k|$ $(k \geqq 0)$, where H_k denotes the Hölder mean of order k. Finally, in part 4, we consider applications of the proceeding results to several well-known means.

1. Before proceeding with the main result of this section it is convenient to prove a few preliminary lemmas.

DEFINITION 6. If a method A has the property that given integers n,m with $n \geqslant m$, there exists an integer ρ and a constant K depending only on the matrix A such that

$$(2) \qquad \left|\sum_{\nu=0}^{m} a_{n\nu}s_\nu\right| \leqq K\left|\sum_{\nu=0}^{\rho} a_{\rho\nu}s_\nu\right| \quad (0 \leqslant \rho \leqslant m \leqslant n)$$

then A is said to have a mean value theorem.

LEMMA 1. *If A has a mean value theorem and if $s_n \in A$, then $a_{nn}s_n = O(1)$.*

PROOF.[2] We simply note that

$$a_{nn}s_n = \sum_{\nu=0}^{n} a_{n\nu}s_\nu - \sum_{\nu=0}^{n-1} a_{n\nu}s_\nu,$$

hence

$$|a_{nn}s_n| \leqq \left|\sum_{\nu=0}^{n} a_{n\nu}s_\nu\right| + \left|\sum_{\nu=0}^{n-1} a_{n\nu}s_\nu\right| \leqq (1+K)\sup_n |\sigma_n| < \infty.$$

LEMMA 2. *If for $A = (a_{n\nu})$ we have* (i) $a_{n\nu} \searrow 0$ *as* $n \nearrow \infty$ *and* (ii)

2) This proof has been given earlier by Peyerimhoff [10].

$\sum_{\nu=0}^{n} a_{n\nu} = 1$ $(n \geqslant 0)$, *then it follows that (a)* $\hat{a}_{n\nu} \nearrow$ *as* $\nu \nearrow (\nu \leqq n,\ n \geqq 1)$, *(b)* $\sum_{n=\nu}^{\infty} \hat{a}_{n\nu} = 1$, *(c)* $\sum_{n=\nu}^{\infty} |\hat{a}_{n\nu} - \hat{a}_{n,\nu+1}| = 2a_{\nu\nu}$, *and* (d) $\hat{a}_{n0} = 0 (n > 0)$.

PROOF. (a) Using formula (1) we have

$$\hat{a}_{n\nu} - \hat{a}_{n,\nu+1} = a_{n\nu} - a_{n-1,\nu} \leqslant 0.$$

(b)
$$\sum_{n=\nu}^{\infty} \hat{a}_{n\nu} = \lim_{N\to\infty} \sum_{n=\nu}^{N} \sum_{\alpha=\nu}^{n} (a_{n\alpha} - a_{n-1,\alpha})$$

$$= \lim_{N\to\infty} \sum_{\alpha=\nu}^{N} \sum_{n=\alpha}^{N} (a_{n\alpha} - a_{n-1,\alpha}) = \lim_{N\to\infty} \sum_{\alpha=\nu}^{N} a_{N\alpha}$$

$$= \lim_{N\to\infty} \left(\sum_{\alpha=0}^{N} a_{N\alpha} - \sum_{\alpha=0}^{\nu-1} a_{N\alpha} \right) = 1 - 0 = 1$$

(c) Using part (a) we see that

$$\sum_{n=\nu}^{\infty} |\hat{a}_{n\nu} - \hat{a}_{n,\nu+1}| = a_{\nu\nu} - \sum_{n=\nu+1}^{\infty} (a_{n\nu} - a_{n-1,\nu}) = 2a_{\nu\nu}.$$

(d)
$$\hat{a}_{n0} = \sum_{\nu=0}^{n} (a_{n\nu} - a_{n-1,\nu}) = 1 - 1 = 0 \, (n > 0).$$

LEMMA 3. *If* $b_\nu \nearrow$, $b_0 = 0$, *and if A has a mean value theorem, then*

$$\left| \sum_{\nu=0}^{\rho} b_\nu a_{\rho\nu} s_\nu \right| \leqq 2K\, b_\rho \sup_n |\sigma_n| \quad (K' = \text{Max}\ (K, 1)).$$

PROOF. By a partial summation we have

$$\sum_{\nu=0}^{\rho} b_\nu a_{\rho\nu} s_\nu = \sum_{\nu=0}^{\rho-1} \Delta b_\nu \sum_{\mu=0}^{\nu} a_{\rho\mu} s_\mu + b_\rho \sigma_\rho$$

and hence we obtain the estimation

$$\left| \sum_{\nu=0}^{\rho} b_\nu a_{\rho\nu} s_\nu \right| \leqq K' \sup_n |\sigma_n| \left\{ -\sum_{\nu=0}^{\rho-1} \Delta b_\nu + b_\rho \right\}$$

$$\leqq 2K' b_\rho \sup |\sigma_n|.$$

In the following theorem we will use the notation

$$\epsilon_\nu(\bar{A},\alpha) = \sum_{n=\nu}^{\infty} \alpha_n \bar{a}_{n\nu}, \text{ where } \alpha = \{\alpha_n\} \text{ with} \sum_{n=0}^{\infty} |\alpha_n| < \infty.$$

In order that $\epsilon_\nu(\bar{A}, \alpha)$ always exist we assume that A has bounded columns, which is always the case when A is regular. $\epsilon_\nu(A, \alpha)$ is defined similarly, and we have the obvious identity

$$(3) \qquad \epsilon_\nu(A, \alpha) = \Delta\epsilon_\nu(\bar{A}, \alpha).$$

THEOREM 1. *Suppose the method A has the properties*

$$\text{(i)} \qquad a_{n\nu} > 0 \ (\nu \leqq n)$$

$$\text{(ii)} \qquad a_{n\nu} \searrow 0 \ \text{as } n \nearrow \infty$$

$$\text{(iii)} \qquad \frac{a_{\rho\nu}}{a_{n\nu}} \hat{a}_{n\nu} \nearrow \text{as } \nu \nearrow (\rho > n)$$

$$\text{(iv)} \qquad \sum_{\nu=0}^{n} a_{n\nu} = 1$$

(v) *Mean value theorem* (2).

Then necessary and sufficient conditions for a sequence $\{\epsilon_\nu\}$ to be such that $\epsilon_\nu \in (A, |A|)_r$ are

$$\text{(a)} \qquad \epsilon_\nu = \epsilon_\nu(\bar{A}, \alpha),$$

$$\text{(b)} \qquad \sum_{\nu=0}^{\infty} |\epsilon_\nu| < \infty .$$

Before proceeding with the proof it is important to note that conditions (i), (ii) and (iv) imply both regularity and absolute regularity.

PROOF. For the sufficiency we must show that $\sum_{n=0}^{\infty} |\beta_n| < \infty$ whenever $\sum a_n \in A$, where $\beta_n = \sum_{\nu=0}^{n} \hat{a}_{n\nu} a_\nu \epsilon_\nu$ with $\{\epsilon_\nu\}$ satisfying (a) and (b). A partial summation and an application of (3) gives us the formula

$$\beta_n = \sum_{\nu=0}^{n} \hat{a}_{n\nu} \epsilon_\nu(A, \alpha) s_\nu + \sum_{\nu=0}^{n} (\hat{a}_{n\nu} - \hat{a}_{n,\nu+1}) \epsilon_{\nu+1}(\bar{A}, \alpha) s_\nu$$

$$= \beta_n^{(1)} + \beta_n^{(2)}.$$

First we consider $\beta_n^{(2)}$.

$$\sum_{n=0}^{\infty} |\beta_n^{(2)}| \leqq \sum_{\nu=0}^{\infty} |\epsilon_{\nu+1}(\bar{A}, \alpha)s_\nu| \sum_{n=\nu}^{\infty} |\hat{a}_{n\nu} - \hat{a}_{n,\nu+1}|$$

An application of Lemma 2(c) and Lemma 1 gives

$$\sum_{n=0}^{\infty} |\beta_n^{(2)}| \leqq 2 \sum_{\nu=0}^{\infty} |\epsilon_{\nu+1}(\bar{A},\alpha)| \cdot |a_{\nu\nu}s_\nu| \leqslant 2M \sum_{\nu=0}^{\infty} |\epsilon_{\nu+1}(\bar{A}, \alpha)| < \infty.$$

To establish the absolute convergence of $\beta_n^{(1)}$ we use the representation for $\epsilon_\nu(A, \alpha)$.

$$\beta_n^{(1)} = \sum_{\nu=0}^{n} \hat{a}_{n\nu}s_\nu \sum_{\rho=\nu}^{n} a_{\rho\nu}\alpha_\rho + \sum_{\nu=0}^{n} \hat{a}_{n\nu}s_\nu \sum_{\rho=n+1}^{\infty} a_{\rho\nu}\alpha_\rho$$

$$= \sum_{\rho=0}^{n} \alpha_\rho \sum_{\nu=0}^{\rho} \hat{a}_{n\nu}a_{\rho\nu}s_\nu + \sum_{\rho=n+1}^{\infty} \alpha_\rho \sum_{\nu=0}^{n} \left(\frac{a_{\rho\nu}}{a_{n\nu}} \hat{a}_{n\nu}\right) a_{n\nu}s_\nu.$$

This gives the estimation

$$\sum_{n=0}^{\infty} |\beta_n^{(1)}| \leqq \sum_{n=0}^{\infty} \sum_{\rho=0}^{n} |\alpha_\rho| \left|\sum_{\nu=0}^{\rho} \hat{a}_{n\nu} a_{\rho\nu}s_\nu\right| + \sum_{n=0}^{\infty} \sum_{\rho=n+1}^{\infty} |\alpha_\rho| \left|\sum_{\nu=0}^{n} \left(\frac{a_{\rho\nu}}{a_{n\nu}} \hat{a}_{n\nu}\right) a_{n\nu}s_\nu\right|.$$

Because of conditions (i)$-$(iv) and Lemma 2 we may apply Lemma 3 to both summations, thus

$$\sum_{n=0}^{\infty} \left|\beta_n^{(1)}\right| \leqq 2K' \sup_n |\sigma_n| \left\{\sum_{n=0}^{\infty} \sum_{\rho=0}^{n} |\alpha_\rho| \hat{a}_{n\rho} + \sum_{n=0}^{\infty} \sum_{\rho=n+1}^{\infty} |\alpha_\rho| \frac{\hat{a}_{nn}a_{\rho n}}{a_{nn}}\right\}$$

$$\leqq O(1) \left\{\sum_{\rho=0}^{\infty} |\alpha_\rho| \sum_{n=\rho}^{\infty} \hat{a}_{n\rho} + \sum_{\rho=1}^{\infty} |\alpha_\rho| \sum_{n=0}^{\rho-1} a_{\rho n}\right\} < \infty.$$

Hence the conditions (a) and (b) imply $\epsilon_\nu \in (A, |A|)_r$.

Now suppose that $\epsilon_\nu \in (A, |A|)_r$ and consider condition (b). Our assumption is then that the convergence of σ_n implies $\sum_{n=0}^{\infty} |\beta_n| < \infty$. Since $\bar{A}$ is normal we can introduce the inverse matrix and write

$$\beta_n = \sum_{\nu=0}^{n} \hat{a}_{n\nu}\epsilon_\nu a_\nu = \sum_{\nu=0}^{n} \hat{a}_{n\nu}\epsilon_\nu \sum_{\mu=0}^{\nu} \bar{a}'_{\nu\mu}\sigma_\mu$$

$$= \sum_{\mu=0}^{n} \sigma_\mu \sum_{\nu=\mu}^{n} \hat{a}_{n\nu}\bar{a}'_{\nu\mu}\epsilon_\nu = \sum_{\mu=0}^{n} A_{n\mu}\sigma_\mu$$

where $A_{n\mu} = \sum_{\nu=\mu}^{n} \hat{a}_{n\nu}\bar{a}'_{\nu\mu}\epsilon_\nu$. With this interpretation the matrix $(A_{n\mu})$ has the property that it transforms every convergent sequence (hence every null sequence) into an absolutely convergent series. A theorem of Chow and Peyerimhoff [11] gives the necessary condition $\sum_{n=0}^{\infty} |A_{nn}| = \sum_{n=0}^{\infty} |\epsilon_n| < \infty$.
Thus the condition (b) is necessary.

To show the necessity of (a) we first observe that $\epsilon_\nu \in (A, |A|)_r$ implies $\epsilon_\nu \in A_r$. Peyerimhoff [10] has shown that when A is normal and regular it is then necessary that $\epsilon_\nu = c + \epsilon_\nu(\bar{A}, \alpha)$. The necessity of condition (b) and the sufficiency of conditions (a) and (b) imply that $c = 0$, hence

$$\epsilon_\nu = \epsilon_\nu(\bar{A}, \alpha) = \sum_{n=\nu}^{\infty} \alpha_n \bar{a}_{n\nu}, \quad \sum_{n=0}^{\infty} |\alpha_n| < \infty.$$

2. In what follows we will generalize Theorem 1 in such a manner that the Cesàro means C_β $(\beta > 0)$ will be included in this generalization. Theorem 1 breaks down for $\beta > 1$ since C_β no longer has a mean value theorem, however the conclusion remains valid.

Let P denote a weighted arithmetical mean with $P_{n\nu} = p_\nu/P_n$, where $P_n = p_0 + p_1 + \cdots p_n$, $p_\nu > 0$, $P_n \to \infty$ and $p_n/P_{n-1} = O(1)$. We shall consider the method $A = BP$.

LEMMA 4. *Let A and B be normal, $\bar{A}$ and $\bar{B}$ have bounded columns, and $B \subseteq A$. Then given $\alpha = \{\alpha_n\}, \sum_{n=0}^{\infty} |\alpha_n| < \infty$ there exists $\beta = \{\beta_n\}, \sum_{n=0}^{\infty} |\beta_n| < \infty$ such that $\epsilon_n(\bar{A},\alpha) = \epsilon_n(\bar{B}, \beta)$.*

LEMMA 5. *Let $A = BP$, $\bar{A}$ and $\bar{B}$ have bounded columns, then*

$$\epsilon_n(\bar{A}, \alpha) - \epsilon_n(\bar{B}, \alpha) = [\epsilon_{n+1}(\bar{A}, \alpha) - \epsilon_n(\bar{A}, \alpha)] \frac{P_{n-1}}{P_n} \ (n \geqq 1).$$

The preceding two lemmas have been proved by Jurkat and Peyerimhoff [8]. We omit the proofs here.

DEFINITION 7. If $s_n \in |B|$ implies $s_{n-1} \in |A|$, we say that $|B| \dot{\subseteq} |A|$.

LEMMA 6. *If A is regular, then*

$$\sum_{n=0}^{\infty} |\epsilon_n(A, \alpha)| < \infty.$$

PROOF. In view of the regularity we have

$$\sum_{n=0}^{\infty} |\epsilon_n(A, \alpha)| \leqq \sum_{n=0}^{\infty} \sum_{\nu=n}^{\infty} |\alpha_\nu| |a_{\nu n}| \sum_{\nu=0}^{\infty} |\alpha_\nu| \sum_{n=0}^{\nu} |a_{\nu n}| < \infty.$$

LEMMA 7. *Let $A = BP$ and suppose B is regular. Then* $\sum_{n=1}^{\infty} \frac{p_n}{P_{n-1}} |\epsilon_n(\overline{B},\alpha)| < \infty$ *if and only if* $\sum_{n=1}^{\infty} \frac{p_n}{P_{n-1}} |\epsilon_n(\overline{A}, \alpha)| < \infty$, *and* $\sum_{n=0}^{\infty} |\epsilon_n(\overline{A}, \alpha)| < \infty$ *implies* $\sum_{n=1}^{\infty} \frac{p_n}{P_{n-1}} |\epsilon_n(\overline{B},\alpha)| < \infty.$

PROOF. These results follow from Lemmas 5 and 6 since

$$\epsilon_n(\overline{A},\alpha) - \epsilon_{n+1}(\overline{A}, \alpha) = \epsilon_n(A, \alpha) = \frac{p_n}{P_{n-1}} [\epsilon_n(\overline{B}, \alpha) - \epsilon_n(\overline{A}, \alpha)].$$

THEOREM 2. *Let B be regular and set $A = BP$. If*

(i) $B \subseteq A$

(ii) $|B| \subseteq |A| \subseteq |AP|$

(iii) $|B| \dot{\subseteq} |B|$, $|A| \dot{\subseteq} |A|$

(iv) $\sum_{n=0}^{\infty} |\epsilon_n(\overline{B}, \alpha)| < \infty$ *implies* $\epsilon_n(\overline{B}, \alpha) \in (B, |B|)_r$

(v) $\sum_{n=1}^{\infty} \frac{p_n}{P_{n-1}} |\epsilon_n(\overline{B}, \alpha)| < \infty$ *implies* $\epsilon_n(\overline{B}, \alpha) \in (B, |A|)_r$

then also

(a) $\sum_{n=0}^{\infty} |\epsilon_n(\overline{A}, \alpha)| < \infty$ *implies* $\epsilon_n(\overline{A}, \alpha) \in (A, |A|)_r$

(b) $\sum_{n=1}^{\infty} \frac{p_n}{P_{n-1}} |\epsilon_n(\overline{A}, \alpha)| < \infty$ *implies* $\epsilon_n(\overline{A}, \alpha) \in (A, |AP|)_r$.

PROOF.

$$\sum_{\nu=0}^{n} \hat{a}_{n\nu} a_\nu \epsilon_\nu(\overline{A}, \alpha) = \sum_{\nu=0}^{n} \frac{\hat{a}_{n\nu} \epsilon_\nu(\overline{A}, \alpha)}{P_{\nu-1}} P_{\nu-1} a_\nu \quad (P_{-1}/P_{-1} = 1)$$

$$= \sum_{\nu=0}^{n} \Delta\left(\frac{\hat{a}_{n\nu}\epsilon_\nu}{P_{\nu-1}}\right) \sum_{\mu=0}^{\nu} P_{\mu-1} a_\mu$$

$$= \sum_{\nu=0}^{n} \epsilon_\nu \Delta\left(\frac{\hat{a}_{n\nu}}{P_{\nu-1}}\right) \sum_{\mu=0}^{\nu} P_{\mu-1} a_\mu + \sum_{\nu=0}^{n} \frac{\hat{a}_{n,\nu+1}}{P_\nu} \Delta\epsilon_\nu \sum_{\mu=0}^{\nu} P_{\mu-1} a_\mu .$$

Using Lemma 5 and the relation $\hat{a}_{n\nu} = P_{\nu-1} \sum_{\mu=\nu}^{n} \frac{\hat{b}_{n\mu} p_\mu}{P_\mu P_{\mu-1}}$ gives the formula

$$\text{(4)} \qquad \sum_{\nu=0}^{n} \hat{a}_{n\nu} a_\nu \epsilon_\nu(\overline{A}, \alpha) = \sum_{\nu=0}^{n} \hat{b}_{n\nu} \epsilon_\nu(\overline{A}, \alpha) \hat{P}_\nu(a_k)$$

$$- \sum_{\nu=0}^{n} \hat{a}_{n\nu} \epsilon_{\nu-1}(\overline{A}, \alpha) \hat{P}_{\nu-1}(a_k) + \sum_{\nu=0}^{n} \hat{a}_{n\nu} \epsilon_{\nu-1}(\overline{B}, \alpha) \hat{P}_{\nu-1}(a_k)$$

$$= \beta_n^{(1)} + \beta_n^{(2)} + \beta_n^{(3)} .$$

Here $\hat{P}_\nu(a_k)$ denotes $\sum_{\mu=0}^{\nu} \hat{p}_{\nu\mu} a_\mu = \sum_{\mu=0}^{\nu} \frac{p_\nu P_{\mu-1}}{P_\nu P_{\nu-1}} a_\mu$ ($\hat{P}_{-1}(a_k) = 0$). Replacing $\hat{A}$ and $\hat{B}$ by $\widehat{AP}$ and $\hat{A}$ respectively gives

$$\text{(5)} \qquad \sum_{\nu=0}^{n} (\widehat{AP})_{n\nu} a_\nu \epsilon_\nu(\overline{A}, \alpha) = \sum_{\nu-0}^{n} \hat{a}_{n\nu} \epsilon_\nu(\overline{A}, \alpha) \hat{P}_\nu(a_k)$$

$$- \sum_{\nu=0}^{n} (\widehat{AP})_{n\nu} \epsilon_{\nu-1}(\overline{A}, \alpha) \hat{P}_{\nu-1}(a_k) + \sum_{\nu=0}^{n} (\widehat{AP})_{n\nu} \epsilon_{\nu-1}(\overline{B}, \alpha) \hat{P}_{\nu-1}(a_k)$$

$$= \alpha_n^{(1)} + \alpha_n^{(2)} + \alpha_\mu^{(3)} .$$

To prove part (a), consider (4) and suppose $\sum_{n=0}^{\infty} |\epsilon_n(\overline{A}, \alpha)| < \infty$. By Lemma 4, $\epsilon_n(\overline{A}, \alpha) = \epsilon_n(\overline{B}, \beta)$ and $\sum_{n=0}^{\infty} |\epsilon_n(\overline{B}, \beta)| < \infty$. If $\sum a_n \in A$, then $\sum \hat{P}_\nu(a_k) \in B$, hence (iv) implies $\sum \epsilon_\nu(\overline{A}, \alpha) \hat{P}_\nu(a_k) \in |B|$ and $\sum_{n=0}^{\infty} |\beta_n^{(1)}| < \infty$. Similarly, $\sum a_n \in A$ implies $\sum \epsilon_\nu(\overline{A}, \alpha) \hat{P}_\nu(a_k) \in |B|$, hence (iii) and (ii) give $\sum \epsilon_{\nu-1}(\overline{A}, \alpha) \hat{P}_{\nu-1}(a_k) \in |A|$, and $\sum_{n=0}^{\infty} |\beta_n^{(2)}| < \infty$. Finally, Lemma 7 implies $\sum_{n=1}^{\infty} \frac{p_n}{P_{n-1}} |\epsilon_n(\overline{B}, \alpha)| < \infty$, hence (v) and (iii) give $\sum \epsilon_{\nu-1}(\overline{B}, \alpha) \hat{P}_{\nu-1}(a_k) \in |A|$ and $\sum_{n=0}^{\infty} |\beta_n^{(3)}| < \infty$. Thus we have $\epsilon_n(\overline{A}, \alpha) \in (A, |A|)_r$. Part (b) is proved in a similar manner using (5).

3. In order to proceed by induction from B to BP^k (k a positive integer), where P is as usual a positive, regular weighted mean with $p_n/P_{n-1} = O(1)$, we must guarantee that

(6) $$B \subseteq BP \subseteq \cdots \subseteq BP^k$$

(7) $$|B| \subseteq |BP| \subseteq \cdots \subseteq |BP^k|$$

and

(8) $$|BP^i| \dot{\subseteq} |BP^i| \quad (i = 0, 1, \cdots, k).$$

In addition, it is necessary to show that condition (v) of Theorem 2 holds. If we assume the above and suppose that B satisfies the conditions of Theorem 1, then we may assert that $\epsilon_n \in (BP^k, |BP^k|)_r$ if and only if

$$\epsilon_n = \epsilon_n(\overline{BP}^k, \alpha) \text{ and} \sum_{n=0}^{\infty} |\epsilon_n| < \infty.$$

Obviously (6) and (7) are satisfied if we only require $B \subseteq BP$ and $|B| \subseteq |BP|$.

If we set $B = C_\beta$ $(0 < \beta \leqq 1)$ and $P = C_1$, then (6), (7), and (8) all hold and B satisfies the conditions of Theorem 1. For condition (5) of Theorem 2 the reader is referred to Bosanquet and Chow [4], Theorem B. The conditions given by Bosanquet and Chow are different from ours, however they are a consequence of ours, as has been proved by Bosanquet and Tatchell [5], Theorems 4 and 5(a). Finally, for $P = C_1$, we have $p_n/P_{n-1} = 1/n$, hence we have the desired result.

4. As applications of Theorem 1 we state the following results without proof.

If P denotes, as usual, a positive, regular weighted mean, then $\epsilon_n \in (P, |P|)_r$ if and only if $\epsilon_n = \epsilon_n(\overline{P}, \alpha)$ and $\sum_{n=0}^{\infty} |\epsilon_n| < \infty$.

Suppose A is a Nörlund mean with $a_{nv} = p_{n-v}/P_n$, $P_n = p_0 + p_1 + \cdots + p_n$. If $p_n > 0$, $p_{n-v}/P_n \searrow 0$ as $n \nearrow \infty$ $(v \leqq n)$, $p_{n+1}/p_n \nearrow$ and $p_n \searrow$, then A satisfies the conditions of Theorem 1.

The discontinuous Riesz means (R^*, n, k), $0 < k < 1$, are included by Theorem 1.

If A is a regular Hausdorff method $H(\mu)$ with $a_{nv} = \binom{n}{v} \int_0^1 t^v (1-t)^{n-v} dg(t)$, and if $g'(t) > 0$, $g''(t) \geqq 0$, $t \dfrac{g''(t)}{g'(t)} \nearrow$ as $t \nearrow (0 < t < 1)$, then the conditions of Theorem 1 are again satisfied. As an example, for C_β we have $g(t) = 1 - (1-t)^\beta$, $g'(t) = \beta(1-t)^{\beta-1}$, $g''(t) = \beta(1-\beta)(1-t)^{\beta-2}$, $t \dfrac{g''(t)}{g'(t)} = (1-\beta)\dfrac{t}{1-t}$ and the conditions are satisfied for $0 < \beta \leqq 1$.

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