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# NOTE ON CONVERGENCE FACTORS

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**1. Introduction.** In this paper I discuss a necessary condition on convergence factors for Riesz summability $(R, \lambda, \kappa)$ (for any $\kappa \geqq 0$), and also necessary and sufficient conditions on convergence factors for generalized Cesàro summability $(C, \lambda, \kappa)$ (where $\kappa$ is an integer); since (under the hypotheses used) $(C, \lambda, \kappa)$ and $(R, \lambda, \kappa)$ are equivalent, this also gives a representation for Riesz convergence factors in the case where $\kappa$ is an integer. Much attention has been given recently in work on Riesz means to the problem of imposing minimal restrictions on the sequence $\lambda$; the restriction considered here will be one which occurs naturally as a necessary condition in some circumstances, and which also appears to be capable of being used to generalize a number of existing results in which heavier restrictions on $\lambda$ have been imposed.

We suppose throughout that $\lambda = \{\lambda_n\}$ is a sequence satisfying

$$0 \leqq \lambda_0 < \lambda_1 < \cdots < \lambda_n \to \infty ,$$

and we shall also employ (when indicated) the condition

$$(1) \qquad \Lambda_{n-1} = O(\Lambda_n), \quad \text{where} \quad \Lambda_n = \lambda_{n+1}/(\lambda_{n+1} - \lambda_n).$$

Given any series[2)] $\Sigma a_n$, denote

$$A^\kappa(\omega) = \sum_{\lambda_\nu < \omega} (\omega - \lambda_\nu)^\kappa a_\nu \ (\kappa \geqq 0), \quad R^\kappa(\omega) = \omega^{-\kappa} A^\kappa(\omega);$$

$$C_n^0 = \sum_{\nu=0}^{n} a_\nu, \quad C_n^p = \sum_{\nu=0}^{n} (\lambda_{n+1} - \lambda_\nu) \cdots (\lambda_{n+p} - \lambda_\nu) a_\nu \quad (p = 1, 2, \cdots),$$

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1) This paper was written while the author was a Fellow at the Summer Research Institute of the Canadian Mathematical Congress, Kingston, Ontario, 1966.

2) When not otherwise specified, limits of summation are assumed to be 0, $\infty$. Also $K$ will denote a constant, independent of the particular variables under consideration, and possibly different at each occurrence. We denote $\Delta b_n = b_n - b_{n+1}$. Finally, $A$ is included in $B$ ($A \subseteq B$) if every series summable-$A$ is also summable-$B$ (to the same value); $A$ and $B$ are equivalent ($A \sim B$) when each is included in the other.

(2) $g_n(x)=\left(1-\frac{x}{\lambda_{n+1}}\right)\cdots\left(1-\frac{x}{\lambda_{n+p}}\right)$ $(0\leqq x<\lambda_{n+1})$, $g_n(x)=0$ $(x\geqq\lambda_{n+1})$,

$$t_n^p=\sum_{\nu=0}^{n} g_n(\lambda_\nu)a_\nu=(\lambda_{n+1}\cdots\lambda_{n+p})^{-1}C_n^p\,.$$

$\Sigma a_n$ is summable $(R,\lambda,\kappa)$ to $s$ when $R^\kappa(\omega)\to s$ as $\omega\to+\infty$, and summable $(C,\lambda,p)$ to $s$ when $t_n^p\to s$ as $n\to\infty$. For properties of $(C,\lambda,\kappa)$ summability see Jurkat [5], Burkill [2], Russell [14], Borwein [1], Meir [16], Bcrwein-Russel [17].

If $A,B$ denote series transformations (or the matrices of such transformations), we shall denote by $[A,B]$ the class of all *summability-factors* $x=\{x_n\}$ such that $\Sigma a_nx_n$ is summable-$B$ for every series $\Sigma a_n$ which is summable-$A$. When $B=I$ (convergence), $[A,I]$ is the class of all *convergence-factors* for $A$-summability. It is trivial that

(3) *if* $C\subseteq A$ *then* $[A,B]\subseteq[C,B]$.

We shall require some properties of divided differences, of which an account can be found in Milne-Thomson [11], Chapter I. For non-negative integers $m,\nu,p$, denote

(4) $$\beta_{m\nu}^{(p)}=\beta_{m\nu}=\prod_{j=m}^{m+p+1}{}'(\lambda_\nu-\lambda_j)$$

where $\Pi'$ indicates that any zero factor corresponding to $j=\nu$ is to be omitted. Given any function $f$ defined in the interval $[\lambda_m,\lambda_{m+p+1}]$, the $(p+1)$th. order divided difference corresponding to the points $\lambda_\nu$ $(m\leqq\nu\leqq m+p+1)$ is

(5) $$f[\lambda_m,\cdots,\lambda_{m+p+1}]=\sum_{\nu=m}^{m+p+1}\frac{f(\lambda_\nu)}{\beta_{m\nu}};$$

if the derivative $f^{(p+1)}$ exists in $(\lambda_m,\lambda_{m+p+1})$ and $f^{(p)}$ is continuous also at the end-points, we have the mean-value theorem

$$f[\lambda_m,\cdots,\lambda_{m+p+1}]=\frac{1}{(p+1)!}f^{(p+1)}(\xi),\ \text{for some } \xi \text{ in } \lambda_m<\xi<\lambda_{m+p+1}\,.$$

Also denote

(6) $$\gamma_{m\nu}^{(p)}=\gamma_{m\nu}=\beta_{mm}/\beta_{m\nu}\,,$$

so that

$$\sum_{\nu=m}^{m+p+1} \gamma_{m\nu} = 0\,; \tag{7}$$

this last result follows from (5) by taking $f(x) \equiv 1$.

LEMMA 1. *Let $p$ be a non-negative integer; if $p \geqq 1$ assume that*

$$\Lambda_{n-1} = O(\Lambda_n)\,. \tag*{[(1)]}$$

*Then*

$$|\gamma_{m\nu}^{(p)}| \leqq K(\Lambda_\nu/\Lambda_m)^p \quad \textit{for} \quad m \leqq \nu \leqq m+p+1\,. \tag{8}$$

PROOF. For $p=0$ we have $|\gamma_{m\nu}^{(0)}|=1$ for $m \leqq \nu \leqq m+1$. If $p$ is a positive integer and $m \leqq \nu \leqq m+p+1$ then, by (4) and (6),

$$|\gamma_{m\nu}^{(p)}| = \frac{(\lambda_{m+1}-\lambda_m)\cdots(\lambda_\nu-\lambda_m)\cdot(\lambda_{\nu+1}-\lambda_m)\cdots(\lambda_{m+p+1}-\lambda_m)}{(\lambda_\nu-\lambda_m)\cdots(\lambda_\nu-\lambda_{\nu-1})\cdot(\lambda_{\nu+1}-\lambda_\nu)\cdots(\lambda_{m+p+1}-\lambda_\nu)}$$

$$\leqq \left(\frac{\lambda_{m+1}-\lambda_m}{\lambda_{m+1}-\lambda_m}\cdot\frac{\lambda_{m+2}-\lambda_m}{\lambda_{m+2}-\lambda_{m+1}}\cdots\frac{\lambda_\nu-\lambda_m}{\lambda_\nu-\lambda_{\nu-1}}\right)\left(\frac{\lambda_{\nu+1}-\lambda_m}{\lambda_{\nu+1}-\lambda_\nu}\right)^{m+p-\nu+1};$$

here we have merely used the fact that $\{\lambda_n\}$ increases, together with the property that if $a<b<x$ then $(x-a)/(x-b)$ decreases as $x$ increases. Now if (1) holds then

$$\frac{\lambda_{m+q}-\lambda_m}{\lambda_{m+q}} \leqq \frac{\lambda_{m+q}-\lambda_{m+q-1}}{\lambda_{m+q}} + \cdots + \frac{\lambda_{m+1}-\lambda_m}{\lambda_{m+1}} = \frac{1}{\Lambda_{m+q-1}} + \cdots + \frac{1}{\Lambda_m} \leqq \frac{K_q}{\Lambda_m} \tag{9}$$

and applying this to our last inequality for $\gamma_{m\nu}$ we obtain, since $\nu \leqq m+p+1$,

$$|\gamma_{m\nu}| \leqq K\left(\frac{\lambda_{m+2}\Lambda_m^{-1}}{\lambda_{m+2}-\lambda_{m+1}}\cdots\frac{\lambda_\nu\Lambda_m^{-1}}{\lambda_\nu-\lambda_{\nu-1}}\right)\left(\frac{\lambda_{\nu+1}\Lambda_m^{-1}}{\lambda_{\nu+1}-\lambda_\nu}\right)^{m+p-\nu+1}$$

$$= K\Lambda_m^{-p}\Lambda_{m+1}\cdots\Lambda_{\nu-1}\Lambda_\nu^{m+p-\nu+1}$$

$$\leqq K\Lambda_m^{-p}\Lambda_\nu^p, \quad \text{by (1)},$$

where $K$ is independent of $\nu$ and $m$; and this is the required result (8).

REMARK. We note, for future reference, the following consequences of this lemma: If $p$ is a positive integer and $\Lambda_{n-1}=O(\Lambda_n)$ then, for $m\leqq\nu\leqq m+p+1$:

$$\frac{(\lambda_{m+p+1}-\lambda_m)\lambda_{m+1}\cdots\lambda_{m+p}}{|\beta_{m\nu}^{(p)}|}=\frac{\lambda_{m+1}\cdots\lambda_{m+p}}{(\lambda_{m+1}-\lambda_m)\cdots(\lambda_{m+p}-\lambda_m)}|\gamma_{m\nu}^{(p)}|$$

$$\leqq\Lambda_m^p\cdot K(\Lambda_\nu/\Lambda_m)^p=K\Lambda_\nu^p\,; \tag{10}$$

$$\frac{\lambda_{m+1}\cdots\lambda_{m+p+1}}{|\beta_{m\nu}^{(p)}|}\leqq K\Lambda_\nu^p\cdot\frac{\lambda_{m+p+1}}{\lambda_{m+p+1}-\lambda_m}\leqq K\Lambda_\nu^p\Lambda_m\leqq K\Lambda_\nu^{p+1}\,; \tag{11}$$

if, in addition, we have $\lambda_\nu\leqq t\leqq\lambda_{m+p+1}$ and $\kappa\geqq p$, then, by (9),

$$(t-\lambda_\nu)/t\leqq(\lambda_{m+p+1}-\lambda_\nu)/\lambda_{m+p+1}\leqq K\Lambda_\nu^{-1}$$

and hence

$$t^{-\kappa}(t-\lambda_\nu)^\kappa\Lambda_m^\kappa|\gamma_{m\nu}^{(p)}|\leqq K(\Lambda_m/\Lambda_\nu)^{\kappa-p}\leqq K\,. \tag{12}$$

**2. Convergence-factors for $(R,\lambda,\kappa)$ summability.** When the matrix $A=(a_{n\nu})$ of a series-to-sequence transformation is *normal* (i.e. $a_{n\nu}=0$ for $\nu>n$, $a_{nn}\neq 0$), the diagonal elements of $A$ provide a limitation on the order of magnitude of the convergence-factors for $A$-summability. The following lemma is a consequence of Jurkat and Peyerimhoff [6], Satz 5.

LEMMA 2a. *If $A$ is normal and $x\in[A,I]$ then $x_n=O(a_{nn})$.*

Normal matrices have a number of attributes which simplify consideration of their summability properties, notably the possession of a unique right inverse (which is also a left inverse). Since the $(R,\lambda,\kappa)$ method is not normal, attempts have been made to define normal methods equivalent to it (for a discussion see [14]), which sometimes necessitate restrictions on $\lambda$. Thus, for example, I have shown in [14], Theorems 4 and 5, that

(13a) $(C,\lambda,p)\subseteq(R,\lambda,p)$ $(p=0,1,2,\cdots)$;

(13b) *if, when $p>2$, (1) holds, then* $(R,\lambda,p)\subseteq(C,\lambda,p)$ $(p=0,1,2,\cdots)$.

(Note: Meir [16] has recently shown that (13 b) holds without restriction on $\lambda$). Again, by restricting $\omega$ in the definition of $(R,\lambda,\kappa)$ to the sequence $\{\lambda_n\}$, we obtain 'discrete' Riesz summability $(R^*,\lambda,\kappa)$ (which is normal), and Jurkat [4] has shown that

(14) $$(R, \lambda, \kappa) \sim (R^*, \lambda, \kappa) \ (0 \leqq \kappa \leqq 1),$$

without restriction on $\lambda$. Since, if $A=(R^*, \lambda, \kappa)$, we have $a_{nn}=\Lambda_n^{-\kappa}$, we obtain from (14) and Lemma 2a the result ([3], Satz 4) that

(15) *if* $0 \leqq \kappa \leqq 1$ *and* $x \in [(R, \lambda, \kappa), I]$ *then* $x_n = O(\Lambda_n^{-\kappa})$.

Maddox [8] gives an example to show that

$$\exists \lambda, x \quad \textit{such that} \quad x \in [(R, \lambda, 2), I] \quad \textit{but} \quad x_n \neq O(\Lambda_n^{-2}),$$

so that we cannot hope to extend (15) to all $\kappa > 1$ without some restriction on $\lambda$. However, Jurkat [3], Sätze 4, 5, has shown that (15) remains true for $\kappa > 1$ when the following conditions are imposed:

(16) $$\text{(a)} \ 0 < m \leqq \frac{\Delta\lambda_n}{\Delta\lambda_{n-1}}, \qquad \text{(b)} \ \frac{\Delta\lambda_n}{\Delta\lambda_{n-1}} \leqq M < \infty.$$

Now if we take $A = (C, \lambda, p)$, we get

$$a_{nn} = \left(\frac{\lambda_{n+1}-\lambda_n}{\lambda_{n+1}}\right) \cdots \left(\frac{\lambda_{n+p}-\lambda_n}{\lambda_{n+p}}\right)$$

and if (1) holds then, by (9), $a_{nn}=O(\Lambda_n^{-p})$; thus, from (13) and Lemma 2a:

*if, when* $p > 1$, (1) *holds, and if* $x \in [(R, \lambda, p), I]$, *then* $x_n = O(\Lambda_n^{-p})$ $(n = 0, 1, 2, \cdots)$.

The question therefore arises as to whether this result remains true with a general $\kappa$ in place of the integer $p$, and without imposing any additional restriction on $\lambda$ besides (1). It will be shown in Corollary 2 that this is in fact the case, thus showing that hypothesis (16a) can be completely removed in Jurkat's theorem ([3], Satz 4), and that (16b) can be replaced by the weaker hypothesis (1).

A further question concerns the conditions under which the sequence $\{\Lambda_n^{-\kappa}\}$ itself can be a convergence-factor for $(R, \lambda, \kappa)$-summability, and in considering this (at least for those values of $\kappa$ for which a normal method is known which is included in $(R, \lambda, \kappa)$ without restriction on $\lambda$) we find that the condition (1) now appears as a necessary condition. It is convenient first to supplement Lemma 2a with

LEMMA 2b. *If $A$ is normal and $x \in [A, I]$ then*

$$x_n a_{n,n-1} = O(a_{nn} a_{n-1,n-1}).$$

PROOF. Denoting by $A^{-1} = (a_{\nu i}^{-1})$ the two-sided inverse of $A$, $\{\sigma_n\}$ the partial sums of $\Sigma a_\nu x_\nu$, and $\{t_n\}$ the $A$-transform of $\Sigma a_\nu$, it is easy to deduce that

$$\sigma_n = \sum_{i=0}^{n} b_{ni} t_i, \quad \text{where} \quad b_{ni} = \sum_{\nu=i}^{n} x_\nu a_{\nu i}^{-1}.$$

Thus $\{\sigma_n\}$ converges whenever $\{t_n\}$ converges (i.e. $x \in [A, I]$) if and only if $(b_{ni})$ is conservative, and in particular it is necessary that

$$\sum_{i=0}^{n} |b_{ni}| \leqq M \quad \text{independently of } n.$$

Hence $|b_{ni}| \leqq M$ for every $n$ and $i$; the choice $i = n$ leads at once to Lemma 2a (since $a_{nn}^{-1} = 1/a_{nn}$) and the choice $i = n-1$ to Lemma 2b, since $a_{n,n-1}^{-1} = -a_{n,n-1}/(a_{nn} a_{n-1,n-1})$.

THEOREM 1. *If $0 < \kappa \leqq 1$, or if $\kappa$ is a positive integer, and if*

$$\{\Lambda_n^{-\kappa}\} \in [(R, \lambda, \kappa), I], \quad \textit{then} \quad \Lambda_{n-1} = O(\Lambda_n).$$

PROOF. Let $x_n = \Lambda_n^{-\kappa}$. Then taking $A = (R^*, \lambda, \kappa)$ $(\kappa > 0)$ we find that

$$\frac{x_n a_{n,n-1}}{a_{nn} a_{n-1,n-1}} > \left(\frac{\Lambda_{n-1}}{\Lambda_n}\right)^\kappa;$$

while taking $A = (C, \lambda, \kappa)$ ($\kappa$ a positive integer) we obtain

$$\frac{x_n a_{n,n-1}}{a_{nn} a_{n-1,n-1}} > \frac{\Lambda_{n-1}}{\Lambda_n}.$$

Using these inequalities in Lemma 2b, together with (13a), (14) and (3), the theorem follows.

While Theorem 1 gives a simple necessary condition in order that $\{\Lambda_n^{-\kappa}\}$ should be an $(R, \lambda, \kappa)$ convergence-factor, it is hardly to be expected that this condition will be sufficient, and in fact Jurkat [3], Satz 3, gives three fairly complicated sufficient conditions for such a result to hold. Taking the most tractable case for purposes of comparison, namely $\kappa=1$, we find that his first condition (which is that $\Lambda_n\nearrow$) then implies the second, while the third automatically holds, so that we have the following:

*In order that* $\{\Lambda_n^{-1}\}$ *should be an* $(R,\lambda,1)$ *convergence-factor, it is necessary that* $\Lambda_{n-1}=O(\Lambda_n)$ *and sufficient that* $\Lambda_{n-1}\leqq\Lambda_n$.

However, the precise necessary and sufficient conditions in order that a sequence $\{x_n\}$ should be an $(R,\lambda,1)$ convergence-factor are already given in Jurkat [3], Satz 1 (the third condition given in this theorem is superfluous, since it can be deduced from the other two, as pointed out by Maddox [10]), and if we put $x_n=\Lambda_n^{-1}$ in this theorem, we obtain:

*In order that* $\{\Lambda_n^{-1}\}$ *should be an* $(R,\lambda,1)$ *convergence-factor, it is necessary and sufficient that*

$$\sum_{\nu=0}^{\infty}\lambda_{\nu+1}\left|\Delta\left(\frac{\Delta\Lambda_\nu^{-1}}{\Delta\lambda_\nu}\right)\right|<\infty .$$

We turn now to the main theorem of this section, the motivation for which has been discussed earlier.

THEOREM 2. *Let* $\kappa>0$. *If* $\kappa>1$ *assume*

$$\Lambda_{n-1}=O(\Lambda_n). \tag{[(1)]}$$

*Then for each unbounded sequence* $\{\theta_n\}$ *of real or complex numbers, there is a series* $\Sigma a_\nu$, *with partial sums* $s_n$, *which is summable* $(R,\lambda,\kappa)$ *to zero, but such that*

$$s_n\neq o(\Lambda_n^\kappa/\theta_n)\,,\quad a_n\neq o(\Lambda_n^\kappa/\theta_n)\,. \tag{17}$$

PROOF. The proof is essentially similar to that of Jurkat [3], but here we use Lemma 1 and also alter the definition of $a_n$, in order to obtain sharper estimates. The choice of the series $\Sigma a_\nu$ depends on an increasing sequence of non-negative integers $\{n_r\}$ chosen inductively as follows: suppose that $n_0, n_1, \cdots, n_{r-1}$ have been chosen and that $a_\nu$ has been defined for $0\leqq\nu\leqq n_{r-1}+p+1$ (where $p$ is the integer such that $p<\kappa\leqq p+1$) in such a way that

$$\sum_{\nu=0}^{n_{r-1}+p+1} a_\nu = 0\,; \tag{18}$$

now choose $n_r$ so that

$$n_r > n_{r-1} + p + 1 \tag{19}$$

$$|\theta_{n_r}| \geqq r \tag{20}$$

$$\left|\omega^{-\kappa} \sum_{\nu=0}^{n_{r-1}+p+1} (\omega-\lambda_\nu)^\kappa a_\nu\right| \leqq \frac{1}{r} \quad \text{for } \omega > \lambda_{n_r}\,. \tag{21}$$

Such a choice of $n_r$ is possible since $\{\theta_n\}$ is unbounded, by hypothesis, and since the left hand side of (21) tends to zero as $\omega \to \infty$, by virtue of (18). Now define

$$a_\nu = 0 \quad \text{for} \quad n_{r-1}+p+1 < \nu < n_r \tag{22}$$

$$a_\nu = \theta_m^{-1} \Lambda_m^\kappa \gamma_{m\nu} \quad \text{for} \quad m \equiv n_r \leqq \nu \leqq n_r + p + 1\,, \tag{23}$$

where $\gamma_{m\nu}$ is given by (6), and it then follows from (7) that (18) holds with $r+1$ in place of $r$. By making the initial choice

$$n_0 = 0, \quad a_\nu = 0 \quad \text{for} \quad n_0 \leqq \nu \leqq n_0 + p + 1,$$

(18)—(23) are then valid for $r = 1, 2, \cdots$.

It is clear that (17) holds with this choice of $\Sigma a_\nu$, for when $\nu = n_r$ we see from (22), (23) and (6) that

$$s_{n_r} = a_{n_r} = \Lambda_{n_r}^\kappa / \theta_{n_r} \quad (r = 1, 2, 3, \cdots)\,.$$

We now show that the series $\Sigma a_\nu$ is summable $(R, \lambda, \kappa)$ to 0. Given $\omega > 0$, let $n$ be the integer-valued function of $\omega$ satisfying $\lambda_n < \omega \leqq \lambda_{n+1}$; then there is an $r$ such that $n_r \leqq n < n_{r+1}$, and hence

$$\begin{aligned} \omega^{-\kappa} A^\kappa(\omega) &= \omega^{-\kappa} \sum_{\nu=0}^{n} (\omega-\lambda_\nu)^\kappa a_\nu \\ &= \omega^{-\kappa} \sum_{\nu=0}^{n_{r-1}+p+1} (\omega-\lambda_\nu)^\kappa a_\nu + \omega^{-\kappa} \sum_{\nu=n_r}^{n} (\omega-\lambda_\nu)^\kappa a_\nu\,, \quad \text{by (22)} \end{aligned}$$

(24) $$= o(1) + S, \quad \text{by (21)},$$

where

(25) $$S = \theta_m^{-1} \Lambda_m^\kappa \omega^{-\kappa} \sum_{\nu=m}^{\min(n, m+p+1)} (\omega - \lambda_\nu)^\kappa \gamma_{m\nu} \ (m = n_r).$$

Suppose first that $m \equiv n_r \leqq n \leqq n_r + p$, so that $\omega \leqq \lambda_{m+p+1}$; if (when $p \geqq 1$) we assume (1), then (12) and (20) show at once that

(26) $$S = O(\theta_{n_r}^{-1}) = o(1) \quad \text{as} \quad \omega \to \infty \ (n_r \leqq n \leqq n_r + p).$$

Alternatively, if $n_r + p + 1 \leqq n < n_{r+1}$ then, by (6) and (25),

$$S = \theta_m^{-1} \Lambda_m^\kappa \omega^{-\kappa} \beta_{mm} \sum_{\nu=m}^{m+p+1} \frac{(\omega - \lambda_\nu)^\kappa}{\beta_{m\nu}}$$

$$\equiv \theta_m^{-1} \Lambda_m^\kappa \omega^{-\kappa} \beta_{mm} d(\omega), \quad \text{say}.$$

Now, by (5), $d(\omega) = e_\omega[\lambda_m, \cdots, \lambda_{m+p+1}]$, where $e_\omega(x) = (\omega - x)^\kappa$ for $\omega > x$; and it follows as in Jurkat [3], p. 270, using the mean-value theorem for divided differences, that $|d(\omega)| \leqq |d(\lambda_{m+p+1})|$ for $\omega > \lambda_{m+p+1}$, whence

$$|S| \leqq |\theta_m^{-1}| \Lambda_m^\kappa \sum_{\nu=m}^{m+p+1} \left( \frac{\lambda_{m+p+1} - \lambda_\nu}{\lambda_{m+p+1}} \right)^\kappa |\gamma_{m\nu}|$$

which gives, by substituting $t = \lambda_{m+p+1}$ in (12),

(27) $$S = O(\theta_{n_r}^{-1}) = o(1) \quad \text{as} \quad \omega \to \infty \quad (n_r + p < n < n_{r+1}).$$

It now follows from (24), (26), (27) that $\omega^{-\kappa} A^\kappa(\omega) = o(1)$ as $\omega \to \infty$; thus $\Sigma a_\nu$ is summable $(R, \lambda, \kappa)$ to zero, and the theorem is proved.

COROLLARY 2. *Let $\kappa > 0$; if $\kappa > 1$ assume that $\Lambda_{n-1} = O(\Lambda_n)$. Then in order that $\Sigma a_n x_n$ should converge whenever $\Sigma a_n$ is summable $(R, \lambda, \kappa)$, it is necessary that $x_n = O(\Lambda_n^{-\kappa})$.*

PROOF. This follows directly from the theorem, as in the proof of [3], Satz 5.

**3. Convergence-factors for $(C, \lambda, \kappa)$ summability.** Although Lemmas 2a, 2b give limitations on the order of magnitude of convergence-factors for

$A$-summability ($A$ normal), some more precise representations are needed in order to obtain necessary and sufficient conditions for these convergence-factors. The following result is due essentially to Peyerimhoff [12] — see also Russell [13], §2.

LEMMA 3. *Let $A$ be normal and* $\lim_{n\to\infty} a_{n\nu}=1$ $(\nu = 0, 1, 2, \cdots)$, *and let* $x \in [A, I]$. *Then*

(28) $$\exists\, \eta,\ \{\eta_k\},\ \textit{with}\ \Sigma|\eta_k| < \infty,\ \textit{such that}\ x_\nu = \eta + \sum_{\kappa=\nu}^{\infty} \eta_k a_{k\nu}\,;$$

*moreover,*

(29) $$\eta_\nu = \sum_{k=\nu}^{\infty} x_k a_{k\nu}^{-1}\,.$$

*If, in addition, $A$ is regular (a $\gamma$-matrix) and* $1/a_{nn} \neq O(1)$, *then* $\eta = 0$.

We obtain the last clause of the lemma as follows: if $A$ is a $\gamma$-matrix then in particular $|a_{k\nu}| \leqq K$ for every $k$ and $\nu$ and so, from (28), $x_\nu = \eta + o(1)$; but if $1/a_{nn} \neq O(1)$ then, by Lemma 2a, there is a sub-sequence of integers such that $x_{\nu_i} = o(1)$; hence $\eta = 0$.

We now apply this lemma to find the form of the convergence-factors for $(C, \lambda, p)$ summability, where $p$ is an integer; there is a related result in Jurkat [5], Satz 12 — there the theorem is a consequence of a number of general results on convergence-factors, and it is interesting to see what can be proved in a direct way with fewer restrictions on $\lambda$.

THEOREM 3. *Let $p$ be a non-negative integer and $g_n(x)$ be defined as in* (2); *let*

(30) $$\Lambda_n \neq O(1)$$

*and when $p \geqq 2$ assume that*

[(1)] $$\Lambda_{n-1} = O(\Lambda_n)\,.$$

*Then* $x \in [(C, \lambda, p), I]$ *if and only if*

(31) $$x_n = O\{g_n(\lambda_n)\}$$

(32) $$\exists\ \{\eta_k\},\ \Sigma|\eta_k| < \infty,\ \textit{such that}\ x_\nu = \sum_{k=\nu}^{\infty} \eta_k\, g_k(\lambda_\nu)\,.$$

PROOF. If $A$ is the $(C, \lambda, p)$ matrix, then $a_{n\nu} = g_n(\lambda_\nu)$ and the necessity of (31) follows at once from Lemma 2a, without restriction on $\lambda$; however, under the hypothesis (1) (imposed for $p \geqq 2$), (9) shows that (31) is equivalent to

(31)′ $$x_n = O(\Lambda_n^{-p})\,.$$

In addition, $A$ is normal and regular and $1/a_{nn} \neq O(1)$ when (30) and (1) are imposed; the necessity of (32) then follows from Lemma 3. [It should be remarked that (30) is a relatively trivial requirement, for if $\Lambda_n = O(1)$ then (see [14], Corollary 3B) $(C, \lambda, p) \sim I$ and the necessary and sufficient conditions for convergence-factors in this case are well-known (see, for example, Hardy [7], Theorem 7), namely: $x \in [I, I]$ if and only if $\Sigma|\Delta x_n| < \infty$.]

It remains to prove the sufficiency of (31) [or (31)′] and (32), and we may suppose, without loss of generality, that

$$t_k^p \equiv \sum_{\nu=0}^{k} g_k(\lambda_\nu)\, a_\nu = o(1) \text{ as } k \to \infty\,.$$

Then, by (32),

(33) $$\begin{aligned}\sum_{\nu=0}^{n} a_\nu\, x_\nu &= \sum_{\nu=0}^{n} a_\nu \left(\sum_{k=\nu}^{n} + \sum_{k=n+1}^{\infty}\right) \eta_k\, g_k(\lambda_\nu)\\ &\equiv \sum_{k=0}^{n} \eta_k\, t_k^p + S_n\,, \quad \text{say.}\end{aligned}$$

Since $t_k^p = o(1)$ and $\Sigma|\eta_k| < \infty$, it follows that $\Sigma \eta_k t_k^p$ converges, so that $\Sigma a_\nu x_\nu$ converges if and only if the sequence $\{S_n\}$ converges. Now

$$S_n = \sum_{k=n+1}^{\infty} \eta_k \sum_{\nu=0}^{n} g_k(\lambda_\nu)\, a_\nu$$

and by partial summation it can be shown (see Russell [14], Lemma 1) that

$$\begin{aligned}\sum_{\nu=0}^{n} g_k(\lambda_\nu)\, a_\nu = &\sum_{r=0}^{p} (-1)^r\, g_k[\lambda_{n+1}, \cdots, \lambda_{n+r+1}]\, C_n^r + \\ &+ (-1)^{p+1} \sum_{\nu=0}^{n} g_k[\lambda_\nu, \cdots, \lambda_{\nu+p+1}](\lambda_{\nu+p+1} - \lambda_\nu)\, C_\nu^p\,.\end{aligned}$$

For $\nu \leqq n < k-p$, $g_k[\lambda_\nu, \cdots, \lambda_{\nu+p+1}]$ is a $(p+1)$th. order divided difference of a polynomial of degree $p$, and hence vanishes; consequently

$$S_n = S_n' + S_n'' \tag{34}$$

where

$$S_n' = \sum_{k=n+1}^{\infty} \eta_k \sum_{r=0}^{p} (-1)^r g_k[\lambda_{n+1}, \cdots, \lambda_{n+r+1}] C_n^r, \tag{35}$$

$$S_n'' = \sum_{k=n+1}^{n+p} \eta_k (-1)^{p+1} \sum_{\nu=k-p}^{n} g_k[\lambda_\nu, \cdots, \lambda_{\nu+p+1}](\lambda_{\nu+p+1}-\lambda_\nu) C_\nu^p. \tag{36}$$

Now $g_k[\lambda_{n+1}, \cdots, \lambda_{n+r+1}] = \sum_{i=n+1}^{n+r+1} \frac{g_k(\lambda_i)}{\beta_{n+1,i}'}$, $\beta_{n+1,i}' = \prod_{j=n+1}^{n+r+1}{}' (\lambda_i - \lambda_j)$, so that

$$\begin{aligned}\sum_{k=n+1}^{\infty} \eta_k g_k[\lambda_{n+1}, \cdots, \lambda_{n+r+1}] &= \sum_{i=n+1}^{n+r+1} \frac{1}{\beta_{n+1,i}'} \sum_{k=n+1}^{\infty} \eta_k g_k(\lambda_i) \\ &= \sum_{i=n+1}^{n+r+1} \frac{x_i}{\beta_{n+1,i}'}\end{aligned}$$

by (32), and since $g_k(\lambda_i) = 0$ for $n+1 \leqq k < i$. Expressing $C_n^r$ in terms of $t_n^r$, it now follows from (35) that

$$S_n' = \sum_{r=0}^{p} (-1)^r t_n^r \sum_{i=n+1}^{n+r+1} \lambda_{n+1} \cdots \lambda_{n+r} x_i / \beta_{n+1,i}'. \tag{37}$$

Now assuming (1) (for $r \geqq 2$), (11) shows (with $r-1$ in place of $p$) that

$$\lambda_{n+1} \cdots \lambda_{n+r} / |\beta_{n+1,i}'| \leqq K_r \Lambda_i^r \ (n+1 \leqq i \leqq n+r+1),$$

and also, by (31)$'$,

$$x_i = O(\Lambda_i^{-p});$$

further (Russell [14], Corollary 3A) $t_n^p = o(1)$ implies

$$t_n^r = o(\Lambda_n^{p-r}) \ (r = 0, 1, \cdots, p).$$

Substitution of these estimates in (37) now gives, by (1),

(38) $$S_n' = \sum_{r=0}^{p} \sum_{i=n+1}^{n+r+1} o(\Lambda_n^{p-r}\Lambda_i^r\Lambda_i^{-p}) = o(1)\,.$$

Turning to $S_n''$, we note first that $S_n''$ vanishes identically for $p=0$ and $p=1$. Now $g_k(\lambda_i)=0$ for $k<i\leqq \nu+p+1$, so that

$$g_k[\lambda_\nu,\cdots,\lambda_{\nu+p+1}] = \sum_{i=\nu}^{k} \frac{g_k(\lambda_i)}{\beta_{\nu i}}\,, \quad \beta_{\nu i} = \prod_{j=\nu}^{\nu+p+1}{}' (\lambda_i-\lambda_j)\,.$$

Now (9) shows, making use of (1), that

$$g_k(\lambda_i) = \frac{(\lambda_{k+1}-\lambda_i)\cdots(\lambda_{k+p}-\lambda_i)}{\lambda_{k+1}\cdots\lambda_{k+p}} \leqq K\Lambda_i^{-p} \quad \text{for} \quad k-p\leqq i\leqq k\,.$$

Also

$$\frac{(\lambda_{\nu+p+1}-\lambda_\nu)\,|C_\nu^p|}{|\beta_{\nu i}|} = \frac{(\lambda_{\nu+p+1}-\lambda_\nu)\lambda_{\nu+1}\cdots\lambda_{\nu+p}\,|t_\nu^p|}{|\beta_{\nu i}|} \leqq K\Lambda_i^p$$

for $\nu\leqq i\leqq \nu+p+1$, by (10) (assuming (1)) and since $\{t_\nu^p\}$ is bounded, by hypothesis. Substituting these estimates into (26), we now find that

(39) $$|S_n''| \leqq K \sum_{k=n+1}^{n+p} |\eta_k| = o(1)\,.$$

Hence, by (34), (38), (39), we have $S_n=o(1)$ and so, by (33), $\Sigma a_\nu x_\nu$ converges to $\Sigma \eta_k t_k^p$; and this proves the theorem.

To write Theorem 3 in an alternative form, an easy calculation shows that the two-sided inverse matrix $A^{-1}=(a_{k\nu}^{-1})$ of the $\gamma$-matrix $A=(C,\lambda,p)$ is given by

(40) $$a_{k\nu}^{-1} = (-1)^{p+1}(\lambda_{\nu+p+1}-\lambda_\nu)\,\lambda_{\nu+1}\cdots\lambda_{\nu+p}/\beta_{\nu k} \quad (\nu\leqq k\leqq \nu+p+1)\,,$$
$$a_{k\nu}^{-1} = 0 \quad \text{otherwise},$$

where $\beta_{\nu k}$ is defined by (4). Thus $A^{-1}$ consists of $p+2$ diagonals containing non-zero elements, with zero elements elsewhere. Now using (29), Theorem 3 takes the form:

THEOREM 3′. *Under the hypotheses of Theorem* 3, $\{x_n\}$ *is a* $(C,\lambda,p)$ *convergence-factor if and only if*

[(31)′] $$x_n = O(\Lambda_n^{-p})$$

(41) $$\sum_{n=0}^{\infty} (\lambda_{n+p+1}-\lambda_n)\lambda_{n+1} \cdots \lambda_{n+p} \left| \sum_{i=0}^{p+1} \frac{x_{n+i}}{\beta_{n,n+i}} \right| < \infty .$$

For $\lambda_n = n$, the inner sum in (41) reduces to $(-1)^{p+1}(p+1)!\ \Delta^{p+1}x_n$, and we obtain as a corollary the well-known Bohr-Hardy-Fekete theorem:

COROLLARY 3′. *Let $p$ be a non-negative integer; then $\Sigma a_n x_n$ converges whenever $\Sigma a_n$ is summable $(C, p)$ if and only if $x_n = O(n^{-p})$ and $\Sigma n^{p+1} |\Delta^{p+1} x_n| < \infty$.*

This result has been generalized in several directions, notably by Bosanquet and Andersen; for further references and a short discussion of convergence-factors for Cesàro summability see Hardy [7], p. 146.

Finally, we remark that in view of 13 (a), (b), Theorems 3 and 3′ also give (for integral $p$) necessary and sufficient conditions in order that $\Sigma a_n x_n$ should converge whenever $\Sigma a_n$ is summable $(R, \lambda, p)$. By restricting $\lambda$ to satisfy (16a) together with $\Lambda_n \nearrow +\infty$ (which implies (16b)) Maddox [8] has been able to obtain a considerably more general result for summability-factors $[(R, \lambda, \kappa), (R, \lambda, \mu)]$; there the summability-factors are expressed in the form of an integral instead of a series form such as (32). The precise relation between the two forms would be quite difficult to determine, though Maddox [9] gives an interesting construction in the case $\lambda_n = n$, i.e. for $[(C, \kappa), (C, \mu)]$ summability-factors. The problem is also mentioned by Jurkat and Peyerimhoff [6], p. 105, in comparing (for $0 < \kappa \leqq 1$) the integral form for $(R, \lambda, \kappa)$ convergence-factors with the series form for $(R^*, \lambda, \kappa)$ convergence-factors. Recently I have been able to show (see [15]) that the conditions on $\lambda$ imposed by Maddox [8], Theorem A, in his result on $[(R, \lambda, \kappa), (R, \lambda, \mu)]$ summability-factors, can be removed entirely in the case $0 \leqq \mu \leqq \kappa \leqq 1$.

## REFERENCES

[1] D. BORWEIN, On a generalized Cesàro summability method of integral order, Tôhoku Math. Journ., (2), 18(1966), 71-73.

[2] H. BURKILL, On Riesz and Riemann summability, Proc. Camb. Phil. Soc., 57(1961), 55-60.

[3] W. B. JURKAT, Über Konvergenzfaktoren bei Rieszschen Mitteln, Math. Zeit., 54(1951), 262-271.

[4] W. B. JURKAT, Über Rieszsche Mittel mit unstetigem Parameter, Math. Zeit., 55(1951), 8-12.

[5] W. B. JURKAT, Über Rieszsche Mittel und verwandte Klassen von Matrixtransformationen, Math. Zeit., 57(1953), 353-394.

[6] W. B. JURKAT AND A. PEYERIMHOFF, Mittelwertsätze bei Matrix- und Integral-transformationen, Math. Zeit., 55(1951), 92–108.
[7] G. H. HARDY, Divergent Series, Oxford University Press (1949).
[8] I. J. MADDOX, Convergence and summability factors for Riesz means, Proc. London Math. Soc. (3), 12(1962), 345–366.
[9] I. J. MADDOX, A note on summability factor theorems, Quart. Journ. Math. (Oxford) (2), 15(1964), 208–216.
[10] I. J. MADDOX, Some inclusion theorems, Proc. Glasgow Math. Assn. 6(1964), 161–168.
[11] L. M. MILNE-THOMSON, The calculus of finite differences, Macmillan, London (1933, reprinted 1960).
[12] A. PEYERIMHOFF, Konvergenz- und Summierbarkeitsfaktoren, Math. Zeit., 55(1951), 23–54.
[13] D. C. RUSSELL, Note on inclusion theorems for infinite matrices, Journ. London Math. Soc., 33(1958), 50–62.
[14] D. C. RUSSELL, On generalized Cesàro means of integral order, Tôhoku Math. Journ. (2), 17(1965), 410–442. Corrigenda; 18(1966), 454–455.
[15] D. C. RUSSELL, On a summability factor theorem for Riesz means, Journ. London Math. Soc. (to appear).
[16] A. MEIR, An inclusion theorem for generalized Cesàro and Riesz means, Canada. Journ. Math. (to appear).
[17] D. BORWEIN AND D. C. RUSSEL, On Riesz and generalized Cesàro summability of arbitrary positive order, Math. Zeit. (to appear).

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