Tôhoku Math. Journ.
Vol. 18, No. 4, 1966

SOME PROPERTIES OF POWERS OF MANIFOLDS

YASURÔ TOMONAGA

(Received April 12, 1966)

Introduction. In this paper we shall show some properties of product manifolds by means of the characteristic classes. For this purpose the Hirzebruch's multiplicative sequences play the basic role. It will be shown that the Fermat's problem is related with topology by means of the multiplicative sequences.

1. Throughout this paper we shall deal with the compact orientable differentiable manifolds and use the following notations:

X_n.......... a compact orientable differentiable manifold of dimension n.

$p_i(\in H^{4i}(X_n, Z))$ ....Pontrjagin class.

$\overline{p}_i(\in H^{4i}(X_n, Z))$ ....dual Pontrjagin class. $\left(1+\sum_{i\geqq 1} p_i\right)\left(1+\sum_{i\geqq 1}\overline{p}_i\right)=1$.

$w_i(\in H^i(X_n, Z_2))$ ....Stiefel-Whitney class.

$\bar{w}_i(\in H^i(X_n, Z_2))$ ....dual Stiefel-Whitney class. $\left(1+\sum_{i\geqq 1} w_i\right)\left(1+\sum_{i\geqq 1}\bar{w}_i\right)=1$.

$p_i'(\in H^{4i}(X_n, Z))$ ....a characteristic class defined by a differential form

$$\alpha\Omega_{[j_1j_2}\Omega_{j_2j_3}\cdots\Omega_{j_{2i}j_1]}$$

where Ω denotes the curvature form and α denotes some constant. ([1])

First we recall the known facts:

(a) Differentiable imbedding.

If $X_n\subset E_{n+r}$, then $\bar{w}_i=0$ $(i\geqq r)$ and $\overline{p}_i=0$ $(2i\geqq r)$,

where E_{n+r} denotes the euclidean space of dimension $n+r$. ([9])

(b) If $X_{4n}\subset E_{8n-2q}$, then $A(X_{4n})\equiv 0 \bmod 2^{q+1}$, and if moreover $q=2 \bmod 4$, then

$$A(X_{4n})\equiv 0 \bmod 2^{q+2},$$

where $A(X_{4n})$ denotes the A-genus of X_{4n}. ([2])

(c) If X_{4n} is connected and almost parallelizable, then

$$p_i = 0 \ (n-1 \geqq i \geqq 1)\,. \ ([3])$$

(d) If X_n admits a continuous field of $(n-k)$-frame, then

$$p_i' = 0 \ (2i \geqq k+1) \text{ and } w_i = 0 \ (i \geqq k+1). \ ([4])$$

(e) If X_n is k-parallelizable, then

$$p_j = 0 \ (k \geqq 4j \geqq 4) \text{ and } w_j = 0 \ (k \geqq j \geqq 1). \ ([5])$$

(f) If X_n admits a Riemannian metric on which the q-sectional curvature is constant, then

$$p_j = 0 \ (2j \geqq q \geqq 2)\,,$$

where q denotes an even integer. ([6])

Next we put

$$(1.1) \qquad 1 + \sum_{i \geqq 1} p_i t^i = \prod_j (1 + t\gamma_j)$$

and define a multiplicative series

$$(1.2) \qquad 1 + \sum_{i \geqq 1} K_i(p_1, \cdots, p_i) t^i = \prod_j (1 + c_1 \gamma_j t + \cdots + c_n \gamma_j^n t^n + \cdots),$$

where c_j's denote some rational numbers. For example the index and the A-genus are expressed as follows:

$$(1.3) \qquad \tau(X_{4n}) = L_n(p_1, \cdots, p_n)[X_{4n}]\,,$$

$$(1.4) \quad 1 + \sum_{i \geqq 1} L_i(p_1, \cdots, p_i) t^i = \prod_j \frac{\sqrt{r_j t}}{\operatorname{tgh}\sqrt{r_j t}} = \prod_j \left(1 + \sum_{k \geqq 1} (-1)^{k-1} \frac{2^{2k}}{(2k)!} B_k \gamma_j^k t^k\right),$$

$$(1.5) \qquad A(X_{4n}) = Q_n(p_1, \cdots, p_n)[X_{4n}]\,,$$

$$(1.6) \quad 1 + \sum_{i \geqq 1} Q_i(p_1, \cdots, p_i) t^i = \prod_j \frac{2\sqrt{r_j t}}{\sinh 2\sqrt{r_j t}} = \prod_j \left(1 + \sum_{k \geqq 1} (-1)^k \frac{2^{2k+1}(2^{2k-1}-1)}{(2k)!} B_k \gamma_j^k t^k\right),$$

where B_k denotes the k-th Bernoulli number. ([7])

It is known that for any pair of manifolds X_{4n} and Y_{4n} the following relations hold:

$$(1.7)\quad K_n(p_1, \cdots, p_n)[X_{4n}+Y_{4n}] = K_n(p_1, \cdots, p_n)[X_{4n}]+K_n(p_1, \cdots, p_n)[Y_{4n}],$$

and

$$(1.8)\quad K_{r+s}(p_1, \cdots, p_{r+s})[X_{4r}\cdot X_{4s}] = K_r(p_1, \cdots, p_r)[X_{4r}]\cdot K_s(p_1, \cdots, p_s)[X_{4s}].$$

Moreover it is known that two manifolds X_n and Y_n are cobordant if and only if all Pontrjagin and Stiefel-Whitney numbers of X_n coincide with those of Y_n. The torsions of the cobordism group are of order 2. ([8])

2.

THEOREM 1. *Let X_{4n} be a connected and almost parallelizable manifold and let Y_{4n} be a manifold with constant q-sectional curvature* ($2n \geqq q \geqq 2$). *Then, in general, X_{4n} and Y_{4n} belong to the different cobordism classes. If $X_{4n} \sim Y_{4n}$, then it necessary that*

$$2X_{4n} \sim 0 \quad \textit{and} \quad 2Y_{4n} \sim 0,$$

where $\sim$ denotes "cobordant".

PROOF. From (c) we have

$$(2.1)\qquad p_i = 0 \quad (n-1 \geqq i \geqq 1)$$

in X_{4n}. On the other hand we have from (f)

$$(2.2)\qquad p_j = 0 \quad (2j \geqq q \geqq 2)$$

in Y_{4n}. Therefore the Pontrjagin numbers of X_{4n} are zero except $p_n[X_{4n}]$, while $p_n[X_{4n}]=0$. Hence, in general, X_{4n} and Y_{4n} differ in their Pontrjagin numbers and they belong to the different cobordism classes. The exceptional case is the one where $p_n[X_{4n}]=0$. Then the cobordism components of X_{4n} consist only of torsions, i.e. $2X_{4n}$ is *bord*. Q.E.D.

Next we consider the product manifolds. Let $2X_{4n} \sim 0$, i.e. the cobordism components of the X_{4n} consist only of torsions. Then the manifold $X_{4n}{}^k = \underbrace{X_{4n}\cdot X_{4n}\cdot \cdots \cdot X_{4n}}_{k}$ is of the same property. The converse is also true, i.e.

(2. 3) $$2X_{4n}^{k} \sim 0 \rightleftarrows 2X_{4n} \sim 0\,,$$

because

$$X_{4n} \sim \sum_{i_1+\cdots+i_k=n} \alpha_{i_1\cdots i_k} P_{2i_1}(c) \cdots P_{2i_k}(c) \quad \text{mod torsion},$$

where $P_i(c)$ denotes the complex projective space of complex dimension i, and α's denote some rational numbers.

THEOREM 2. *Let X_{4n}, Y_{4n} and Z_{4n} be any triplet of manifolds for which*

$$K_n[X_{4n}]\,K_n[Y_{4n}]\,K_n[Z_{4n}] \neq 0$$

where $\{K_n(p_1, \cdots, p_n)\}$ denotes a multiplicative sequence, for example the index, the A-genus etc.. Then the relation of the form

$$X_{4n}^{k} + Y_{4n}^{k} \sim Z_{4n}^{k} \qquad \text{mod } \textit{torsion}$$

doesn't hold, where $k \geqq 3$ and we assume that the equation

$$x^k + y^k = z^k \qquad xyz \neq 0$$

doesn't admit any integral solution.

PROOF. If

(2. 4) $$X_{4n}^{k} + Y_{4n}^{k} \sim Z_{4n}^{k} \quad \text{mod torsion},$$

then we have

(2. 5) $$K_{kn}[X_{4n}^{k}] + K_{kn}[Y_{4n}^{k}] = K_{kn}[Z_{4n}^{k}]$$

which leads to

(2. 6) $$(K_n[X_{4n}])^k + (K_n[Y_{4n}])^k = (K_n[Z_{4n}])^k\,.$$

Since $K_n[X_{4n}]$, $K_n[Y_{4n}]$ and $K_n[Z_{4n}]$ are non-zero rational numbers, the Fermat's equation

(2. 7) $$x^k + y^k = z^k$$

must have an integral solution. Q.E.D.

3. Let a product manifold $X_{4n}{}^k$ be connected and almost parallelizable. Then we have from the formula given in [3]

$$\tau(X_{4n}{}^k) = \frac{2^{2kn}(2^{2kn-1}-1)B_{kn}}{(2kn)!} p_{kn}[X_{4n}{}^k] \tag{3.1}$$

and

$$A(X_{4n}^k) = \frac{2^{4kn}(-B_{kn})}{2(2kn)!} p_{kn}[X_{4n}^k] \quad (A(X_{4n}{}^k) = 2^{4kn}\widehat{A}(X_{4n}{}^k)). \tag{3.2}$$

Hence we have

$$\tau(X_{4n}) = 2^{2n}\left\{\frac{(2^{2kn-1}-1)}{(2kn)!}B_{kn}\right\}^{1/k} p_n[X_{4n}] \tag{3.3}$$

and

$$A(X_{4n}) = 2^{4n}\left\{\frac{-B_{kn}}{2(2kn)!}\right\}^{1/k} p_n[X_{4n}]. \tag{3.4}$$

Of course τ and A are integers. Therefore, if either

$$\left\{\frac{(2^{2kn-1}-1)}{(2kn)!}B_{kn}\right\}^{1/k} \quad \text{or} \quad \left\{\frac{-B_{kn}}{2(2kn)!}\right\}^{1/k}$$

is irrational, then we have

$$p_n[X_{4n}] = 0 \tag{3.5}$$

which leads to

$$p_{4n}[X_{4n}{}^k] = 0. \tag{3.6}$$

Hence we have

$$2X_{4n}{}^k \sim 0, \quad \text{i.e.} \quad 2X_{4n} \sim 0. \tag{3.7}$$

Thus we have the

THEOREM 3. *If $X_{4n}{}^k$ is connected and almost parallelizable and either*

$$\left\{\frac{(2^{2kn-1}-1)}{(2nk)!}B_{kn}\right\}^{1/k} \quad \text{or} \quad \left\{\frac{-B_{kn}}{2(2kn)!}\right\}^{1/k}$$

is irrational, then $2X_{4n} \sim 0$.

Next we consider an equation of the form

$$M_{4k}X_{4n}{}^{m}+M_{4(k+n)}X_{4n}{}^{m-1}+\cdots+M_{4(k+mn)}\sim 0 \quad \text{mod torsion}, \tag{3.8}$$

where $k, m, n \geqq 1$ and M_{4i}'s denote some known manifolds and X_{4n} denotes unknown manifold. Let $K_n(p_1, \cdots, p_n)$ be any multiplicative sequence. Then (3.8) becomes

$$a_0X^m + a_1X^{m-1} + \cdots + a_m = 0 \tag{3.9}$$

where

$$\begin{cases} a_i = K_{k+ni}(p_1, \cdots, p_{k+ni})[M_{4(k+ni)}], & i = 0, 1, \cdots, m, \\ X = K_n(p_1, \cdots, p_n)[X_{4n}]. \end{cases} \tag{3.10}$$

Therefore, if (3.9) doesn't have any rational solution, then the relation (3.8) is impossible. For example the relation

$$X_{4n}{}^2 + 3P_{2k}(c)X_{4k} + P_{4k}(c) \sim 0 \quad \text{mod torsion} \tag{3.11}$$

is impossible, because (3.1) leads to

$$\tau(X_{4k})^2 + 3\tau(X_{4k}) + 1 = 0 \quad ([7]\ \text{p. } 85). \tag{3.12}$$

REFERENCES

[1] S. S. CHERN, On the characteristic classes of Riemannian manifolds, Proc. Nat. Acad. Sci. U. S. A., 33(1947), 78–82.
[2] M. F. ATIYAH AND F. HIRZEBRUCH, Quelques théorèmes de non-plongement pour les variétés différentiables, Bull. Soc. Math. France, 87(1959), 383–396.
[3] J. W. MILNOR AND M. A. KERVAIRE, Bernoulli numbers, homotopy groups, and a theorem of Rohlin, Proc. Int. Cong. Math., (1958), 454–458, Camb. Univ. Press, N. Y. 1960.
[4] Y. TOMONAGA, Manifolds admitting continuous field of frames, Tôhoku Math. Journ., 16(1964), 334–339.
[5] Y. TOMONAGA, On k-parallelizable manifolds, Tôhoku Math. Journ., 17(1965), 1–6.
[6] J. A. THORPE, Sectional curvatures and characteristic classes, Ann. of Math., 80(1964), 429–443.
[7] F. HIRZEBRUCH, Neue topologische Methoden in der algebraischen Geometrie, Springer, 1962.
[8] C. T. C. WALL, Determination of cobordism ring, Ann. of Math., 72(1960), 292–311.
[9] J. MILNOR, Lectures on characteristic classes, Princeton University, 1957.

UTSUNOMIYA UNIVERSITY, JAPAN.