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PARTIALLY CONFORMAL TRANSFORMATIONS WITH RESPECT TO $(m-1)$-DIMENSIONAL DISTRIBUTIONS OF m-DIMENSIONAL RIEMANNIAN MANIFOLDS, II

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This is part II of my preceding paper [*] and contains chapter III. As an application of Lemma 15. 10, we consider a regular, compact K-contact Riemannian manifold M ($\dim M > 3$) and its fibering $M \to M/\zeta$ ([21]), where ζ is an associated vector field with a given contact form. The distribution D is, in this case, an orthogonal distribution to ζ with respect to an associated Riemannian metric. Let u be an infinitesimal $[m-1]^s$-conformal transformation on M, then it induces an infinitesimal conformal transformation u on M/ζ by the Lemma, and it is known that any infinitesimal conformal transformation on a compact almost Kaehlerian manifold is a Killing vector field ([25], [26]). Thus we see that u is an infinitesimal $[m-1]^s$-isometry.

In §17, generalizing Lemma 15.10, we show the invariance of the coefficient α for g of φ^*g on each trajectory of ζ. As a continuation of §15, we study the structure of $\mathfrak{P}^{sc}$ in §18. In §§20$\sim$23, we discuss the properties of $(m-1)$-conformal transformations or infinitesimal $(m-1)$-conformal transformations in analogous way to the usual conformal transformations in Riemannian geometry.

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Chapter III

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Chapter III

17. A property of α. Every $u \in \mathfrak{P}^s$ generates a local 1-parameter group φ_t of local $(m-1)^s$-conformal transformations. In §15 by using Lemma 15.10 we have seen that the coefficient α_t for g of $\varphi_t{}^*g$ is constant on each trajectory of ζ, if ζ_U is a Killing vector field. Generally we prove

LEMMA 17.1. *If ζ_U and $\xi_{'V}$ are Killing vector fields on each U and V and φ is an $(m-1)^s$-conformal transformation of M to N, then $\zeta\alpha=0$ holds.*

PROOF. Taking the Lie derivatives with respect to ζ of the equation $\varphi^*h=\alpha g+\beta w\otimes w$, we have

$$L(\zeta)\varphi^*h = (\zeta\alpha)g + (\zeta\beta)w \otimes w\,.$$

As $L(\zeta)\varphi^*h = \varphi^*(L(\varphi\zeta)h)$ and $\varphi\zeta = \mu\xi$, we have

$$\gamma\varphi^*(d\mu) \otimes w + \gamma w \otimes \varphi^*(d\mu) = (\zeta\alpha)g + (\zeta\beta)w \otimes w\,.$$

Therefore $\zeta\alpha=0$ holds.

By this Lemma, we get

PROPOSITION 17.2. *If ${}^e\zeta$ is complete, regular and ζ_U is a Killing vector field for each U, then every $\varphi \in \Pi^s$ on M induces a conformal transformation on M/ζ.*

18. The structure of Lie algebra $\mathfrak{P}^{sc}$. In §15 we proved that the subgroup Π^{sc} is a Lie transformation group on a manifold on which ${}^e\zeta$ is complete, regular and ζ_U is a Killing vector field for each U. In the proof, we made use of the fact that any infinitesimal $[m-1]^s$-conformal transformation on M induces an infinitesimal conformal transformation on M/ζ. In this section we consider the converse. Let a vector field X^* on M be the lift of a vector field X on M/ζ with respect to w, i.e. it is characterized by $\pi X^*=X$ and $w(X^*)=0$. For any vector fields X and Y on M/ζ, the relations

$$[X^*, \zeta] = 0\,, \qquad (18.1)$$

(18. 2) $$[X^*, Y^*] = [X, Y]^* - dw(X^*, Y^*)\cdot\zeta$$

hold good. As in §15 h denotes a Riemannian metric on M/ζ which satisfies $g=\pi^*h+w\otimes w$.

LEMMA 18.1. *Suppose that* ${}^{\varepsilon}\zeta$ *is complete, regular and* ζ_U *is a Killing vector field on each* U. *If* X *is an infinitesimal conformal transformation such that* $L(X)h=Ah$, A *denoting a scalar function on* M/ζ, *then* X^* *is an infinitesimal* $(m-1)$*-conformal transformation such that*

(18. 3) $$L(X^*)g = ag+w\otimes i(X^*)dw + i(X^*)dw\otimes w +(-a)w\otimes w\,,$$

where $a=A\cdot\pi$ *is a scalar function on* M.

PROOF. Let Y^*, Z^* be lifts of vector fields Y, Z on M/ζ, then we have

$$\begin{aligned}(L(X^*)g)(Y^*, Z^*) &= X^*\cdot g(Y^*, Z^*) - g([X^*, Y^*], Z^*) - g(Y^*, [X^*, Z^*])\\ &= (L(X)h)(Y, Z)\cdot\pi\\ &= ((A\cdot\pi)g)(Y^*, Z^*)\,.\end{aligned}$$

Similarly

$$\begin{aligned}(L(X^*)g)(Y^*, \zeta) &= -g(-dw(X^*, Y^*)\,\zeta, \zeta)\\ &= (i(X^*)\,dw)(Y^*)\,,\\ (L(X^*)g)(\zeta, \zeta) &= 0\,.\end{aligned}$$

These three equations imply (18. 3), since $i(\zeta)dw=0$.

LEMMA 18.2. *In Lemma* 18.1, X^* *is special if and only if* $L(X^*)w=0$.

PROOF. By (18. 3) X^* is special if and only if $i(X^*)dw$ is proportional to w, and this is equivalent to $i(X^*)dw=0$ by virtue of $i(\zeta)dw=0$.

We can prove that if M admits a vector field u such that $w_U(u)$ is constant in each U and $L(u)w=cw$ for a scalar function c, then $c=0$. So we consider the case where $w_U(u)$ is not constant, let ${}^{\varepsilon}f=\{f_U\}$ be a family of scalar functions f_U such that ${}^{\varepsilon}f^{\varepsilon}\zeta$ is a vector field. Then

(18. 4) $$L(f\zeta)g = w\otimes df + df\otimes w\,.$$

Thus if X is an infinitesimal conformal transformation on M/ζ, then $u=X^*+f\zeta$ is an infinitesimal $(m-1)$-conformal transformation:

$$
(18.5) \qquad L(u)g = ag + w \otimes \{i(X^*)dw + df - \zeta f\cdot w\} + \{i(X^*)dw + df-\zeta f\cdot w\} \otimes w + (2\zeta f-a)w \otimes w .
$$

And we have

$$
(18.6) \qquad L(u)w = i(X^*)dw + df .
$$

Thus, in order that a vector field $X^*+f\zeta$ belongs to $\mathfrak{P}^{sc}$, it is necessary and sufficient that f is a solution of the equation

$$
(18.7) \qquad i(X^*)dw + df - cw = 0
$$

for some constant c. Suppose that D is completely integrable and $M\to M/\zeta$ has a global section S which is an integral submanifold of D. As in this case the equation (18.7) is equivalent to $\zeta f=c$ and $Y^*f=0$ for any vector field Y on M/ζ, we can solve ${}^{\varepsilon}f$ by giving the initial condition (constants) on S. Notice here that the complete integrability of D is equivalent to the fact that ζ is a parallel field.

From Lemmas 18.1 and 18.2 the next Proposition follows.

PROPOSITION 18.3. *If ${}^{\varepsilon}\zeta$ is parallel, regular and complete, then for any infinitesimal conformal transformation X on M/ζ, X^* is an infinitesimal $[m-1]^s$-conformal transformation such that $L(X^*)w = 0$.*

Let $\mathfrak{C}$ be a Lie algebra of all infinitesimal conformal transformations on M/ζ and $\mathfrak{C}^*$ be one composed of lifts of all ellements of $\mathfrak{C}$, and we get

THEOREM 18.4. *Assume that ${}^{\varepsilon}\zeta$ is parallel, regular and complete, then we have the direct decomposition*

$$
\mathfrak{P}^{sc} \approx \mathfrak{C}^* + \mathfrak{R} ,
$$

where $\mathfrak{R}$ is one of the followings:

(a) *If ${}^{\varepsilon}\zeta$ does not define a vector field on M, $\mathfrak{R}=\{0\}$, or $\{r^{\varepsilon}f^{\varepsilon}\zeta\,; r\in R\}$.*
(b) *If ${}^{\varepsilon}\zeta$ defines a vector field ζ on M, $\mathfrak{R}=\{r\zeta\,; r\in R\}$ or $\{r\zeta+sf\zeta\,; r,s\in R\}$.*

In (a) *or* (b), ${}^{\varepsilon}f$ *is a family of certain functions* f_U *on* U, *and* f *is a certain function on* M.

PROOF. This decomposition is exactly given by $u = (\pi u)^* + w(u)\cdot\zeta$ for $u \in \mathfrak{P}^{sc}$, where we have $(\pi u)^* \in \mathfrak{C}^*$ by Proposition 18.3 and $w(u)\cdot\zeta$ belongs to some $\mathfrak{K}$.

REMARK. Under the same conditions as in Theorem 18.4, we have the decomposition $\mathfrak{P}^s \approx \mathfrak{C}^* + \mathfrak{K}$, where $\mathfrak{K}$ is spanned by vectors ${}^{\varepsilon}f{}^{\varepsilon}\zeta$ for any family $\{f_U\}$ of functions on U which satisfy $Y^* f_U = 0$ for any vector field Y on M/ζ. $\mathfrak{K}$ is generally infinite dimensional.

19. Volume preserving $[m-1]^s$-conformal transformations. Assume that a compact M has a point x such that the integral curve i.e. leaf $l(x)$ of ζ passing through x is closed, and let φ be an $[m-1]^s$-conformal transformation. Then we have $\varphi\zeta = \mu\zeta$, $\mu^2 \cdot \varphi = \alpha + \beta$. We assume that $\alpha + \beta$ is constant and smaller than 1, then the length of $\varphi^k l(x)$ approaches to 0 as $k \to \infty$. As M is compact this can not happen, so $\alpha + \beta$ must be 1. By virtue of (10. 1) and this, we can conclude the following

THEOREM 19.1. *Let* φ *be an* $[m-1]^s$*-conformal transformation which preserves the volume element of a compact* M. *If* $\alpha + \beta$ *is constant and* M *has a closed leaf of* ${}^{\varepsilon}\zeta$, *then* φ *is an isometry.*

Concerning an infinitesimal transformation we have

THEOREM 19.2. *Let* u *be an infinitesimal* $[m-1]^s$*-conformal transformation which preserves the volume element of a compact* M *with properties* (i) *and* (ii). *If* c *is constant, then* u *is a Killing vector field.*

PROOF. We have $2c = a + b = 0$ (Theorem 16.2) and as u is volume preserving, $am + b = 0$ holds, and so u is a Killing vector field.

LEMMA 19.3. *Suppose that* ζ_U *is a Killing vector field for each* U *and* M *has a closed leaf of* ${}^{\varepsilon}\zeta$. *If* u *satisfies* $L(u)\zeta = -c\zeta$ *for some constant* c, *then* $c = 0$.

THEOREM 19.4. *Suppose that* ζ_U *is a Killing vector field for each* U *and* M *has a closed leaf of* ${}^{\varepsilon}\zeta$. *If an infinitesimal* $[m-1]^s$*-conformal transformation preserves the volume element and* c *is constant, then* u *is a Killing vector field.*

PROOFS OF LEMMA 19.3 AND THEOREM 19.4. Let l be a closed leaf of ζ. We take a tublar neighborhood W of l as in §15. Then, under the assumption that ζ_U is a Killing vector field, each leaf of ζ contained in W has the same length as l. On the other hand, as c is constant, the function $\mu_t^2 = (\alpha_t+\beta_t)\cdot\varphi_t^{-1}$ in $\varphi_t\zeta = \mu_t\zeta$ is also constant for each small t. And so μ_t^2 must be 1, namely $c=0$ holds, combining this with $am+b=0$ we have $a=b=0$.

20. A characterization of infinitesimal $(m-1)^s$-conformal transformations on compact Riemannian manifolds. Analogously to the case of infinitesimal conformal transformations (see p. 128, [7]), we construct an integral formula and we get necessary and sufficient conditions for an infinitesimal transformation to define an infinitesimal $(m-1)^s$-conformal transformation on a compact Riemannian manifold. By the same letter u we also denote the covariant vector field $u_i = g_{ij}u^j$. Now we define a $(0, 2)$-tensor field $S=S(u)$ as follows:

$$S_{ij} = u_{i,j} + u_{j,i} - m^{-1}(2u^r{}_{,r}-b)\,g_{ij} - bw_iw_j\,, \tag{20.1}$$

where we have put

$$b = b(u) = 2(m-1)^{-1}(m\,u_{i,j}\,w^iw^j - u^r{}_{,r})\,.$$

First we have

$$S_{ij}\,g^{ij} = 0\,,\quad S_{ij}\,w^iw^j = 0\,. \tag{20.2}$$

Differentiating (20.1) covariantly, we get

$$S_{ij}{}^{,i} = u_{j,i}{}^{i} + R_{ij}u^i + (1-2m^{-1})\,u^i{}_{,ij} + m^{-1}b_j - (bw^iw_j)_{,i}\,, \tag{20.3}$$

where we used the Ricci identity: $u^i{}_{,ji} - u^i{}_{,ij} = R_{ij}u^i$. Let Q be the operator $Q:\, u_i \to 2R_{ij}u^j$, then we have

$$S_{ij}{}^{,i} = [Qu - \Delta u - (1-2m^{-1})\,d\delta u + m^{-1}db]_j - (bw^iw_j)_{,i}\,, \tag{20.4}$$

where $\Delta = d\delta+\delta d$. As S_{ij} is a symmetric tensor, we obtain

$$(S_{ij}u^j)^{,i} = S_{ij}{}^{,i}u^j + 2^{-1}S_{ij}(u^{i,j}+u^{j,i})\,. \tag{20.5}$$

By (20.2), the second term of the right hand side is equal to the inner product of S, i.e. $2^{-1}(S_{ij}S^{ij})$. Now we get

LEMMA 20.1. *Let M be a compact orientable Riemannian manifold, then we have the following integral formula*

$$<S(u), S(u)> = <u, \Delta u - Qu + (1-2m^{-1})d\delta u - m^{-1}db + \zeta b\cdot w - b\delta w\cdot w + b\nabla_\zeta w>$$

for any 1*-form u on M.*

PROPOSITION 20.2. *In order that a vector field u defines an infinitesimal $(m-1)^s$-conformal transformation on a compact M, it is necessary and sufficient that u is a solution of the equation*

$$(20.6)\quad \Delta u - Qu + (1-2m^{-1})d\delta u - m^{-1}db + \zeta b\cdot w - b\delta w\cdot w + b\nabla_\zeta w = 0$$

where $b = 2(m-1)^{-1}(m(\nabla_\zeta u)(\zeta)+\delta u)$.

PROOF. We may assume that M is oriented, because otherwise we can consider the double covering manifold. If u defines an infinitesimal $(m-1)^s$-conformal transformation on M, we have (20.6) by (20.4). Conversely if u satisfies (20.6), by Lemma 20.1 $S=0$ holds. Equivalently u is an infinitesimal $(m-1)^s$-conformal transformation.

PROPOSITION 20.3. *Let M be a compact Riemannian manifold with properties* (i) *and* (ii), *and suppose that u satisfies $L(u)w_U = cw_U$ for some constant c, then u is an infinitesimal $[m-1]^s$-conformal transformation if and only if*

$$(20.7)\qquad \Delta u - Qu + (m-1)^{-1}\{(m-3)\,d\delta u + 2i(\zeta)d\delta u\cdot w\} = 0\,.$$

PROOF. As M has properties (i) and (ii), c must be zero by Theorem 16.2, equivalently we see that $u_{i,j}w^iw^j = 0$ holds. Then we have $b = 2(m-1)^{-1}\delta u$, thus (20.7) is equivalent to (20.6).

THEOREM 20.4. *Let M be a compact* 3*-dimensional Riemannian manifold such that ζ_U is a Killing vector field on each U. Then Π^{sc} is a Lie group.*

PROOF. Let $u \in \mathfrak{P}^{sc}$, then we have $L(u)w=0$ and $L(u)\zeta=0$. On the other hand, as ζ_U is a Killing vector field, we have $i(\zeta)d\delta u = L(\zeta)\delta u = \delta L(\zeta)u = 0$. Therefore by Proposition 20.3, any $u \in \mathfrak{P}^{sc}$ satisfies $\Delta u = Qu$. This system of differential equations is of elliptic type, and $\mathfrak{P}^{sc}$ is finite dimensional [24].

21. The case of negative Ricci curvature. Assume that u is an infinitesimal $[m-1]^s$-conformal transformation which satisfies $L(u)w=0$ on a compact and orientable M. Then as the relations $u_{i,j}w^iw^j=c=0$ hold, by (20. 1) we have

$$u_{i,j}+u_{j,i}=2(m-1)^{-1}u^r{}_{,r}g_{ij}-2(m-1)^{-1}u^r{}_{,r}w_iw_j\,. \tag{21. 1}$$

Contracting (21. 1) with $u^{i,j}$ we get

$$u^{i,j}u_{j,i}=-u_{i,j}u^{i,j}+2(m-1)^{-1}(u^r{}_{,r})^2\,. \tag{21. 2}$$

On the other hand, it is known that

$$<R_1(u,u)+u^{i,j}u_{j,i}-(u^r{}_{,r})^2, 1>=0$$

in any compact orientable Riemannian manifold. Substituting (21. 2) into the last equation, we get

$$<R_1(u,u), 1>=2<\nabla u, \nabla u>+(m-1)^{-1}(m-3)<\delta u, \delta u>\,,$$

from which we can conclude the following

THEOREM 21.1. *If M is compact and the Ricci curvature is negative, then any infinitesimal $[m-1]^s$-conformal transformation u such that $L(u)w=0$ is a parallel field. If the Ricci curvature is negative definite, then there is no non-trivial infinitesimal $[m-1]^s$-conformal transformation satisfying $L(u)w=0$.*

22. The relations of scalar curvatures. The Lie derivatives of the scalar curvature by an infinitesimal $(m-1)$-conformal transformation is written as (14. 6) and it satisfies (14. 7) which is a simple relation. However the relation of ${}^{\varphi}R$ and R for an $(m-1)$-conformal transformation φ is not so simple, so we impose some assumptions on manifolds M and transformations φ. One of utilities of the relations of the scalar curvatures ${}^{\varphi}R$ and R is to obtain the analogous theorems to Theorems 16.10 and 16.12. Accordingly we take up two cases (a) and (b) in this section.

(a) ${}^{\varepsilon}\zeta$, ${}^{\delta}\eta$ are parallel and φ is an $(m-1)^s$-conformal transformation of M to N. Under these assumptions, we have $\zeta\alpha=0$ by Lemma 17.1. Then from (4. 6) we have

$$(22.1)\quad W^i_{jk} = \frac{1}{2\alpha}(\alpha_j\delta^i_k + \alpha_k\delta^i_j - \alpha^i g_{jk}) - \frac{1}{2\alpha}\beta^i w_j w_k + \frac{w^i}{2\alpha(\alpha+\beta)}\{\beta(\zeta\beta)\,w_j w_k - \beta(\alpha_j w_k + \alpha_k w_j) + \alpha(\beta_j w_k + \beta_k w_j)\}$$

from which one can deduce

$$(22.2)\quad W^i_{ji} = \frac{m}{2\alpha}\alpha_j + \frac{1}{2\alpha(\alpha+\beta)}(\alpha\beta_j - \beta\alpha_j)\,,$$

$$(22.3)\quad W^i_{jk}g^{jk} = \frac{2-m}{2\alpha}\alpha^i - \frac{1}{2\alpha}\beta^i + \frac{2\alpha+\beta}{2\alpha(\alpha+\beta)}\zeta\beta\cdot w^i\,,$$

$$(22.4)\quad W^i_{jk}w^j w^k = -\frac{1}{2\alpha}(\alpha^i+\beta^i) + \frac{2\alpha+\beta}{2\alpha(\alpha+\beta)}\zeta\beta\cdot w^i\,.$$

Substituting these into (6.4), after calculation we get

$$(22.5)\quad {}^{\varphi}R = {}'R\cdot\varphi = \frac{R}{\alpha} - \frac{1}{4\alpha^3(\alpha+\beta)^2}\{(m-1)(m-6)\alpha^2 + 2(m^2-8m+11)\alpha\beta + (m-2)(m-7)\beta^2\}(d\alpha, d\alpha) - \frac{1}{2\alpha^2(\alpha+\beta)^2}\{(m-5)\alpha + (m-3)\beta\}(d\alpha, d\beta) + \frac{1}{2\alpha(\alpha+\beta)^2}(d\beta, d\beta) + \frac{1}{\alpha^2(\alpha+\beta)}\{(m-1)\alpha + (m-2)\beta\}\,\delta d\alpha + \frac{1}{\alpha(\alpha+\beta)}\delta\, d\beta - \frac{1}{2\alpha(\alpha+\beta)^2}(\zeta\beta)^2 + \frac{1}{\alpha(\alpha+\beta)}\zeta\zeta\beta\,.$$

Now, as φ is an $(m-1)^s$-conformal transformation, we have $\varphi^*\eta = \gamma w$ where $\gamma^2 = \alpha+\beta$. Taking the exterior derivatives, we have $d\gamma\wedge w = 0$, since w and η are parallel. So $d\gamma$ is proportional to w and we get $d\alpha + d\beta = (\zeta\beta)w$. Then the next two relations are immediate consequences.

$$(22.6)\quad (d\alpha, d\beta) = -(d\alpha, d\alpha)\,,$$

$$(22.7)\quad (d\beta, d\beta) = (d\alpha, d\alpha) + (\zeta\beta)^2\,.$$

Also from $d\alpha + d\beta = (\zeta\beta)\,w$, it follows that

$$\delta d\alpha + \delta d\beta = -\zeta\zeta\beta . \tag{22.8}$$

Thus, by substituting (22.6), (22.7) and (22.8) into (22.5), we can eliminate β from (22.5), and we have

PROPOSITION 22.1. *If ${}^{\varepsilon}\zeta$ and ${}^{\delta}\eta$ are parallel and φ is an $(m-1)^s$-conformal transformation, the scalar curvatures satisfy*

$$\alpha\, {}^{\varphi}R - R = -\frac{(m-2)(m-7)}{4\alpha^2}(d\alpha, d\alpha) + \frac{m-2}{\alpha}\delta\, d\alpha . \tag{22.9}$$

(*b*) ζ_U, $\xi_{'V}$ are Killing vector fields and φ is an $(m-1)^s$-conformal transformation having constant γ^2. In this case, $d\alpha + d\beta = 0$ holds. As $\zeta\alpha = 0$, we have also $\zeta\beta = 0$. Then (4.6) is

$$\begin{aligned} W^i_{jk} = {} & \frac{1}{2\alpha}(\alpha_j \delta^i_k + \alpha_k \delta^i_j - \alpha^i g_{jk} + \alpha^i w_j w_k) \\ & + \frac{\beta}{\alpha}(w_k w^i{}_{,j} + w_j w^i{}_{,k}) - \frac{w^i}{2\alpha}(\alpha_j w_k + \alpha_k w_j) . \end{aligned} \tag{22.10}$$

And by contractions

$$W^i_{ji} = (m-1)(2\alpha)^{-1}\alpha_j , \tag{22.11}$$

$$W^i_{jk} g^{jk} = -(m-3)(2\alpha)^{-1}\alpha^i , \tag{22.12}$$

$$W^i_{jk} w^j w^k = 0 . \tag{22.13}$$

Then by (6.4) we have

PROPOSITION 22.2. *If ζ_U and $\xi_{'V}$ are Killing vector field and φ is an $(m-1)^s$-conformal transformation of M to N such that $\varphi^*\eta_{'V} = \gamma_{'VU} w_U$ for some constant $\gamma_{'VU}$, then we have*

$$\alpha\, {}^{\varphi}R - R = -\frac{(m-2)(m-7)}{4\alpha^2}(d\alpha, d\alpha) + \frac{m-2}{\alpha}\delta d\alpha - \frac{\beta}{\alpha} R_1({}^{\varepsilon}\zeta, {}^{\varepsilon}\zeta) . \tag{22.14}$$

Now we study the analogous properties to the results by M. Obata [10].

THEOREM 22.3. *If M is compact and of non-positive (non-negative resp.) scalar curvature and N of non-negative (non-positive resp.) scalar* *curvature, and if* ${}^{\varepsilon}\zeta$ *is parallel, then there is no* $(m-1)^s$*-conformal transformation of* M *to* N *for which* ${}^{\delta}\eta$ *is also parallel, unless both scalar curvatures vanish. And if both scalar curvatures vanish, every* $(m-1)^s$*-conformal transformation of* M *to* N *for which* ${}^{\delta}\eta$ *is also parallel is an* $(m-1)^s$*-homothety.*

PROOF. If we put $\phi=(1/2)\log\alpha$, we obtain

$$(d\alpha, d\alpha) = 4\alpha^2(d\phi, d\phi)\,, \tag{22.15}$$

$$\delta\, d\alpha = 2\alpha\delta\, d\phi - 4\alpha(d\phi, d\phi)\,. \tag{22.16}$$

And (22.9) turns to

$$\alpha\, {}^{\varphi}R - R = -(m-2)(m-3)(d\phi, d\phi) + 2(m-2)\delta\, d\phi\,. \tag{22.17}$$

Assume that M is compact orientable, then integration of (22.17) gives

$$<\alpha\, {}^{\varphi}R - R, 1> = -(m-2)(m-3)<d\phi, d\phi> \leqq 0\,,$$

from which we have the first part and second part ($m>3$) of the Theorem. To prove the second part ($m=3$) we use (22.17) again.

THEOREM 22.4. *Let* M *and* N *be compact Riemannian manifolds of non-positive scalar curvatures which are not identically equal to zero and assume that* ${}^{\varepsilon}\zeta$ *and* ${}^{\delta}\eta$ *are parallel field, then the* $(m-1)^s$*-conformal transformation* φ *of* M *to* N *is an* $(m-1)^s$*-homothety if and only if* $'R\cdot\varphi = e^{-2\mu}R$ *for some constant* μ.

PROOF. If φ is an $(m-1)^s$-homothety, we have $'R\cdot\varphi = e^{-2\phi}R$ by (22.9). Conversely, assume that $'R\cdot\varphi=e^{-2\mu}R$ for some constant μ and M compact orientable, then

$$(e^{2(\phi-\mu)}-1)R = -(m-2)(m-3)(d\phi, d\phi) + 2(m-2)\delta\, d\phi$$

holds. Contracting the last equation with $(e^{2m(\phi-\mu)}-1)$, and integrating over M we have

$$\begin{aligned} <(e^{2(\phi-\mu)}-1)R, e^{2m(\phi-\mu)}-1> &= (m-2)(m-3)<d\phi, d\phi> \\ &\quad + 3(m+1)(m-2)<e^{m(\phi-\mu)}d\phi, e^{m(\phi-\mu)}d\phi>. \end{aligned} \tag{22.18}$$

Thus ϕ must be constant.

THEOREM 22.5. *Under the same assumption as in Theorem* 22.4, *the* $(m-1)^s$*-conformal transformation* φ *is an* $(m-1)^s$*-isometry if and only if* φ *preserves the scalar curvature.*

PROOF. This is a special case $\phi=\mu=0$ in Theorem 22.4.

23. The case of constant scalar curvature. From Theorems 22.3 and 22.5, one deduces the following

THEOREM 23.1. *Suppose that* M *and* N *are compact and of non-positive constant scalar curvature and* ${}^{\varepsilon}\zeta$ *is parallel field. Then every* $(m-1)^s$*-conformal transformation of* M *to* N *for which* ${}^{\delta}\xi$ *is parallel is an* $(m-1)^s$*-homothety.*

COROLLARY 23.2. *Suppose that* M *is compact and of non-positive constant scalar curvature and* ${}^{\varepsilon}\zeta$ *is parallel. Then every* $[m-1]^s$*-conformal transformation of* M *is an* $[m-1]^s$*-isometry.*

Corresponding to Theorem 16.12, we prove

THEOREM 23.3. *Assume that* M *is compact, of non-positive constant scalar curvature and admits a closed leaf of* ${}^{\varepsilon}\zeta$, *and assume that* ζ_U *is a Killing vector field on each* U. *Then any* $[m-1]^s$*-conformal transformation* φ *of* M *onto itself satisfying* $\varphi^* w_V=\gamma_{VU}w_U$ *for some constant* γ_{VU} *is an isometry.*

PROOF. By the argument in §19, one get $\gamma_{VU}^2=1$ namely $\alpha+\beta=1$. Then, by (22.15) and (22.16), (22.14) can be written as

$$(23.1)\quad (\alpha-1)R=-(m-2)(m-3)(d\phi,d\phi)+2(m-2)\delta d\phi-\alpha^{-1}(1-\alpha)R_1(\zeta,\zeta).$$

Multiplying (23.1) by α^m-1 and integrating over M which is assumed to be compact orientable, we have

$$(23.2)\quad \langle(\alpha-1)R,\alpha^m-1\rangle=(m-2)(m-3)\langle d\phi,d\phi\rangle$$
$$+3(m+1)(m-2)\langle e^{m\phi}d\phi,e^{m\phi}d\phi\rangle$$
$$+\langle\alpha^{-1}(\alpha-1)R_1(\zeta,\zeta),\alpha^m-1\rangle.$$

As $R_1(\zeta,\zeta)$ is non-negative by Lemma 16.8, ϕ or α is constant. By Corollary 10.3, the relations $\alpha=1$ and $\beta=0$ hold, so φ is an isometry.

24. Infinitesimal $(m-1)^s$-conformal transformations which leave the Ricci curvature invariant. Some relations obtained in §14 are referred in this section. Let u be an infinitesimal $(m-1)^s$-conformal transformation on M. Transvecting (14. 3) with g^{jk} and $w^j w^k$ respectively, we have the following two reations

$$(24.1)\qquad g^{jk}L(u)R_{jk} = (1-m)a^r{}_{,r} - b^r{}_{,r} + \zeta\zeta b + 2\zeta b \cdot w^r{}_{,r} + w_{r,j}b^r w^j + b\{(w^r{}_{,r}w^j)_{,j} + (w^{j,r}w_r)_{,j}\}\,,$$

$$(24.2)\qquad 2w^j w^k L(u)R_{jk} = (2-m)a_{j,k}w^j w^k - a^r{}_{,r} - b^r{}_{,r} + b_{j,k}w^j w^k + 2\zeta b\cdot w^r{}_{,r} + 2w_{r,k}b^r w^k + 2b\{w^r{}_{,kr}w^k - w_k{}^{,r}{}_r w^k - w_{j,k}w^k w^j{}_{,r}w^r\}\,.$$

THEOREM 24.1. *Assume that M is compact, ζ_U is a Killing vector field on each U and the scalar curvature R is positive constant. If an infinitesimal $(m-1)^s$-conformal transformation u leaves the Ricci curvature invariant, then it is an infinitesimal $(m-1)^s$-isometry.*

PROOF. From (24. 1) and (24. 2) it follows that

$$(24.3)\qquad (\mathrm{m}-1)\delta da + \delta db + \zeta\zeta b = 0\,,$$

$$(24.4)\qquad \delta da + \delta db + \zeta\zeta b + 4bw^{k,r}w_{k,r} = 0\,.$$

On the other hand, (14. 5) shows that $L(u)g^{jk}\cdot R_{jk} = -aR-bT = 0$, where $T=R_{jk}w^j w^k = w^{j,k}w_{j,k}$. Then by (24. 3) and (24. 4), we get $(2-m)\delta da = 4aR$. So if M is orientable we have $-(m-2)<da, da>=4<a^2R, 1>$. This completes the proof.

25. Appendices.

(*a*) Let u be an infinitesimal $(m-1)$-conformal transformation, transvecting (13. 1) with $w^i w^j$ we get $2u_{i,j}w^i w^j = a+b$. If M is orientable, compact and has properties (i) and (ii), the integration of $2(u_i w^i w^j)_{,j} = a+b$ over M gives $<a+b, 1>=0$. Thus combining this and (16. 1), we have

LEMMA 25.1. *Let M be a compact orientable Riemannian manifold with properties* (i) *and* (ii), *and u be an infinitesimal $(m-1)$-conformal transformation, then*

$$<a, 1> = 0\,,\ <b, 1> = 0$$

hold good. (*cf. Theorem* 16.2)

COROLLARY 25.2. *In a compact M with properties* (i) *and* (ii), *every infinitesimal* $(m-1)$-*homothety is an infinitesimal* $(m-1)$-*isometry.*

(*b*) The orthogonality of u and a geodesic.

THEOREM 25.3. *Assume that u is an infinitesimal* $(m-1)$-*isometry and l is a geodesic which is also an integral curve of the distribution D. Then the inner product of u and a unit tangent vector field X on l to l is constant. Particularly, if u is orthogonal to l at one point of l, then u is orthogonal to l at every point of l.*

PROOF. Since X is a unit tangent vector to a geodesic we have $\nabla_X X|_l =0$. Difierentiating $g(u, X)$ along l we get

$$\nabla_X(g(u, X)) = g(\nabla_X u, X) + g(u, \nabla_X X) .$$

The first term of the right hand side is equal to $u_{i,j}X^iX^j$. As u is an infinitesimal $(m-1)$-isometry and as $w_iX^i = 0$ holds, we have $u_{i,j}X^iX^j = 0$. Thus we have $\nabla_X(g(u, X)) = 0$ on l, so $g(u, X)$ is constant on l.

(c) The functions α_t, β_t and γ_t. Let x_0 be an arbitrary point of M and u be infinitesimal $[m-1]^s$-conformal transformation. And take a neighborhoods U and V $(V\subset U)$ of x_0, where we consider a local 1-parameter group of local transformations $\varphi_t: V \to \varphi_t V \subset U$ $(|t| < q(x_0))$ generated by u as in §15. We have seen that every φ_t is an $[m-1]^s$-conformal transformation:

$$\varphi_t{}^*g = \alpha_t g + \beta_t w \otimes w , \tag{25.1}$$

$$\varphi_t{}^*w = \gamma_t w,\ \gamma_t^2 = \alpha_t + \beta_t . \tag{25.2}$$

We define functions α, β and γ on $(-q(x_0), q(x_0)) \times V$ by $\alpha(t, x)=\alpha_t(x)$, $\beta(t, x) =\beta_t(x)$ and $\gamma(t, x)=\gamma_t(x)$, $t \in (-q(x_0), q(x_0))$, $x \in V$. Then α and β satisfy the following differential equations

$$\frac{\partial\alpha}{\partial t}(t, x) = \alpha(t, x)(a\cdot\varphi_t)(x) \tag{25.3}$$

$$\frac{\partial\beta}{\partial t}(t, x) = \beta(t, x)(a\cdot\varphi_t)(x) + b(\varphi_t x)\{\alpha(t, x) + \beta(t, x)\} . \tag{25.4}$$

We give here a proof for (25.4). From (9.4) we have

$$\beta(t+s, x) = \alpha_s(\varphi_t x)\beta_t(x) + \beta_s(\varphi_t x)(\alpha_t(x) + \beta_t(x)).$$

Therefore we get

$$\beta(t+s, x) - \beta(t, x) = \beta_t(x)\{\alpha_s(\varphi_t x)-1\} + \beta_s(\varphi_t x)\{\alpha_t(x) + \beta_t(x)\}.$$

Then (25.4) follows.

LEMMA 25.4. *Solutions of* (25.3) *and* (25.4) *are*

$$\alpha(t, x) = \exp\left(\int_0^t a(\varphi_s x)\, ds\right), \tag{25.5}$$

$$\beta(t, x) = \exp\left(\int_0^t (a+b)(\varphi_s x)\, ds\right) - \exp\left(\int_0^t a(\varphi_s x)\, ds\right). \tag{25.6}$$

COROLLARY 25.5. *Let u be an infinitesimal* $[m-1]^s$*-conformal transformation, if a and b are constant, we have*

$$\alpha(t, x) = e^{at}, \quad \beta(t, x) = e^{(a+b)t} - e^{at}, \quad \gamma(t, x) = e^{ct} = e^{\frac{1}{2}(a+b)t}.$$

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