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SOME REMARKS ON ANDÔ'S THEOREMS

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1. In [1] T. Andô has proved the following result.

THEOREM A. *Let T be a compact operator on a Hilbert space H. Then every subspace which is invariant under T reduces T if and only if T is a normal operator.*

The purpose of this paper is to remark that the above theorem is generalised to an operator such as T^m is compact for some integer $m \geqq 0$ and to prove a related result.

2. In the sequel, an operator means a bounded linear operator on a Hilbert space H. We denote by $\sigma(T)$ the spectrum and by $\sigma_p(T)$ the point spectrum of an operator T. $\mathfrak{N}_T(\lambda)$ means the λ-th proper subspace of an operator T and $P_{\mathfrak{M}}$ is the orthogonal projection onto a closed subspace $\mathfrak{M} \subset H$.

The following lemma is essentially proved in [5], but we give a proof for convenience' sake.

LEMMA 1. *Let T be an operator such as T^m is compact for some integer $m \geqq 0$. Then $\mu \in \sigma(T) \cap \{\lambda : |\lambda| = \|T\|\}$ implies $\mu \in \sigma_p(T)$.*

PROOF. If $\mu \in \sigma(T)$ and $|\mu| = \|T\|$, there exists a sequence $\{x_n\}$ of unit vectors in H such as $\|Tx_n - \mu x_n\| \to 0$ $(n \to \infty)$. Since T^m is a compact operator, we may assume that (if necessary, by choosing a suitable sub-sequence) the sequence $\{T^m x_n\}$ converges to a certain vector $x \in H$. Then we have

$$\|T^{m-1}x_n - \frac{1}{\mu} T^m x_n\| \leqq \frac{\|T^{m-1}\|}{|\mu|} \|Tx_n - \mu x_n\| \to 0 \quad (n \to \infty)$$

and

$$\|T^{m-1}x_n - \frac{1}{\mu} x\| \leqq \|T^{m-1}x_n - \frac{1}{\mu} T^m x_n\| + \frac{1}{|\mu|} \|T^m x_n - x\| \to 0 \quad (n \to \infty).$$

Continuing the above argument successively, we can conclude that $\{x_n\}$ converges to a certain non-zero vector $x_0 \in H$. Hence we have

$$\|Tx_0-\mu x_0\| \leqq \|Tx_0-Tx_n\| + \|Tx_n-\mu x_n\| + \|\mu x_n-\mu x_0\|$$
$$\leqq 2\|T\| \, \|x_n-x_0\| + \|Tx_n-\mu x_n\| \to 0 \quad (n \to \infty).$$

Therefore $Tx_0=\mu x_0$, which completes the proof.

THEOREM 1. *Let T be a normal operator such as T^m is compact for some integer $m \geqq 0$ and $\mathfrak{M} \subset H$ a subspace which is invariant under T, then $\mathfrak{M}$ reduces T.*

PROOF. Since T is a normal operator, $\{\mathfrak{N}_T(\lambda) : \lambda \in \sigma_p(T)\}$ is a mutually orthogonal family of reducing subspaces. Let P and $P_\lambda(\lambda \in \sigma_p(T))$ be the orthogonal projections onto $\mathfrak{M}$ and $\mathfrak{N}_T(\lambda)$ respectively, $H_0 = \sum_{\lambda \in \sigma_p(T)} \oplus \mathfrak{N}_T(\lambda)$, $H_1 = H_0^\perp$ and Q the orthogonal projection onto H_1. Then T_1, the restriction of T onto H_1, is 0. For, if $T_1 \neq 0$ there is a $\mu \in \sigma(T_1)$ such as $\|T_1\| = |\mu|$ by normality of T_1, and $\mu \in \sigma_p(T_1)$ by Lemma 1 and this is a contradiction. From now the argument is quite similar to that of Andô [1]. By the ergodic theorem,

$$\frac{1}{n}\sum_{k=1}^{n} (\lambda^{-1}T)^k \to P_\lambda \quad \text{(in the strong topology)}$$

for each $\lambda \in \sigma_p(T)$. Similarly,

$$\frac{1}{n}\sum_{k=1}^{n} (\lambda^{-1}PTP)^k \to Q_\lambda \quad \text{(in the strong topology)}$$

for each $\lambda \in \sigma_p(T)$, where Q_λ is a projection. Since $(PTP)^k = PT^kP$ $(k=1, 2, \cdots)$, we have $Q_\lambda = PP_\lambda P$ for each $\lambda \in \sigma_p(T)$, and hence $PP_\lambda P$ is a projection. Thus we have, for each $x \in H$,

$$\begin{aligned} \|PP_\lambda Px - P_\lambda Px\|^2 &= (PP_\lambda Px, PP_\lambda Px) - (PP_\lambda Px, P_\lambda Px) \\ &\qquad - (P_\lambda Px, PP_\lambda Px) + (P_\lambda Px, P_\lambda Px) \\ &= (Q_\lambda x, x) - (Q_\lambda x, x) - (Q_\lambda x, x) + (Q_\lambda x, x) = 0. \end{aligned}$$

Hence, $P_\lambda P = PP_\lambda$ for each $\lambda \in \sigma_p(T)$. It follows that $T^*\mathfrak{M} \subset \mathfrak{M}$. For, if $x \in \mathfrak{M}$,

$$T^*x = \sum_{\lambda\in\sigma_p(T)} T^*P_\lambda Px + T_1^*(QPx) = \sum_{\lambda\in\sigma_p(T)} T^*P_\lambda Px \quad (\text{since } T_1 = 0)$$

$$= \sum_{\lambda\in\sigma_p(T)} \bar{\lambda}P_\lambda Px \quad (\text{since } T^*z = \bar{\lambda}z \text{ for } z\in\mathfrak{N}_T(\lambda))$$

$$= \sum_{\lambda\in\sigma_p(T)} P\bar{\lambda}P_\lambda x = PT^*\Big(\sum_{\lambda\in\sigma_p(T)} P_\lambda x\Big) \in \mathfrak{M}\,.$$

Therefore the theorem is proved.

REMARK. In Lemma 1 and Theorem 1 the assumption of compactness of T^m is replaced by the assumption of compactness of operator $T^{*p_1}T^{q_1}\cdots T^{*p_r}T^{q_r}$ for some non-negative integers $p_1, q_1, \cdots, p_r, q_r$.

The following theorem is the converse of Theorem 1.

THEOREM 2. *Let T be an operator such as T^m is compact for some integer $m \geqq 0$. If every subspace which is invariant under T reduces T, then T is necessarily a normal operator.*

PROOF. Since each $\mathfrak{N}_T(\lambda)$ $(\lambda\in\sigma_p(T))$ is invariant under T, $\mathfrak{N}_T(\lambda)$ is a reducing subspace for each $\lambda\in\sigma_p(T)$, and we have $TT^*x=\lambda T^*x$ for $x\in\mathfrak{N}_T(\lambda)$. It follows that

$$\|T^*x-\bar{\lambda}x\|^2 = (TT^*x, x) - \lambda(T^*x, x) - \bar{\lambda}(Tx, x) + |\lambda|^2\|x\|^2$$

$$= \lambda(T^*x, x) - \lambda(T^*x, x) - |\lambda|^2\|x\|^2 + |\lambda|^2\|x\|^2$$

$$= 0\,.$$

Thus $\{\mathfrak{N}_T(\lambda) : \lambda\in\sigma_p(T)\}$ is a mutually orthogonal family of reducing subspaces and the restriction of T onto $H_0 = \sum_{\lambda\in\sigma_p(T)}\oplus\mathfrak{N}_T(\lambda)$ is a normal operator. Let Q be the orthogonal projection onto $H_1 = H_0^\perp$. It is sufficient to show that $TQ = 0$. Now, suppose the contrary. Then $T^mQ = (TQ)^m \neq 0$. In fact, if $(TQ)^m = 0$, $(TQ)(TQ)^{m-1}x = 0$ for $x\in H_1$ and $0\in\sigma_p(T)$, which is a contradiction. By this fact, Andô's discussion [1 : p. 339] is fairly applicable to the compact operator $(TQ)^m$ since a polynomially compact operator*) has a non-trivial invariant subspace by a recent result [3], and we can conclude that $TQ=0$. For the

*) An operator T is called polynomially compact if $p(T)$ is a compact operator for some polynomial $p(.)$.

sake of completeness, we shall state the detail of the proof. Consider the family $\boldsymbol{F}$ of all subspaces $\mathfrak{M}\subset H_1$ which are invariant under T_1, the restriction of T onto H_1, and satisfy the condition $\|P_{\mathfrak{m}}T_1^m\| = \|T_1^m\|$. Then we can see that the family $\boldsymbol{F}$ contains a minimal member $\mathfrak{M}_0$. This is an immediate consequence of Zorn's lemma and Lemma 2 which will be proved later. Of course, $\mathfrak{M}_0 \neq (0)$ since $T_1^m \neq 0$. If $\dim \mathfrak{M}_0 \geqq 2$, $\mathfrak{M}_0$ contains a non-trivial subspace $\mathfrak{M}$ which is invariant under T_1 by [3]. $\mathfrak{M}$ reduces T_1 by hypothesis and we have

$$\|T_1^m\| = \|P_{\mathfrak{m}_0}T_1^m\| = \mathrm{Max}\{\|P_{\mathfrak{m}}T_1^m\|, \|(P_{\mathfrak{m}_0}-P_{\mathfrak{m}})\,T_1^m\|\}\,.$$

It follows that either $\mathfrak{M}$ or $\mathfrak{M}_0\cap\mathfrak{M}^{\perp}\subsetneqq\mathfrak{M}_0$ is a member of $\boldsymbol{F}$, and this contradicts the minimality of $\mathfrak{M}_0$. In case $\dim\mathfrak{M}_0 = 1$, $\mathfrak{M}_0 = \{\alpha x\colon \alpha \text{ complex}\}$ for some unit vector $x \in H_1$ and $Tx=\lambda x$ for some complex number λ, which is also a contradiction. At any rate, it does not happen that $T_1 \neq 0$ and the proof of the theorem is finished if Lemma 2 is proved.

The following lemma shows that the family $\boldsymbol{F}$ in the proof of Theorem 2 is inductive and assures the existence of $\mathfrak{M}_0$.

LEMMA 2. *Let A be a compact operator on a Hilbert space H and $\{\mathfrak{M}_\alpha\}$ be a totally ordered family (by inclusion) of subspaces each of which is invariant under A and satisfies $\|P_{\mathfrak{m}_\alpha}A\| = \|A\|$. Then $\|P_{\mathfrak{n}}A\| = \|A\|$ where $\mathfrak{N}=\bigcap_\alpha \mathfrak{M}_\alpha$.*

PROOF. Let $\varepsilon > 0$ be given. For each α, there exists a unit vector $x_\alpha \in H$ such as

$$\|(P_{\mathfrak{n}}-P_{\mathfrak{m}_\alpha})Ax_\alpha\| > \|(P_{\mathfrak{n}}-P_{\mathfrak{m}_\alpha})A\| - \frac{\varepsilon}{4}.$$

Since $\{x_\alpha\}$ is a bounded set and A is a compact operator, we can choose a subnet $\{x_{\alpha_\nu}\}$ and an $x \in H$ such that $\{Ax_{\alpha_\nu}\}$ converges to x strongly. As $P_{\mathfrak{n}}$ is a strong limit of $P_{\mathfrak{m}_\alpha}$ there exists a ν such that

$$\|(P_{\mathfrak{m}_{\alpha_\nu}}-P_{\mathfrak{n}})x\| < \frac{\varepsilon}{4}, \quad \|Ax_{\alpha_\nu}-x\| < \frac{\varepsilon}{4}.$$

Then we have

$$\begin{aligned}\|(P_{\mathfrak{n}}-P_{\mathfrak{m}_{\alpha_\nu}})A\| &< \|(P_{\mathfrak{n}}-P_{\mathfrak{m}_{\alpha_\nu}})Ax_{\alpha_\nu}\| + \frac{\varepsilon}{4}\\ &\leqq \|(P_{\mathfrak{n}}-P_{\alpha_{\alpha_\nu}})(Ax_{\alpha_\nu}-x)\| + \|(P_{\mathfrak{n}}-P_{\mathfrak{m}_{\alpha_\nu}})x\| + \frac{\varepsilon}{4}\end{aligned}$$

$$\leqq 2\|Ax_{\alpha_\nu}-x\| + \|(P_{\mathfrak{n}}-P_{\mathfrak{m}_{\alpha_\nu}})x\| + \frac{\varepsilon}{4} < \varepsilon .$$

As $\varepsilon > 0$ is arbitrary, we have

$$\|P_{\mathfrak{n}}A\| = \lim_{\nu}\|P_{\mathfrak{m}_{\alpha_\nu}}A\| = \|A\|$$

which completes the proof.

3. In connection with the results in section 2 and [5] we shall prove a theorem. Following V. Istrătescu [4], an operator T is called of class (N) if $\|T^2x\| \geqq \|Tx\|^2$ for all unit vectors $x \in H$. Then the following theorem is a special case of the result in [5].

THFOREM B. *Let T be an operator of class (N) such as T^m is compact for some integer $m \geqq 0$. Then T is necessarily a normal operator.*

The following lemma is an easy exercise (see [2]) and we omit the proof.

LEMMA 3. *Let T be a hyponormal operator and P a projection. If $PTP=TP$ and TP is normal, $TP=PT$.*

From Theorem B and Lemma 3, we have the following theorem.

THEOREM 3. *Let T be a hyponormal operator and $\mathfrak{M}$ a subspace which is invariant under T. If $T^mP_{\mathfrak{m}}$ is a compact operator for some integer $m \geqq 0$, then $\mathfrak{M}$ reduces T.*

PROOF. For a unit vector $x \in H$,

$$\|TP_{\mathfrak{m}}x\|^2 = (T^*TP_{\mathfrak{m}}x, x) \leqq \|T^*TP_{\mathfrak{m}}x\|$$
$$\leqq \|T^2P_{\mathfrak{m}}x\| = \|(PT_{\mathfrak{m}})^2x\| .$$

Hence $TP_{\mathfrak{m}}$ is an operator of class (N). On the other hand, $T^mP_{\mathfrak{m}} = (TP_{\mathfrak{m}})^m$ is a compact operator by the hypothesis and so $TP_{\mathfrak{m}}$ is a normal operator by Theorem B. Therefore $TP_{\mathfrak{m}}=P_{\mathfrak{m}}T$ by Lemma 3.

ADDENDUM. Professor T. Andô has kindly remarked in a private communication that when T is a normal operator T^m is a compact operator if and only if T is a compact operator and hence our Theorem 1 is an immediate consequence of Theorem A.

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