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THE RADIUS OF UNIVALENCE OF CERTAIN ANALYTIC FUNCTIONS

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In a recent paper [1] J. Krzyz and M. Reade have proved the following two theorems.

THEOREM 1. *Suppose that*

$$f(z) = z + a_2 z^2 + \cdots\cdots$$

and

$$g(z) = z + b_2 z^2 + \cdots\cdots$$

are analytic in the unit disc D and suppose that $\mathrm{Re}\left[\frac{f(z)}{g(z)}\right] > 0$ *in D. If* $g(z)$ *is univalent in D, then* $f(z)$ *is univalent and star-like in the disc* $|z| < 2-\sqrt{3}$.

This result is sharp.

THEOREM 2. *Suppose that*

$$f(z) = z + a_2 z^2 + \cdots\cdots$$

and

$$g(z) = z + b_2 z^2 + \cdots\cdots$$

are regular in the unit disc D, and suppose that $\left|\frac{f(z)}{g(z)} - 1\right| < 1$ *in D. If* $g(z)$ *is univalent in D, then* $f(z)$ *is univalent and star-like in* $|z| < \frac{1}{3}$.

This result is sharp.

In this note we shall prove the following theorem which generalizes the above two theorems.

THEOREM 3. *Suppose that*

$$f(z) = z + a_2 z^2 + \cdots\cdots\cdots$$

and

$$g(z) = z + b_2 z^2 + \cdots\cdots\cdots$$

are regular in the unit disc D, and suppose that $\left|\frac{f(z)}{g(z)} - \alpha\right| < \alpha \left(\alpha > \frac{1}{2}\right)$ *in D. If* $g(z)$ *is univalent in D, then* $f(z)$ *is univalent and star-like in* $|z| < \min\left(r, \tanh\left(\frac{1}{2}\right)\right)$, *where r is the smallest positive root of the equation*

$$\left(1 - \frac{1}{\alpha}\right)x^3 - 3\left(1 - \frac{1}{\alpha}\right)x^2 - 3x + 1 = 0.$$

The number r is sharp in the sense that $f(z)$ *is not star-like in a larger circle.*

The proof of the theorem depends on the following lemma due to J. Krzyz and M. Reade [1].

LEMMA. *Let*

$$g(z) = z + b_2 z^2 + \cdots\cdots$$

be analytic and univalent in the unit disc D. Then the inequality

$$\text{(1)} \qquad \operatorname{Re}\left[\frac{zg'(z)}{g(z)}\right] \geqq \frac{1-|z|}{1+|z|}$$

holds for $|z| \leqq \tanh\frac{1}{2} = 0.46212\cdots$.

The bound (1) is sharp for each z, it is attained by a rotation of the Koebe's function $K(z) = \frac{z}{(1-z)^2}$.

PROOF OF THEOREM 3. Let

$$\text{(2)} \qquad \psi(z) = \frac{1}{\alpha}\frac{f(z)}{g(z)} - 1,$$

then

$$\psi(0) = \frac{1}{\alpha} - 1. \tag{3}$$

Let

$$\psi_1(z) = \frac{\psi(z) - \psi(0)}{1 - \psi(0)\psi(z)}, \tag{4}$$

then

$$\psi_1(0) = 0 \quad \text{and} \quad |\psi_1(z)| < 1.$$

Therefore,

$$\psi_1(z) = z\varphi(z), \tag{5}$$

here $\varphi(z)$ is analytic for $|z|<1$ and $|\varphi(z)| \leqq 1$. From (2), (3) and (4) we get

$$\frac{f(z)}{g(z)} = \frac{1 + z\varphi(z)}{1 + \left(\frac{1}{\alpha} - 1\right) z\varphi(z)}. \tag{6}$$

By differentiating and simplifying (6) we have

$$\frac{zf'(z)}{f(z)} = \frac{zg'(z)}{g(z)} + \frac{z\left(2 - \frac{1}{\alpha}\right)(\varphi(z) + z\varphi'(z))}{(1 + z\varphi(z))\left(1 + \left(\frac{1}{\alpha} - 1\right) z\varphi(z)\right)}. \tag{7}$$

Taking the real parts on both sides of (7) and using (1) gives

$$\operatorname{Re}\left[\frac{zf'(z)}{f(z)}\right] \geqq \frac{1 - |z|}{1 + |z|} - \left| \frac{\left(2 - \frac{1}{\alpha}\right)(z\varphi(z) + z^2\varphi'(z))}{(1 + z\varphi(z)\left(1 + \left(\frac{1}{\alpha} - 1\right) z\varphi(z)\right)} \right|. \tag{8}$$

Now, for $\frac{1}{2} < \alpha < 1$,

$$\left|(1 + z\varphi(z))\left(1 + \left(\frac{1}{\alpha} - 1\right) z\varphi(z)\right)\right| \geqq (1 - |z\varphi(z)|)\left(1 - \left(\frac{1}{\alpha} - 1\right)|z\varphi(z)|\right)$$

$$= 1 - \frac{|z\varphi(z)|}{\alpha} + \left(\frac{1}{\alpha} - 1\right)|z\varphi(z)|^2.$$

On the other hand, for $\alpha \geqq 1$,

$$\left|(1+z\varphi(z))\left(1+\left(\frac{1}{\alpha}-1\right)z\varphi(z)\right)\right| = \left|1+\frac{z\varphi(z)}{\alpha}+\left(\frac{1}{\alpha}-1\right)z^2\varphi^2(z)\right|$$

$$\geqq 1-\frac{|z\varphi(z)|}{\alpha}-\left(1-\frac{1}{\alpha}\right)|z\varphi(z)|^2.$$

Hence, in either case, we have

$$(9)\quad \left|(1+z\varphi(z))\left(1+\left(\frac{1}{\alpha}-1\right)z\varphi(z)\right)\right| \geqq 1-\frac{1}{\alpha}|z\varphi(z)|+\left(\frac{1}{\alpha}-1\right)|z\varphi(z)|^2.$$

(8) gives in connection with (9)

$$\operatorname{Re}\left[\frac{zf'(z)}{f(z)}\right] \geqq \frac{1-|z|}{1+|z|}-\frac{\left(2-\frac{1}{\alpha}\right)\left[|z\varphi(z)|+|z|^2\frac{1-|\varphi(z)|^2}{1-|z|^2}\right]}{1-\frac{1}{\alpha}|z\varphi(z)|+\left(\frac{1}{\alpha}-1\right)|z\varphi(z)|^2},$$

where we used the estimate $|\varphi'(z)| \leqq \frac{1-|\varphi(z)|^2}{1-|z|^2}$ [2, p. 18].

After some simplification we get

$$(10)\quad \operatorname{Re}\left[\frac{zf'(z)}{f(z)}\right] \geqq \frac{1-|z|}{1+|z|}-\frac{\left(2-\frac{1}{\alpha}\right)}{1-|z|^2}\left\{\frac{|z\varphi(z)|(1-|z|^2)+|z|^2(1-|\varphi(z)|^2)}{1-\frac{1}{\alpha}|z\varphi(z)|+\left(\frac{1}{\alpha}-1\right)|z\varphi(z)|^2}\right\}.$$

It can be easily proved that

$$(11)\quad \frac{|z\varphi(z)|(1-|z|^2)+|z|^2(1-|\varphi(z)|^2)}{1-\frac{1}{\alpha}|z\varphi(z)|+\left(\frac{1}{\alpha}-1\right)|z\varphi(z)|^2} \leqq \frac{|z|(1-|z|^2)}{1-\frac{1}{\alpha}|z|+\left(\frac{1}{\alpha}-1\right)|z|^2}.$$

Hence from (10) and (11) we have

$$(12)\quad \operatorname{Re}\left[\frac{zf'(z)}{f(z)}\right] \geqq \frac{1-|z|}{1+|z|}-\frac{\left(2-\frac{1}{\alpha}\right)|z|}{1-\frac{1}{\alpha}|z|+\left(\frac{1}{\alpha}-1\right)|z|^2}$$

which holds for $|z| \leqq \tanh \frac{1}{2}$.

The function $f(z)$ will be star-like if $\operatorname{Re}\left[\frac{zf'(z)}{f(z)}\right] > 0$, that is if

$$\frac{1-|z|}{1+|z|} - \frac{\left(2-\frac{1}{\alpha}\right)|z|}{1-\frac{1}{\alpha}|z| + \left(\frac{1}{\alpha}-1\right)|z|^2} > 0,$$

or, if

$$\left(1-\frac{1}{\alpha}\right)|z|^3 - 3\left(1-\frac{1}{\alpha}\right)|z|^2 - 3|z| + 1 > 0. \tag{13}$$

Let

$$P(x) = \left(1-\frac{1}{\alpha}\right)x^3 - 3\left(1-\frac{1}{\alpha}\right)x^2 - 3x + 1.$$

Now we observe that $P(0) = 1 > 0$ and $P(1) = -2\left(2-\frac{1}{\alpha}\right) < 0$. Therefore the smallest positive root of the equation $P(x) = 0$, lies between 0 and 1. If we denote this root by r, it follows that the inequality (13) holds for $x = |z| < r$. Hence, we conclude that $f(z)$ is univalent and star-like for $|z| < \min\left(r, \tanh\left(\frac{1}{2}\right)\right)$.

We proceed to show that the result is best possible. Take

$$g(z) = \frac{z}{(1-z)^2},$$

$$f(z) = g(z)\left[\frac{1+z}{1+\left(\frac{1}{\alpha}-1\right)z}\right].$$

Now

$$\left|\frac{1}{\alpha}\frac{f(z)}{g(z)} - 1\right| = \left|\frac{1}{\alpha}\frac{1+z}{1+\left(\frac{1}{\alpha}-1\right)z} - 1\right| = \left|\frac{\left(\frac{1}{\alpha}-1\right)+z}{1+\left(\frac{1}{\alpha}-1\right)z}\right| < 1,$$

when $|z| < 1$ and $\left|\frac{1}{\alpha}-1\right| < 1$.

Also

$$\frac{zf'(z)}{f(z)} = z\left[\frac{1}{z} + \frac{1}{1+z} + \frac{2}{1-z} - \frac{\left(\frac{1}{\alpha}-1\right)}{1+\left(\frac{1}{\alpha}-1\right)z}\right] = 0 .$$

When

$$\left(1-\frac{1}{\alpha}\right)z^3 - 3\left(1-\frac{1}{\alpha}\right)z^2 - 3z + 1 = 0 .$$

It follows that $\frac{zf'(z)}{f(z)}$ vanishes at $z=r$, showing that the number r can not be improved.

This completes the proof of the theorem.

REMARK: Theorems 1 and 2 are special cases of Theorem 3 and follow from it by taking $\alpha=\infty$ and 1 respectively.

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