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COMPLEX-VALUED DIFFERENTIAL FORMS ON NORMAL CONTACT RIEMANNIAN MANIFOLDS

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Introduction. Almost contact manifolds have, as is well known, an aspect of the odd-dimensional version of almost complex manifolds, and especially normal contact Riemannian manifolds are looked upon as what correspond to Kähler manifolds. The purpose of this paper is to develop a theory on a normal contact Riemannian manifold parallel to that of Kähler manifold through the researches of complex-valued differential forms on the former.

After introducing several operators in the beginning section, in §2 we shall see that a trigrade structure, corresponding to the bigrade one in almost complex manifold, is naturally induced in the algebra of complex-valued forms on a contact Riemannian manifold. In §3 normal contact Riemannian manifolds are discussed from our standpoint of view and §4 is devoted to the investigations of harmonic forms on a compact normal contact Riemannian manifolds. The main result in this section is Theorem 4.4 which asserts the evenness of the r-th Betti numbers of the manifold for certain values of r. Some further researches are pursued in the last section.

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1. Preliminaries. Given an m-dimensional differentiable manifold M, we denote by $V(M)$ the space of complex-valued vector fields on M, by $A(M)$ that of complex-valued forms on M and by Π_r $(r=0, 1, \cdots, m)$ the projection of $A(M)$ onto the subspace $A_r(M)$ of r-forms.

Let M be a contact Riemannian manifold with the structure (η, g). We denote the associated vector field by ξ and the $(1,1)$-tensor field by ϕ as usual. These are related in the following manner:

$$(1.1) \qquad \begin{cases} g(\xi, X) = \eta(X), \quad \eta(\xi) = 1, \quad \phi\xi = 0, \quad \eta(\phi X) = 0\,, \\ 2g(X, \phi Y) = d\eta(X, Y), \quad \phi^2 X = -X + \eta(X)\xi\,, \end{cases}$$

where $X, Y \in V(M)$. For further properties, see [3].

The Riemannian metric g induces in each $A_r(M)$ a scalar product $g(\ ,\ )$: $A_r(M)\times A_r(M)\to A_0(M)$, which is defined by

$$g(\alpha, \beta) = \frac{1}{r!} g^{i_1 j_1} \cdots g^{i_r j_r} \alpha_{i_1\cdots i_r} \beta_{j_1\cdots j_r}$$

for r-forms α and β. Moreover, since a contact manifold is always orientable, through the integral over M an (Hermitian) inner product is defined in the usual way: one of α and β having compact support,

$$<\alpha, \beta> = \int_M \alpha\wedge *\bar{\beta} = \int_M g(\alpha, \bar{\beta}) * 1\,,$$

where $*$ is the star operator by means of the metric g. For a linear operator T in $A(M)$, we denote by T^* the adjoint of T with respect to this inner product. T^* (if it exists) is determined by $<T\alpha, \beta> = <\alpha, T^*\beta>$, and the correspondence $T\to T^*$ is (i) conjugate linear: $(aS+bT)^* = \bar{a}S^*+\bar{b}T^*$ (a and b are complex numbers), (ii) anti-homomorphic: $(ST)^* = T^*S^*$ and (iii) involutive: $T^{**}=T$.

Now, we shall introduce several operators in $A(M)$. We define

$$l= e(\eta), \quad \lambda = l^*, \quad L = e(\varphi), \quad \Lambda = L^* \quad \left(\varphi = \frac{1}{2} d\eta\right),$$

where $e(\alpha)$ denotes the exterior product by a form α: $e(\alpha)\beta = \alpha\wedge\beta$. The following identities are almost trivial:

$$(1.2) \qquad \begin{cases} lL-Ll = 0\,, & dL-Ld = 0\,, & dl+ld = 2L\,, \\ \lambda\Lambda-\Lambda\lambda = 0\,, & \delta\Lambda-\Lambda\delta = 0\,, & \delta\lambda+\lambda\delta = 2\Lambda\,, \\ \lambda L-L\lambda = 0\,, & l\Lambda-\Lambda l = 0\,, & l\lambda+\lambda l = 1\,. \end{cases}$$

The last three follow from the fact that λ is an anti-dervation.

Let us introduce another operator Φ, which is defined by

$$(\Phi\alpha)(X_1,\cdots,X_r) = \sum_{\mu=1}^{r} \alpha(X_1,\cdots,\phi X_\mu,\cdots,X_r)\,, \qquad X_1,\cdots,X_r \in V(M)$$

for an r-form α. First, we shall show that Φ is a derivation:

$$(1.3) \qquad \Phi(\alpha\wedge\beta) = \Phi\alpha\wedge\beta + \alpha\wedge\Phi\beta\,.$$

Clearly it suffices to verify it for the case $\deg\alpha=1$. Let β be an r-form.

Then, for $X_1, \cdots, X_{r+1} \in V(M)$ we have

$$\begin{aligned}\Phi(\alpha\wedge\beta)(X_1,\cdots,X_{r+1}) &= \sum_{\mu=1}^{r+1}\Big\{\sum_{\nu\neq\mu}(-1)^{\nu-1}\alpha(X_\nu)\beta(X_1,\cdots,\widehat{X}_\nu,\cdots,\phi X_\mu,\cdots,X_{r+1}) \\ &\qquad + (-1)^{\mu-1}\alpha(\phi X_\mu)\,\beta(X_1,\cdots,\widehat{X}_\mu,\cdots,X_{r+1})\Big\} \\ &= \sum_{\mu=1}^{r+1}(-1)^{\mu-1}(\Phi\alpha)(X_\mu)\,\beta(X_1,\cdots,\widehat{X}_\mu,\cdots,X_{r+1}) \\ &\quad + \sum_{\nu=1}^{r+1}(-1)^{\nu-1}\alpha(X_\nu)\sum_{\mu\neq\nu}\beta(X_1,\cdots,\widehat{X}_\nu,\cdots,\phi X_\mu,\cdots,X_{r+1}) \\ &= (\Phi\alpha\wedge\beta)(X_1,\cdots,X_{r+1}) + (\alpha\wedge\Phi\beta)(X_1,\cdots,X_{r+1}),\end{aligned}$$

where $\widehat{X}_\nu$ means that X_ν is omitted. The proof for the case $\deg\alpha\geqq 2$ is achieved by induction. We also observe that Φ is skew-Hermitian:

$$\Phi^* = -\Phi. \tag{1.4}$$

For, we get easily $g(\alpha, \Phi\beta) = -g(\Phi\alpha, \beta)$ from the local expression, and then (1.4) follows immediately. As a consequence of (1.3), (1.4) and simple facts $\Phi\eta = \Phi\varphi = 0$, we obtain

PROPOSITION 1.1. *Φ commutes with l, λ, L and Λ.*

Denote by $\theta(\xi)$ (or briefly θ) the Lie derivation with respect to ξ. In a K-contact Riemannian manifold (i.e. a contact Riemannian manifold such that ξ is a Killing vector field) this θ, considered as a linear operator in $A(M)$, also satisfies

$$\theta^* = -\theta. \tag{1.5}$$

In fact, as we have $\theta(\xi)g = 0$ in this case,

$$\theta(\xi)(g(\alpha, \beta)) = g(\theta\alpha, \beta) + g(\alpha, \theta\beta) \tag{1.6}$$

holds for any two forms α and β of the same degree. But, for any function f with compact support we see

$$<\theta(\xi)f, 1> = <\xi(f), 1> = <df, \eta> = <f, \delta\eta> = 0,$$

since $\delta\eta$ vanishes in a contact Riemannian manifold (cf. [3], 9-3). Hence, if one of α and β is of compact support, integrating (1. 6) over M, we get the required result: $<\theta\alpha, \beta> + <\alpha, \theta\beta> = 0$.

Clearly θ commutes with d and λ because of a formula $\theta = d\lambda + \lambda d$ and identities $d^2 = \lambda^2 = 0$. θ is also permutable with l and L since $\theta\eta = \theta\varphi = 0$ are satisfied in a contact Riemannian manifold. Moreover, if the manifold is K-contact, θ commutes with δ, Λ and Φ. The commutativities of θ with δ and Λ are apparent from (1. 5) and those with d and L. To see that θ commutes with Φ, we note that Φ is defined by means of ϕ and $\theta\phi = 0$ holds in a K-contact Riemannian manifold. So, The Lie derivative of Φ vanishes and this implies our assertion. Summarizing above, we have

PROPOSITION 1.2. *In a contact Riemannian manifold, the Lie derivation θ with respect to ξ, considered as a linear operator in $A(M)$, commutes with d, l, λ and L. In a K-contact Riemannian manifold, θ commutes with any of d, δ, l, λ, L, Λ and Φ.*

2. **The trigrade structure in $A(M)$.** Let M be a $(2n+1)$-dimensional contact Riemannian manifold with a structure (η, g, ξ, ϕ). The tensor field ϕ, regarded as a linear operator in $V(M)$, induces a direct sum decomposition of $V(M)$: $V(M) = V_0 + V_i + V_{-i}$, where V_ε $(\varepsilon = 0, i, -i)$ is the eigenspace of ϕ belonging to its eigenvalue ε, and the projections of $V(M)$ onto V_0, V_i and V_{-i} are given by

$$P_0(X) = \eta(X)\,\xi\,, \quad P(X) = \frac{1}{2}\{X - \eta(X)\,\xi - i\phi X\}, \quad \overline{P}(X) = \frac{1}{2}\{X - \eta(X)\,\xi + i\phi X\}$$

for $X \in V(M)$, respectively (cf. [1]). Clearly they satisfy

$$(2.1) \qquad \begin{cases} P_0 + P + \overline{P} = 1, \quad P^2 = P, \quad \overline{P}^2 = \overline{P}, \quad P\overline{P} = \overline{P}P = 0, \\ P\xi = 0, \quad \overline{P}\xi = 0, \quad \phi P = P\phi = iP, \quad \phi\overline{P} = \overline{P}\phi = -iP, \\ \qquad g(PX, Y) = g(X, \overline{P}Y) \quad \text{for} \quad X, Y \in V(M)\,. \end{cases}$$

Now we shall introduce a set of operators $\Pi_{u,p,q}$ $(0 \leqq u \leqq 1\,;\ 0 \leqq p, q \leqq n)$ in $A(M)$. We define $\Pi_{0,p,q}$ by

$$(\Pi_{0,p,q}\alpha)(X_1, \cdots, X_{p+q}) = \frac{1}{p!q!}\sum_\sigma \operatorname{sgn}(\sigma)\alpha(PX_{\sigma(1)}, \cdots, PX_{\sigma(p)}, \overline{P}X_{\sigma(p+1)}, \cdots, \overline{P}X_{\sigma(p+q)})$$

for $\alpha \in A_{p+q}(M)$ and $X_1, \cdots, X_{p+q} \in V(M)$, where the summation on σ is taken over all substitutions of $(1, 2, \cdots, p+q)$, and $\Pi_{1,p,q}$ by

$$\Pi_{1,p,q} = l\,\Pi_{0,p,q}\lambda\,. \tag{2.2}$$

For the sake of convenience, we put $\Pi_{u,p,q}\alpha=0$ if $\deg\alpha\neq u+p+q$. Then, the actions of all $\Pi_{u,p,q}$ are extended to the whole $A(M)$.

These operators $\Pi_{u,p,q}$ are projective; that is, they satisfy

$$\Pi_{u,p,q}\Pi_{v,r,s} = \begin{cases} 0 & \text{if} \quad (u,p,q)\neq(v,r,s) \\ \Pi_{u,p,q} & \text{if} \quad (u,p,q)=(v,r,s)\,, \end{cases} \tag{2.3}$$

$$\sum_{0\leqq u\leqq 1;0\leqq p,q\leqq n} \Pi_{u,p,q} = 1\,. \tag{2.4}$$

(2. 3) is verified by making use of (2. 1) and (1. 2), while (2. 4) is got by expanding $\alpha(X_1, \cdots, X_r) = \alpha(P_0X_1+PX_1+\overline{P}X_1, \cdots, P_0X_r+PX_r+\overline{P}X_r)$.

The complex conjugate and the adjoint of $\Pi_{u,p,q}$ are given by

$$\overline{\Pi}_{u,p,q} = \Pi_{u,q,p}\,, \tag{2.5}$$

$$\Pi^*_{u,p,q} = \Pi_{u,p,q}\,. \tag{2.6}$$

Their proofs are quite simple.

If we put $A_{u,p,q}=\Pi_{u,p,q}A(M)$, by virtue of (2. 4) we have

$$A(M) = \sum_{0\leqq u\leqq 1;0\leqq p,q\leqq n} A_{u,p,q} \qquad \text{(direct sum)}.$$

Moreover, it is easy to see

$$A_{u,p,q}\wedge A_{v,r,s} \subset A_{u+v,p+r,q+s}\,,$$

where we understand that $A_{u+v,p+r,q+s}=(0)$ if one of indices exceeds its proper range. Thus, we have

THEOREM 2.1. *In a $(2n+1)$-dimensional contact Riemannian manifold M, the algebra $A(M)$ of complex-valued forms on M is naturally endowed with a trigrade structure such that one grade is of dimension* 1 *and other two are of dimension n.*

A form α of $A_{u,p,q}$ is said to be *of type* (u, p, q). It is characterized by

$\Pi_{u,p,q}\alpha=\alpha$. In particular, η is clearly of type $(1,0,0)$, while φ is of type $(0,1,1)$ since we can show $\Pi_{0,1,1}\varphi=\varphi$ easily.

An operator T in $A(M)$ is said to be *of type* (v,r,s) if it maps $A_{u,p,q}$ into $A_{u+v,p+r,q+s}$ for every triple (u,p,q). Clearly T is of type (v,r,s) if and only if

$$T\Pi_{u,p,q}=\Pi_{u+v,p+r,q+s}T$$

holds for every (u,p,q). Taking its adjoint, we see immediately that if T is of type (v,r,s), then T^* is of type $(-v,-r,-s)$.

Let us examine the types of our operators. Those of l, λ, L and Λ are apparent, and thence we have

$$\text{(2.7)}\qquad \begin{cases} l\Pi_{0,p,q}=\Pi_{1,p,q}l\,, & l\Pi_{1,p,q}=0\,, \quad \Pi_{0,p,q}l=0\,,\\ \lambda\Pi_{1,p,q}=\Pi_{0,p,q}\lambda\,, & \lambda\Pi_{0,p,q}=0\,, \quad \Pi_{1,p,q}\lambda=0,\\ L\Pi_{u,p,q}=\Pi_{u,p+1,q+1}L\,, & \Lambda\Pi_{u,p,q}=\Pi_{u,p-1,q-1}\Lambda\,. \end{cases}$$

Φ is of type $(0,0,0)$. This follows from

$$\text{(2.8)}\qquad \Pi_{u,p,q}\Phi=(p-q)i\,\Pi_{u,p,q}=\Phi\Pi_{u,p,q}\,.$$

To get (2.8), let α be a $(p+q)$-form. Then, we have

$$\begin{aligned}
&(\Pi_{0,p,q}\Phi\alpha)(X_1,\cdots,X_{p+q})\\
&=\frac{1}{p!q!}\sum_\sigma \operatorname{sgn}(\sigma)\Big\{\sum_{\mu=1}^{p}\alpha(PX_{\sigma(1)},\cdots,\phi PX_{\sigma(\mu)},\cdots,PX_{\sigma(p)},\bar{P}X_{\sigma(p+1)},\cdots,\bar{P}X_{\sigma(p+q)})\\
&\qquad\qquad+\sum_{\mu=p+1}^{p+q}\alpha(PX_{\sigma(1)},\cdots,PX_{\sigma(p)},\bar{P}X_{\sigma(p+1)},\cdots,\phi\bar{P}X_{\sigma(\mu)},\cdots,\bar{P}X_{\sigma(p+q)})\Big\}\\
&=\frac{1}{p!q!}\sum_\sigma \operatorname{sgn}(\sigma)\Big\{\sum_{\mu=1}^{p}\alpha(\cdots,iPX_{\sigma(\mu)},\cdots)+\sum_{\mu=p+1}^{p+q}\alpha(\cdots,-i\bar{P}X_{\sigma(\mu)},\cdots)\Big\}\\
&=(p-q)\,i(\Pi_{0,p,q}\alpha)(X_1,\cdots,X_{p+q})
\end{aligned}$$

for $X_1,\cdots,X_{p+q}\in V(M)$. Hence we see $\Pi_{0,p,q}\Phi=(p-q)i\Pi_{0,p,q}$, and from (2.2) we have a similar relation $\Pi_{1,p,q}\Phi=(p-q)i\Pi_{1,p,q}$. Thus the first equality of (2.8) is obtained. The second equality is merely the adjoint of the first.

In a K-contact Riemannian manifold, θ is also of type $(0,0,0)$. This is verified from the fact that the Lie derivative $\theta(\xi)\Pi_{u,p,q}$ of $\Pi_{u,p,q}$, which is defined by means of η,ξ and ϕ, vanishes in the present case.

3. Normal contact Riemannian manifolds. A contact Riemannian manifold M is said to be normal if the so-called torsion tensor field N, which is defined by

$$N(X,Y)=[X,Y]+\phi[\phi X,Y]+\phi[X,\phi Y]-[\phi X,\phi Y]-\{X(\eta(Y))-Y(\eta(X))\}\xi$$

vanishes identically on M. This condition can be described in terms of the decompositions of $V(M)$ and $A(M)$.

THEOREM 3.1. *In a contact Riemannian manifold, the following three conditions are equivalent*:

(a) *the torsion tensor field* N *vanishes*,

(b) $[\xi, V_i]\subset V_i$ *and* $[V_i, V_i]\subset V_i$,

(c) $dA_{0,p,q}\subset A_{1,p,q}\oplus A_{0,p+1,q}\oplus A_{0,p,q+1}$.

PROOF. The equivalence of (a) and (b) was shown by S. Sasaki and C. J. Hsu [4], and that of (a) and $(c)_{p=1,q=0}$ was got by M. Kurita [2]. Though the analyticity is assumed in both papers, this is not essential so long as the conditions are stated in these forms. So here we have only to prove (c) under the assumption $(c)_{p=1,q=0}$. To do this, notice that any form of $A_{p,q}$ (for brevity we often denote $A_{0,p,q}$ by $A_{p,q}$ and $A_{1,p,q}$ by $A'_{p,q}$) can be expressed locally as a sum of simple forms $\alpha_1\wedge\cdots\wedge\alpha_p\wedge\beta_1\wedge\cdots\wedge\beta_p$ such that $\alpha_1,\cdots,\alpha_p\in A_{1,0}$ and $\beta_1,\cdots,\beta_q\in A_{0,1}$. For such a form $\alpha_1\wedge\cdots\wedge\alpha_p\wedge\beta_1\wedge\cdots\wedge\beta_q$ of $A_{p,q}$, making use of $(c)_{p=1,q=0}$ and its complex conjugate $(c)_{p=0,q=1}$, we have

$$\begin{aligned}d(\alpha_1\wedge\cdots\wedge\alpha_p\wedge\beta_1\wedge\cdots\wedge\beta_q)&\in dA_{1,0}\wedge A_{p-1,q}+dA_{0,1}\wedge A_{p,q-1}\\&\subset(A_{2,0}\oplus A_{1,1}\oplus A'_{1,0})\wedge A_{p-1,q}+(A_{0,2}\oplus A_{1,1}\oplus A'_{0,1})\wedge A_{p,q-1}\\&\subset A_{p+1,q}\oplus A_{p,q+1}\oplus A'_{p,q}.\end{aligned}$$

This completes the proof.

On the other hand, it is known [3] that a contact Riemannian manifold is normal if and only if

$$\nabla_k\phi_{ij}=\eta_i g_{jk}-\eta_j g_{jk} \tag{3.1}$$

is satisfied, where ∇_k denotes the covariant differentiation by means of the metric g. Making use of this fact, here we shall give another condition to be normal in terms of our operators.

THEOREM 3.2. *In a normal contact Riemannian manifold of dimension* $2n+1$, *we have*

$$d\Lambda-\Lambda d=\Phi\delta-\delta\Phi-2\sum_{r=0}^{2n+1}(n+1-r)\lambda\Pi_r. \tag{3.2}$$

Conversely, if (3.2) *is valid in a contact Riemannian manifold. then the manifold is normal.*

PROOF. We proceed by tensor calculus. Let α be an r-form in a normal contact Riemannian manifold, then we have

$$\begin{aligned}
&(d\Lambda\alpha-\Lambda d\alpha)_{i_2\cdots i_r}\\
&=\sum_{\mu=2}^{r}(-1)^{\mu}\nabla_{i_\mu}\Big(\frac{1}{2}\phi^{hk}\alpha_{hki_2\cdots\hat{i}_\mu\cdots i_r}\Big)\\
&\qquad-\frac{1}{2}\phi^{hk}\Big(\nabla_h\alpha_{ki_2\cdots i_r}-\nabla_k\alpha_{hi_2\cdots i_r}+\sum_{\mu=2}^{r}(-1)^{\mu}\nabla_{i_\mu}\alpha_{hki_2\cdots\hat{i}_\mu\cdots i_r}\Big)\\
&=\frac{1}{2}\sum_{\mu=2}^{r}(\nabla_{i_\mu}\phi^{hk})\,\alpha_{hi_2\cdots i_{\mu-1}ki_{\mu+1}\cdots i_r}-\phi^{hk}\nabla_h\alpha_{ki_2\cdots i_r}\\
&=\frac{1}{2}\sum_{\mu=2}^{r}(\xi^h\delta^k_{i_\mu}-\xi^k\delta^h_{i_\mu})\,\alpha_{hi_2\cdots i_{\mu-1}ki_{\mu+1}\cdots i_r}-\phi^{hk}\nabla_h\alpha_{ki_2\cdots i_r}\\
&=(r-1)(\lambda\alpha)_{i_2\cdots i_r}-\phi^{hk}\nabla_h\alpha_{ki_2\cdots i_r}
\end{aligned}$$

and

$$\begin{aligned}
&(\delta\Phi\alpha-\Phi\delta\alpha)_{i_2\cdots i_r}\\
&=-\nabla^k\Big(\phi^h{}_k\alpha_{hi_2\cdots i_r}+\sum_{\mu=2}^{r}\phi^h{}_{i_\mu}\alpha_{ki_2\cdots i_{\mu-1}hi_{\mu+1}\cdots i_r}\Big)+\sum_{\mu=2}^{r}\phi^h{}_{i_\mu}\nabla^k\alpha_{ki_2\cdots i_{\mu-1}hi_{\mu+1}\cdots i_r}\\
&=-(\nabla^k\phi^h{}_k)\,\alpha_{hi_2\cdots i_r}-\phi^h{}_k\nabla^k\alpha_{hi_2\cdots i_r}-\sum_{\mu=2}^{r}(\nabla^k\phi^h{}_{i_\mu})\,\alpha_{ki_2\cdots i_{\mu-1}hi_{\mu+1}\cdots i_r}\\
&=-2n\xi^h\alpha_{hi_2\cdots i_r}+\phi^{hk}\nabla_h\alpha_{ki_2\cdots i_r}-\sum_{\mu=2}^{r}(\xi^h\delta^k_{i_\mu}-\eta_{i_\mu}g^{hk})\,\alpha_{ki_2\cdots i_{\mu-1}hi_{\mu+1}\cdots i_r}\\
&=(-2n+r-1)(\lambda\alpha)_{i_2\cdots i_r}+\phi^{hk}\nabla_h\alpha_{ki_2\cdots i_r}.
\end{aligned}$$

Combining these two equalities, we get (3.2). Conversely, if (3.2) is valid in a contact Riemannian manifold, evaluating (3. 2) on 2-forms, we easily attain to (3. 1). Hence the manifold is normal. Q.E.D.

Clearly Theorem 3.2 remains true if we replace (3. 2) by its adjoint

$$\delta L - L\delta = d\Phi - \Phi d + 2\sum_{r=0}^{2n+1} (n-r)\, l\,\Pi_r . \tag{3. 3}$$

Making use of (3. 2) and (3. 3), we can calculate the commutators of the Laplacian Δ and various operators:

PROPOSITION 3.3. *In a normal contact Riemannian manifold of dimension* $2n+1$, *we have*

$$\Delta\lambda - \lambda\Delta = 2\{\Phi\delta - \delta\Phi - 2\sum(n+1-r)\,\lambda\,\Pi_r\} , \tag{3. 4}$$

$$\Delta\Lambda - \Lambda\Delta = -4\sum(n+1-r)\,\Lambda\,\Pi_r - 2\lambda\delta , \tag{3. 5}$$

$$\Delta\Phi - \Phi\Delta = -2(\theta - \lambda d + l\delta) , \tag{3. 6}$$

$$\Delta l - l\Delta = 2\{d\Phi - \Phi d + 2\sum(n-r)\, l\,\Pi_r\} , \tag{3. 7}$$

$$\Delta L - L\Delta = 4\sum(n-r)\, L\,\Pi_r - 2ld . \tag{3. 8}$$

4. Harmonic forms. In this section we concern with harmonic forms on a *compact* normal contact Riemannian manifold. About this subject, S. Tachibana [5] got some fundamental results:

THEOREM 4.1 (S. TACHIBANA). *Let α be a harmonic r-form on a compact normal contact Riemannian manifold of dimension* $2n+1$. *Then,*

(i) *if* $r \leqq n$, $\lambda\alpha = 0$,

(ii) *if* $r \leqq n+1$, $\Lambda\alpha = 0$,

(iii) $\Phi\alpha$ *is again harmonic.*

These can be verified by using (3. 4)~(3. 6) as well. If we use (3. 7) and (3. 8) instead, we get dual results:

COROLLARY 4.2. *α being the same as in the above theorem,*

(i) *if* $r \geqq n+1$, $l\alpha = 0$,

(ii) *if* $r \geqq n$, $L\alpha = 0$.

Now we shall show that if α is harmonic, each $\Pi_{u,p,q}\alpha$ is also harmonic (or zero). Let α be a harmonic r-form and assume that $r \leqq n$ for the moment. Then, every $\Pi_{1,p,q}\alpha$ vanishes because of (2. 2) and Theorem 4.1 (i). Apply Φ^{μ} $(\mu=0, 1, \cdots, r)$ on $\alpha=\sum_{p+q=r} \Pi_{0,p,q}\alpha$. Making use of (2. 8) repeatedly, we have a system of linear equations in $r+1$ variables $\Pi_{0,t,r-t}\alpha$ $(t=0, 1, \cdots, r)$:

$$\alpha = \sum_{t=0}^{r} \Pi_{0,t,r-t}\alpha$$

$$\Phi\alpha = \sum_{t=0}^{r} (2t-r)\, i\, \Pi_{0,t,r-t}\alpha$$

$$\cdots\cdots\cdots\cdots\cdots$$

$$\Phi^{r}\alpha = \sum_{t=0}^{r} [(2t-r)\, i]^{r}\, \Pi_{0,t,r-t}\alpha.$$

The determinant of the coefficients differs from zero, as is easily seen, and so each $\Pi_{0,t,r-t}\alpha$ is expressed as a linear combination (with constant coefficients) of harmonic forms $\alpha, \Phi\alpha, \cdots, \Phi^{r}\alpha$. Hence $\Pi_{0,t,r-t}\alpha$ is harmonic. For a harmonic r-form α such that $r \geqq n+1$, by a similar way we see that $\Pi_{0,p,q}\alpha=0$ and $\Pi_{1,p,q}\alpha$ is harmonic. Summarizing above, we have

THEOREM 4.3. *In a compact normal contact Riemannian manifold of dimension* $2n+1$, *various components of simple type* $\Pi_{u,p,q}\alpha$ *of a harmonic form* α *are all harmonic (or zero). In particular,* $\Pi_{0,p,q}\alpha=0$ *if* $p+q \geqq n+1$ *and* $\Pi_{1,p,q}\alpha=0$ *if* $p+q \leqq n-1$.

Denote by $H_{r}(M)$ the space of (complex-valued) harmonic r-forms on the manifold M and by $H_{u,p,q}$ that of harmonic forms of type (u, p, q). If $r \leqq n$, from the above theorem we have

$$H_{r}(M) = \sum_{p+q=r} H_{0,p,q} \qquad \text{(direct sum)}.$$

Since the complex dimension $\dim_{C} H_{r}(M)$ of $H_{r}(M)$ is equal to the r-th Betti number $b_{r}(M)$ of M, noting that $H_{u,p,q}$ and $H_{u,q,p}$ are (conjugate) isomorphic, for any odd dimension r $(\leqq n)$ we have

$$b_{r}(M) = \dim_{C} H_{r}(M) = \sum_{p+q=r} \dim_{C} H_{0,p,q} = 2 \sum_{t=0}^{(r-1)/2} \dim_{C} H_{0,t,r-t}.$$

Thus, $b_r(M)$ is necessarily even. Consequently, taking account of the Poincaré duality, we get

THEOREM 4.4. *The r-th Betti number of a compact normal contact Riemannian manifold of dimension* $2n+1$ *is even, if* r *is odd and* $\leqq n$ *or if* r *is even and* $\geqq n+1$.

REMARK. For the purpose to prove this result only, the following argument may be simpler. If we set $(\phi\alpha)(X_1, \cdots, X_r)=\alpha(\phi X_1, \cdots, \phi X_r)$ for $\alpha \in A_r(M)$, we easily see $\phi^2\alpha = (-1)^r \alpha$ for harmonic $r(\leqq n)$-form α. On the other hand, it is known [5] that if α is harmonic, so is $\phi\alpha$. Therefore, the ϕ defines a complex structure in the real vector space $H_r^R(M)$ of real harmonic r-forms on M, and hence the real dimension of $H_r^R(M)$, which is equal to $b_r(M)$, must be even.

5. The decomposition of *d*. Throughout this section we assume that the manifold M in consideration is always a normal contact one.

The operator d is not of simple type; to clarify this situation, we recall Theorem 3.1 (c). Differentiating $A_{1,p,q}=\eta\wedge A_{0,p,q}$, we have a similar relation $dA_{1,p,q}\subset A_{1,p+1,q} \oplus A_{1,p,q+1} \oplus A_{0,p+1,q+1}$, and these relations suggest to define

$$d_1 = \sum_{u,p,q} \Pi_{u,p+1,q}\, d\,\Pi_{u,p,q}\,, \qquad \bar{d}_1 = \sum_{u,p,q} \Pi_{u,p,q+1}\, d\,\Pi_{u,p,q}\,,$$

$$d_2 = \sum_{p,q} \Pi_{1,p,q}\, d\,\Pi_{0,p,q}\,, \qquad d_3 = \sum_{p,q} \Pi_{0,p+1,q+1}\, d\,\Pi_{1,p,q}\,.$$

Then, we have a decomposition $d=d_1+\bar{d}_1+d_2+d_3$ and each of d_1, $\bar{d}_1$, d_2 and d_3 is of simple type. Clearly d_1 and $\bar{d}_1$ are complex conjugate with each other, while both d_2 and d_3 are real operators. The last two have another expressions

$$(5.1) \qquad d_2 = l\theta\,, \qquad d_3 = 2L\lambda\,.$$

In fact, $d_2=\sum\Pi_{1,p,q}d\,\Pi_{0,p,q}=\sum l\,\Pi_{0,p,q}\lambda\, d\,\Pi_{0,p,q}=l\sum\Pi_{0,p,q}\theta\,\Pi_{0,p,q}=l\sum\Pi_{u,p,q}\theta=l\theta$. The latter is verified in a similar manner.

Denote the adjoints of d_1, $\bar{d}_1$, d_2 and d_3 by δ_1, $\bar{\delta}_1$, δ_2 and δ_3 respectively. They are explicitly given by

$$\delta_1 = \sum\Pi_{u,p-1,q}\delta\,\Pi_{u,p,q}\,, \qquad \bar{\delta}_1 = \sum\Pi_{u,p,q-1}\delta\,\Pi_{u,p,q}\,,$$

$$\delta_2 = \sum\Pi_{0,p,q}\delta\,\Pi_{1,p,q} = -\theta\lambda\,, \quad \delta_3 = \sum\Pi_{1,p-1,q-1}\delta\,\Pi_{0,p,q} = 2l\Lambda\,.$$

Now we shall show a result analoguous to a well-known formula in Kähler manifolds.

PROPOSITION 5.1. *In a normal contact Riemannian manifold, we have*

$$d_1\Lambda - \Lambda d_1 = i\bar{\delta}_1\,, \quad \bar{d}_1\Lambda - \Lambda\bar{d}_1 = -i\delta_1\,. \tag{5.2}$$

PROOF. With the aid of (3.2) and (2.8) we can proceed as

$$\begin{aligned} d_1\Lambda - \Lambda d_1 &= \Sigma\,\Pi_{u,p,q-1}(d\Lambda-\Lambda d)\,\Pi_{u,p,q} \\ &= \Sigma\,\Pi_{u,p,q-1}\{\Phi\delta-\delta\Phi-2(n+1-u-p-q)\lambda\}\,\Pi_{u,p,q} \\ &= \Sigma\{(p-q+1)\,i\,\Pi_{u,p,q-1}\cdot\delta\,\Pi_{u,p,q}-\Pi_{u,p,q-1}\delta\cdot(p-q)\,i\,\Pi_{u,p,q} \\ &= i\bar{\delta}_1\,. \end{aligned}$$

The latter is merely the complex conjugate of the former. Q.E.D,

The commutators of d_2 and d_3 with Λ are given by

$$d_2\Lambda - \Lambda d_2 = 0\,, \quad d_3\Lambda - \Lambda d_3 = -2\sum_r (n+1-r)\lambda\Pi_r\,. \tag{5.3}$$

The first is clear from $(5.1)_1$ and the second follows from $(5.1)_2$ and

$$\Lambda L - L\Lambda = \sum (n-r)\,\Pi_r + 2l\lambda\,, \tag{5.4}$$

which is obtained by using (3.2) or directly by tensor calculus. Summing up (5.2) and (5.3), we get

$$d\Lambda - \Lambda d = -i(\delta_1-\bar{\delta}_1) - 2\sum (n+1-r)\,\lambda\,\Pi_r\,. \tag{5.5}$$

As an application of this formula, we shall show

PROPOSITION 5.2. *In a normal contact Riemannian manifold, any closed form of type* $(0, p, 0)$ *is harmonic.*

PROOF. Let α be a form of type $(0, p, 0)$. By considerations on type, it is easily seen that $\lambda\alpha=0$, $\Lambda\alpha=0$ and $\bar{\delta}_1\alpha=\delta_2\alpha=\delta_3\alpha=0$. Therefore, if α is closed besides, it follows from (5.5) that $\delta_1\alpha=0$ and hence $\delta\alpha=0$. Consequently α must be harmonic. Q.E.D.

Now, assume that M is compact. Then any harmonic $r(\leqq n)$-form α satisfies $\lambda\alpha=0$, $\Lambda\alpha=0$ and $\theta\alpha=0$. These conditions are perhaps not sufficient for α to be harmonic. Then — how far are they from the full condition to be harmonic ? To answer this, we make use of the following identities

$$(5.6)\qquad \begin{cases} d_1\delta_1+\delta_1 d_1 = \overline{d}_1\overline{\delta}_1+\overline{\delta}_1\overline{d}_1+2i(\Lambda L-L\Lambda)\theta\,, \\ d_2\delta_2+\delta_2 d_2 = -\theta^2\,, \\ d_3\delta_3+\delta_3 d_3 = 4\{\sum(n+1-r)l\lambda\Pi_r+L\Lambda\}\,. \end{cases}$$

The last two equalities are immediate from (5.1) and (5.4), while the first is verified with the aid of (5.2) and

$$d_1\overline{d}_1 + \overline{d}_1 d_1 + d_2 d_3 + d_3 d_2 = 0\,,$$

which is one of the identities obtained by comparing various types in the expansion of $(d_1+\overline{d}_1+d_2+d_3)^2=0$. From the last two of (5.6) we have two equivalences

$$d_2\alpha = \delta_2\alpha = 0 \Longleftrightarrow \theta\alpha = 0\,,$$

$$d_3\alpha = \delta_3\alpha = 0 \Longleftrightarrow \lambda\alpha = \Lambda\alpha = 0 \qquad (\deg\alpha\leqq n)\,.$$

On the other hand, obviously a form α is harmonic if and only if it satisfies $d_1\alpha=\overline{d}_1\alpha=d_2\alpha=d_3\alpha=0$ and $\delta_1\alpha=\overline{\delta}_1\alpha=\delta_2\alpha=\delta_3\alpha=0$. Hence, taking account of the first equality of (5.6), we have

PROPOSITION 5.3. *In a $(2n+1)$-dimensional compact normal contact Riemannian manifold, an $r(\leqq n)$-form α satisfying $\lambda\alpha=0$, $\Lambda\alpha=0$ and $\theta\alpha=0$ is harmonic if and only if it satisfies $d_1\alpha=\delta_1\alpha=0$ (or $\overline{d}_1\alpha=\overline{\delta}_1\alpha=0$).*

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