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SPECTRAL RESOLUTION OF A HYPONORMAL OPERATOR WITH THE SPECTRUM ON A CURVE

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1. Introduction. In [6], Stampfli proved that if T is a hyponormal operator (i.e. $T^*T \geqq TT^*$) and if the spectrum of T lies on a rectifiable smooth Jordan curve and does not separate the plane, then T is normal, by using the localization technique of Dunford.

The purpose of this note is to extend this result as follows by constructing the resolution of the identity directly:

If T is hyponormal and if the spectrum of T lies on a Jordan curve which consists of a finite number of rectifiable smooth arcs (it may well be the case that the spectrum seperates the plane), then T is normal.

It is known that a hyponormal operator satisfies a certain growth condition on the resolvent (as Def. 1). This growth condition guarantees the single-valued maximal analytic continuations of resolvents under some spectral conditions. Then, in section 3, extending the method of J. Schwartz [4], we shall show the existence of proper invariant subspaces. Next, in section 4, we shall prove that for a hyponormal operator, these subspaces are reducing subspaces, in particular, spectral subspaces. By piecing these subspaces together to form a resolution of the identity, we conclude our theorem.

2. Some preliminaries. Throughout this note, an operator means a bounded linear operator on a Hilbert space H. $\sigma(T)$, $\sigma_p(T)$, $\sigma_c(T)$ and $\sigma_r(T)$ denote the spectrum, the point spectrum, the continuous spectrum and the residual spectrum of an operator T, respectively.

DEFINITION 1. An operator T on H satisfies the condition (A) if for each $z \in \rho(T)$, $\|(T-zI)^{-1}\| \leqq \{d(z,\sigma(T))\}^{-1}$ where $\rho(T)$ denotes the resolvent set of T and $d(z,\sigma(T))$ denotes the distance between z and the spectrum $\sigma(T)$.

DEFINITION 2. An operator T on H satisfies the condition (B) if its spectrum $\sigma(T)$ lies on a Jordan curve C which consists of a finite number of rectifiable smooth arcs (it may well be the case that the spectrum separates the plane).

By the statement that γ is a smooth arc, we shall understand that γ has a parametrization $\zeta = g(s)$, $0 \leqq s \leqq l(\gamma)$, in terms of arc length s, and that $g(s)$, $g'(s)$ and $g''(s)$ are continuous.

For convenience' sake, throughout this note, we assume that the curve C defined as above is positively oriented and, for arbitrary fixed ζ_0 on C, C has a parametrization $\zeta = g(s)$, $0 \leqq s \leqq l(C)$, in terms of arc length s from ζ_0, $g(0) = \zeta_0$, $g(s) = g(s+l(C))$, and $g(s)$ is continuous on C and $g'(s)$, $g''(s)$ are continuous except the points $\zeta_k = g(s_k)$, $s_k < s_{k+1}$, $k = 1, 2, \cdots, n$ on C. It is clear that the existence of the one-sided limits $g'_+(s_k)$, $g'_-(s_k)$, $g''_+(s_k)$ and $g''_-(s_k)$, $k=1, 2, \cdots, n$ by the definition of C (each arc is smooth).

DEFINITION 3. For a bounded closed subset Y of the plane, a point $p \in Y$ is semi-bare if there is a circle through p such that no points of Y lie inside this circle.

LEMMA 1. *Each point on the curve C defined as above is a semi-bare point.*

PROOF. By the smoothness of each arc, for each $\zeta \in C$, there is the tangent of C at ζ (of course, for the case $\zeta = \zeta_k$, we consider the one-sided limits). And hence, by the simpleness of the curve C, there is a circle tangent to C at ζ such that no points of C lie inside this circle. This completes the proof.

THEOREM 1. *If an operator T on H satisfies the conditions* (A) *and* (B), *then $\sigma_r(T)=\phi$ and $\sigma_r(T^*)=\phi$, and it can be expressed uniquely as a direct sum $T=T_1 \oplus T_2$ defined on a product space $H=H_1 \oplus H_2$ where H_1 is spanned by all the proper vectors of T such that*:

(a) *T_1 is normal and $\sigma(T_1)=$ the closure of $\sigma_p(T_1)$*
(b) *$\sigma(T_2)=\sigma_c(T_2)$*
(c) *T is normal if and only if T_2 is normal.*

PROOF. If $\zeta \in \sigma_r(T)$, then by Lemma 1, there exists a $\zeta_0 \in \rho(T)$ such that $d(\zeta_0, \sigma(T)) = |\zeta-\zeta_0|$, and so, by the condition (A), we have $\|(T-\zeta_0 I)^{-1}\| = |\zeta-\zeta_0|^{-1}$. On the other hand, $\zeta \in \sigma_r(T)$ implies $\bar{\zeta} \in \sigma_p(T^*)$, then $\bar{\zeta}-\bar{\zeta}_0 \in \sigma_p(T^*-\bar{\zeta}_0 I)$ and $(\bar{\zeta}-\bar{\zeta}_0)^{-1} \in \sigma_p((T^*-\bar{\zeta}_0 I)^{-1})$; hence, $(\zeta-\zeta_0)^{-1} \in \sigma_p((T-\zeta_0 I)^{-1}) \cup \sigma_r((T-\zeta_0 I)^{-1})$. However by [3: Theorem 4 (i)], $\sigma_r((T-\zeta_0 I)^{-1}) \cap \{z: |z| = \|(T-\zeta_0 I)^{-1}\|\} = \phi$. Therefore $(\zeta-\zeta_0)^{-1} \in \sigma_p((T-\zeta_0 I)^{-1})$. This implies $\zeta \in \sigma_p(T)$. This is a contradiction. i.e. $\sigma_r(T) = \phi$. $\sigma_r(T^*) = \phi$ may be proved in just the same way.

Next, let $Tx=\zeta x$, $x \neq 0$, then by Lemma 1 and by the condition (A), there exists a $\zeta_0 \in \rho(T)$ such that $\|(T-\zeta_0 I)^{-1}\| = |\zeta-\zeta_0|^{-1}$ and $(T-\zeta_0 I)^{-1}x = (\zeta-\zeta_0)^{-1}x$; hence by [3 : Theorem 3], we have $(T^*-\bar{\zeta}_0 I)^{-1}x = (\bar{\zeta}-\bar{\zeta}_0)^{-1}x$. i.e. $T^*x=\bar{\zeta}x$. This means the proper subspace $\mathfrak{N}_\zeta(T)$ of T belonging to ζ (i.e. $\mathfrak{N}_\zeta(T)=\{x: Tx=\zeta x\}$) is a reducing subspace of H. And also this implies that each proper subspaces of T blonging to distinct proper values are mutually orthogonal; because let $Tx_1 = \zeta_1 x_1$, $Tx_2 = \zeta_2 x_2$, $x_1 \neq 0$, $x_2 \neq 0$, $\zeta_1 \neq \zeta_2$, then $\zeta_1(x_1, x_2) = (\zeta_1 x_1, x_2) = (Tx_1, x_2) = (x_1, T^* x_2) = (x_1, \bar{\zeta}_2 x_2) = \zeta_2(x_1, x_2)$ and hence $(x_1, x_2) = 0$.

Let H_1 be the direct sum $\bigoplus_{\zeta \in \sigma_p(T)} \mathfrak{N}_\zeta(T)$ of all the proper subspaces of T belonging to the point spectrum, then H_1 is a reducing subspace of H, and clearly, the restriction T_1 of T on H_1 is normal. Hence $\sigma_p(T) = \sigma_p(T_1)$ and $\sigma_r(T_1)$ is empty.

Consider any complex number ζ which is not in the colsure of $\sigma_p(T_1)$. Let $d > 0$ be such that $|\zeta - z| \geqq d$ for all $z \in$ the closure of $\sigma_p(T_1)$. Then, for any $x \in H_1$, $\|(T_1-\zeta I)x\|^2 = \|(T_1-\zeta I) \bigoplus_{\lambda \in \sigma_p(T)} x_\lambda\|^2 = \|\bigoplus_{\lambda \in \sigma_p(T)} (T_1-\zeta I)x_\lambda\|^2 = \sum_{\lambda \in \sigma_p(T)} \{|\lambda-\zeta|^2 \|x_\lambda\|^2\} \geqq \sum_{\lambda \in \sigma_p(T)} d^2 \|x_\lambda\|^2 = d^2 \|\bigoplus_{\lambda \in \sigma_p(T)} x_\lambda\|^2 = d^2 \cdot \|x\|^2$. Therefore the bounded inverse of $(T_1-\zeta I)$ exists for every such ζ. i.e. $\zeta \in \rho(T_1)$. This means $\sigma(T_1) \subset$ the closure of $\sigma_p(T_1)$.

Next $\sigma_r(T)=\phi$ and $\sigma_r(T^*)=\phi$ imply $\sigma_p(T) = \overline{\sigma_p(T^*)}$. And this means $\sigma_p(T_2)=\phi$ and $\sigma_r(T_2)=\phi$, because $\sigma_p(T_2) \subset \sigma_p(T)$ and $\sigma_p(T^*_2) \subset \sigma_p(T^*)$. Therefore $\sigma(T_2)=\sigma_c(T_2)$.

The last assertion of this theorem is clear by the above discussion.

REMARK. In [1], C. H. Meng proved the same result as Theorem 1 under the following conditions instead of the conditions (A) and (B);

(1) the closure of the numerical range of T is exactly convex hull $\Sigma(T)$ of the spectrum of T.

(2) the spectrum of T lies on a convex curve.

It is known that the condition (1) is equivalent to $\|(T-\zeta I)^{-1}\| \leqq \{d(\zeta, \Sigma(T))\}^{-1}$ for all $\zeta \notin \Sigma(T)$ where $\Sigma(T)$ denotes the convex hull of $\sigma(T)$ (see [2]). It is easy to see that by the condition (2), each $\zeta \in \sigma(T)$ is a semi-bare point. Therefore we can prove thit result by the same method as in Theorem 1.

LEMMA 2. *Let C be as Lemma 1. Then for each pairs of the points $\zeta_\alpha = g(s_\alpha)$, $s_j < s_\alpha < s_{j+1}$, $\zeta_\beta = g(s_\beta)$, $s_k < s_\beta < s_{k+1}$, $s_\alpha < s_\beta$ on C and any sufficiently small positive number ε, we have a closed simple connected domain $D(s_\alpha, s_\beta)$ containing the subarc $(g(s_\alpha), g(s_\beta))$ of C in its interior such that*:

(a) $\partial D(s_\alpha, s_\beta)$ *(boundary of $D(s_\alpha, s_\beta)$) is a rectifiable Jordan curve which* *intersects with* C *at* ζ_α *and* ζ_β *only.*

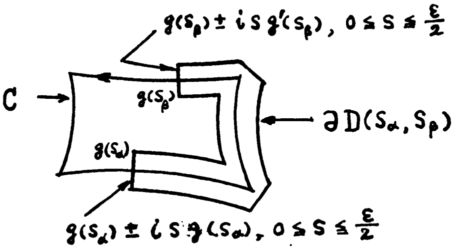
(b) *for each* $\zeta \in \partial D(s_\alpha, s_\beta) \cap \{\zeta : |\zeta - g(s_\alpha)| < \varepsilon/4\}$, $d(\zeta, C) = |\zeta - g(s_\alpha)|$ *and also for each* $\zeta \in \partial D(s_\alpha, s_\beta) \cap \{\zeta : |\zeta - g(s_\beta)| < \varepsilon/4\}$, $d(\zeta, C) = |\zeta - g(s_\beta)|$.

(c) $\max_{\zeta \in \partial D(s_\alpha, s_\beta)} d(\zeta, \mathrm{arc}\,[g(s_\alpha), g(s_\beta)]) < \varepsilon$.

PROOF. The smoothness guarantees the existence of $g'(s)$ at s_α and s_β, and that for each smooth arc $[g(s_k), g(s_{k+1})]$, $k = 1, 2, \cdots, n$, the minimum ρ_k of the radii of curvature is non-zero; hence $\rho_0 = \min_{1 \leq k \leq n} \rho_k$ is non-zero.

Let $d_1 = d(g(s_\alpha), C - \mathrm{arc}\,[g(s_j), g(s_{j+1})])$, and $d_2 = d(g(s_\beta), C - \mathrm{arc}\,[g(s_k), g(s_{k+1})])$ and let ε be so small that $\varepsilon < \min(\rho_0, d_1/2, d_2/2)$. Then we can construct the simple closed rectifiable curve indicated in the following figure (*) which contains the segments $g(s_\alpha) \pm i \cdot s \cdot g'(s_\alpha)$, $0 \leqq s \leqq \varepsilon/2$ and $g(s_\beta) \pm i \cdot s \cdot g'(s_\beta)$, $0 \leqq s \leqq \varepsilon/2$ as its subarcs and which for each ζ on this curve, $\max_{\zeta \in \text{this cvrve}} d(\zeta, \mathrm{arc}\,[g(s_\alpha), g(s_\beta)]) < \varepsilon$.

Let $D(s_\alpha, s_\beta)$ be the domain surrounded by this curve.



Figure(*)

THEOREM 2. *Let* T *be an operator on* H *which satisfies the conditions* (A) *and* (B). *If for each* $x \in H$ *and* $D(s_\alpha, s_\beta)$ *being given in Lemma* 2, *we define the vector-valued function* $f_x(\zeta)$ *on* $\partial D(s_\alpha, s_\beta)$ *as follows*;

$$f_x(\zeta) = \begin{cases} (\zeta - g(s_\alpha))^2 (\zeta - g(s_\beta))^2 (T - \zeta I)^{-1} x, & \\ & \text{if } \zeta \neq g(s_\alpha) \text{ and } \zeta \neq g(s_\beta), \\ 0, & \text{if } \zeta = g(s_\alpha) \text{ or } \zeta = g(s_\beta), \end{cases}$$

then $f_x(\zeta)$ *is strongly continuous on* $\partial D(s_\alpha, s_\beta)$.

PROOF. Clearly we have only to show the continuity at $g(s_\alpha)$ and $g(s_\beta)$. But this is also clear by the condition (A) and by Lemma 2 (b).

3. Existence of invariant subspaces. For fixed $x \in H$, $(T - \zeta I)^{-1} x$ is an analytic vector-valued function on $\rho(T)$. In this section we consider the analytic continuation of $(T - \zeta I)^{-1} x$ defined as follows;

DEFINITION 4. A vector-valued functicn $x(\zeta)$ is an analytic continuation of $(T-\zeta I)^{-1}x$ if $x(\zeta)$ is defined on an open set $D(x)$ containing $\rho(T)$, analytic on $D(x)$ and $x(\zeta)=(T-\zeta I)^{-1}x$ whenever $\zeta \in \rho(T)$.

LEMMA 3. *Let T be an operator which satisfies the condition* (B) *and let $\sigma(T)=\sigma_c(T)$. Then for fixed $x \in H$, $(T-\zeta I)^{-1}x$ has the single-valued maximal analytic continuation $x_e(\zeta)$ on $D_e(x)$ if it has an analytic continuation, and then $x_e(\zeta)=(T-\zeta I)^{-1}x$ for all $\zeta \in D_e(x)$.*

Since, in this case, $\sigma(T^*)=\sigma_c(T^*)$, $(T^*-\bar{\zeta}I)^{-1}x$ has also the single-valued maximal analytic continuation if it has an analytic continuation.

PROOF. Let $x(\zeta)$ be an analytic continuation of $(T-\zeta I)^{-1}x$ on $D(x)$. By the condition (B), for each $\zeta \in D(x)$, we can choose a sequence $\{\zeta_\alpha\}$, $\zeta_\alpha \in \rho(T)$ such that $\zeta_\alpha \to \zeta$. Then we have $(T-\zeta_\alpha I)x(\zeta_\alpha)=x$ for all ζ_α by the definition 4. Hence, $\|x-(T-\zeta I)x(\zeta)\| = \|(T-\zeta_\alpha I)x(\zeta_\alpha)-(T-\zeta I)x(\zeta)\| \leqq \|(T-\zeta_\alpha I)\| \cdot \|x(\zeta_\alpha)-x(\zeta)\| + \|(T-\zeta_\alpha I)-(T-\zeta I)\| \cdot \|x(\zeta)\| = \|T-\zeta_\alpha I\| \cdot \|x(\zeta_\alpha) -x(\zeta)\| + |\zeta_\alpha-\zeta| \cdot \|x(\zeta)\| \to 0$ as $\zeta_\alpha \to \zeta$ for each $\zeta \in D(x)$. Because $\zeta \in \sigma(T) = \sigma_c(T)$, $(T-\zeta I)$ is one to one and $x(\zeta)=(T-\zeta I)^{-1}x$ for all $\zeta \in D(x) \cdots\cdots (1)$.

Next, let $x_1(\zeta)$ and $x_2(\zeta)$ be two analytic continuations of $(T-\zeta I)^{-1}x$ on $D(x_1)$ and $D(x_2)$ respectively, then for each $\zeta \in D(x_1) \cap D(x_2)$, $(T-\zeta I)(x_1(\zeta) - x_2(\zeta)) = (T-\zeta I)x_1(\zeta) - (T-\zeta I)x_2(\zeta) = x-x=0$ by (1). On the other hand, $\zeta \notin \sigma_p(T)$. Hence, $x_1(\zeta)=x_2(\zeta)$ on $D(x_1) \cap D(x_2) \cdots\cdots (2)$.

We consider the family $\{x_\alpha(\zeta);\, \alpha \in N\}$ of all the analytic continuations $x_\alpha(\zeta)$ of $(T-\zeta I)^{-1}\, x$ on $D(x_\alpha)$, respectively. And we define $x_e(\zeta)=x_\alpha(\zeta)$ if $\zeta \in D(x_\alpha)$, then $x_e(\zeta)$ is analytic on $D_e(x)=\bigcup_{\alpha \in N} D(x_\alpha)$; hence $x_e(\zeta)$ is clearly the maximal analytic continuation of $(T-\zeta I)^{-1}x$. And by (2), $x_e(\zeta)$ is single-valued. By (1), $x_e(\zeta)=(T-\zeta I)^{-1}x$ for all $\zeta \in D_e(x)$.

DEFINITION 5. $R(\zeta : T, x)$, $\rho(T:x)$ and $\sigma(T:x)$ denote the maximal single-valued analytic continuation of $(T-\zeta I)^{-1}x$, the set $\{\zeta : R(\zeta : T, x)$ is analytic at $\zeta\}$ and its complenent, respectively.

LEMMA 4. *Let T be an operator which satisfies the condition* (B) *and let $\sigma(T)=\sigma_c(T)$. If $\sigma(T:x) \cap \overline{\sigma(T^*:y)} = \emptyset$ (the bar indicates the complex conjugate), then $(x,y)=0$.*

PROOF. By Lemma 3, $R(\zeta : T, x) = (T-\zeta I)^{-1}x$ on $\rho(T:x)$ and $R(\bar{\zeta} : T^*, x) = (T^*-\bar{\zeta}I)^{-1}x$ on $\rho(T^*:x)$.

Let $f(\zeta)=((T-\zeta I)^{-1}x, y) = (x, (T^*-\bar{\zeta}I)^{-1}y) = \overline{((T^*-\bar{\zeta}I)^{-1}y, x)}$, then $f(\zeta)$ is analytic at $\zeta \notin \sigma(T:x)$ and also at $\zeta \notin \overline{\sigma(T^*:y)}$. And hence $f(\zeta)$ is analytic everywhere. On the other hand, it is known that $\|(T-\zeta I)^{-1}\| \leqq \{d(\zeta, \widetilde{W(T)})\}^{-1}$ whenever $\zeta \notin \widetilde{W(T)}$, where $\widetilde{W(T)}$ denotes the closure of the numerical range of T (i.e. $W(T) = \{(Tx, x) : x \in H, \|x\| = 1\}$; see [8]), and hence $f(\zeta)$ vanishes at infinity. Therefore $f(\zeta)$ must be identically zero. However $f(\zeta) = \sum_{n=0}^{\infty} -(T^n x, y)\zeta^{-(n+1)}$, hence all coefficients of ζ^n must be zero, in particular $(x, y) = 0$.

Using the same method as in [4], we have the following two therems.

THEOREM 3. *Suppose T be an operator which satisfies the conditions* (A) *and* (B), *and suppose $\sigma(T) = \sigma_c(T)$. For each pair of the points $\zeta_\alpha = g(s_\alpha)$, $\zeta_\beta = g(s_\beta)$; $s_\alpha < s_\beta$ on C, let*

$$H(s_\alpha, s_\beta) = \{x \in H : \sigma(T : x) \subset \operatorname{arc}(g(s_\alpha), g(s_\beta)]\}\ , \quad \textit{and let}$$

$$H^*(s_\alpha, s_\beta) = \{x \in H : \overline{\sigma(T^* : x)} \subset \operatorname{arc}(g(s_\alpha), g(s_\beta)]\}\ .$$

Then $H(s_\alpha, s_\beta)$ and $H^*(s_\alpha, s_\beta)$ are closed linear subspaces of H, invariant under T and T^* respectively; moreover, $H(s_\alpha, s_\beta)$ and $H^*(s_\beta, s_\alpha + l(C))$, and also $H^*(s_\alpha, s_\beta)$ and $H(s_\beta, s_\alpha + l(C))$ are mutually orthogonal.

PROOF. Because both of the invariantness under T and the linearity of $H(s_\alpha, s_\beta)$ are clear, we have only to prove the closedness of $H(s_\alpha, s_\beta)$.

Let $x_n \to x$, $x_n \in H(s_\alpha, s_\beta)$ and let $R(\zeta : T, x_n)$ be the maximal single-valued analytic continuation of $(T-\zeta I)^{-1} x_n$, then

$$R(\zeta : T, x_n) = (T-\zeta I)^{-1} x_n \to (T-\zeta I)^{-1} x \quad \text{for all} \quad \zeta \in \rho(T)\,.$$

For any sufficiently small positive number ε', let $D(s_\beta + \varepsilon', s_\alpha + l(C) + \varepsilon')$ be a closed simple connected domain containing the subarc $(g(s_\beta + \varepsilon),$ $g(s_\alpha + l(C) + \varepsilon'))$ of C as given in Lemma 2. Then $R(\zeta : T, x_n)$ are analytic in $\operatorname{Int}(D(s_\beta + \varepsilon', s_\alpha + l(C) + \varepsilon'))$.

Next we define the vector-valued function $g_n(\zeta)$ on $D(s_\beta + \varepsilon', s_\alpha + l(C) + \varepsilon')$ as follows;

$$g_n(\zeta) = \begin{cases} (\zeta - g(s_\beta + \varepsilon))^2 (\zeta - g(s_\alpha + l(C) + \varepsilon'))^2 R(\zeta : T, x_n) \\ \qquad \textit{if } \zeta \neq g(s_\beta + \varepsilon') \text{ and } \zeta \neq g(s_\alpha + l(C) + \varepsilon') \\ 0 \qquad \textit{if } \zeta = g(s_\beta + \varepsilon) \text{ or } \zeta = g(s_\alpha + l(C) + \varepsilon') \end{cases}$$

Then $g_n(\zeta)$ are analytic in $\mathrm{Int}(D(s_\beta+\varepsilon, s_\alpha + l(C)+\varepsilon'))$ and strongly continuous on $\partial D(s_\beta+\varepsilon', s_\alpha+l(C)+\varepsilon')$ by Theorem 2. By the maximum modulus principle, $\{g_n(\zeta)\}$ is a uniform Cauchy sequence with respect to ζ; hence its limit function $g_0(\zeta)$ is analytic in $\mathrm{Int}(D(s_\beta+\varepsilon', s_\alpha+l(C)+\varepsilon'))$ and $(\zeta - g(s_\beta+\varepsilon'))^{-2}\cdot(\zeta-g(s_\alpha+l(C)+\varepsilon'))^{-2} g_0(\zeta)$ is also analytic in $\mathrm{Int}(D(s_\beta+\varepsilon', s_\alpha+l(C)+\varepsilon'))$. Clearly this is an analytic continuation of $(T-\zeta I)^{-1}x$ onto the $\mathrm{arc}(g(s_\beta+\varepsilon'), g(s_\alpha+l(C)+\varepsilon'))$, i.e. $\sigma(T:x) \subset \mathrm{arc}[g(s_\alpha + \varepsilon'), g(s_\beta + \varepsilon')]$. Because we can choose ε' arbitrarily small, we have $\sigma(T:x) \subset \mathrm{arc}(g(s_\alpha), g(s_\beta)]$; hence $x \in H(s_\alpha, s_\beta)$.

The closedness of $H^*(s_\alpha, s_\beta)$ may be proved in just the same way, and the last statement is a consequence of Lemma 4.

THEOREM 4. *Suppose T be an operator which satisfies the conditions* (A) *and* (B) *and suppose $\sigma(T) = \sigma_c(T)$. Let $H(s_\alpha, s_\beta)$, $H(s_\beta, s_\alpha + l(C))$ and $D(s_\beta+\varepsilon', s_\alpha + l(C) + \varepsilon')$ be as same as in Theorem* 3, *and for arbitrary fixed $x \in H$, define as*

$$x(\zeta) = \begin{cases} (\zeta - g(s_\beta+\varepsilon'))^2(\zeta - g(s_\alpha + l(C)+\varepsilon'))^2(T-\zeta I)^{-1}x, \\ \quad \text{if } \zeta \in \partial D(s_\beta+\varepsilon', s_\alpha+l(C)+\varepsilon') - \{g(s_\beta+\varepsilon'), g(s_\alpha+l(C)+\varepsilon')\}, \\ 0, \quad \text{if } \zeta = g(s_\beta+\varepsilon') \text{ or } \zeta = g(s_\alpha+l(C)+\varepsilon'). \end{cases}$$

Then, if $b(z)$ is any numerical-valued function, analytic in the interior of the unit disk and continuous on its boundary Γ and if τ is the conformal mapping from $D(s_\beta+\varepsilon', s_\alpha+l(C)+\varepsilon')$ to the unit disk (*the simple connectedness of $D(s_\beta+\varepsilon, s_\alpha+l(C)+\varepsilon')$ guarantees the existence of this mapping*), *the contour integral*

$$y = \int_\Gamma b(z)\, x(\tau^{-1}(z))\, dz \tag{1}$$

belongs to the space $H(s_\beta, s_\alpha+l(C))$. Moreover, unless x belongs to the space $H(s_\alpha, s_\beta)$, there exists a numerical-valued function $b(z)$ analytic in the interior of the unit disk and continuous on Γ such that the vector y defined by (1) *is different from zero.*

PROOF. By Theorem 2 and by the definition of the conformal mapping, $x(\tau^{-1}(z))$ is continuous on Γ. And by the resolvent equation, for any $\mu \in \rho(T) \cap \mathrm{Ext}(D(s_\beta+\varepsilon', s_\alpha+1(C)+\varepsilon'))$,

$$(T-\mu I)^{-1}x(\zeta) = (\zeta-\mu)^{-1}x(\zeta) \\ - (\zeta-\mu)^{-1}(\zeta-g(s_\beta+\varepsilon'))^2(\zeta-g(s_\alpha+l(C)+\varepsilon'))^2(T-\mu I)^{-1}x.$$

Then, by Cauchy's theorem,

$$(T-\mu I)^{-1} y = \int_{\Gamma} \frac{b(z)x(\tau^{-1}(z))}{\tau^{-1}(z)-\mu} dz$$

$$- \int_{\Gamma} \frac{b(z)(\tau^{-1}(z)-g(s_\beta+\varepsilon'))^2(\tau^{-1}(z)-g(s_\alpha+l(C)+\varepsilon'))^2(T-\mu I)^{-1} x}{\tau^{-1}(z)-\mu} dz$$

$$= \int_{\Gamma} \frac{b(z)x(\tau^{-1}(z))}{\tau^{-1}(z)-\mu} dz \qquad (2)$$

Since the final expression of (2) is plainly analytic in $\mathrm{Ext}(D(s_\beta+\varepsilon', s_\alpha+l(C)+\varepsilon'))$, it follows at once $\sigma(T:y)\subset \mathrm{arc}[g(s_\beta+\varepsilon'), g(s_\alpha+l(C)+\varepsilon')]$. Because ε' is arbitrary, $\sigma(T:y)\subset \mathrm{arc}(g(s_\beta), g(s_\alpha+l(C))]$. i.e. $y \in H(s_\beta, s_\alpha+l(C))$.

Next, we assume that the vector y defined by (1) is zero for each $b(z)$ which is analytic in the interior of the unit disk and continuous on its boundary Γ. Then,

$$\int_{\Gamma} b(z)x(\tau^{-1}(z))\, dz = 0 \quad \text{for all such } b(z).$$

Hence the vector-valued function $x(\tau^{-1}(z))$ defined on Γ must be the boundary value of a vector-valued function analytic in the interior of the unit disk and continuous on Γ. Therefore $x(\zeta)$ must be continuable in $\mathrm{Int}(D(s_\beta+\varepsilon', s_\alpha + l(C) + \varepsilon'))$. And hence $(T-\zeta I)^{-1}x$ must be continuable onto the arc$(g(s_\beta+\varepsilon'), g(s_\alpha + l(C) + \varepsilon'))$. Since ε' is arbitrary small, $\sigma(T:x) \subset \mathrm{arc}(g(s_\alpha), g(s_\beta)]$. i.e. $x \in H(s_\alpha, s_\beta)$.

As a consequence of above two theorem we have

THEOREM 5. *If an operator T on H with $\sigma(T) = \sigma_c(T)$ satisfies the conditions* (A) *and* (B), *then there exist non-trivial closed linear subspaces which are invariant under T.*

PROOF. By Theorem 3, we have only to prove that $H(s_\alpha, s_\beta)$ and $H(s_\beta, s_\alpha+l(C))$ are non-trivial. We may assume $\sigma(T)$ lies on both arcs $(g(s_\alpha), g(s_\beta)]$ and $(g(s_\beta), g(s_\alpha+l(C))]$, because we can choose the pairs of points $\zeta_\alpha=g(s_\alpha)$ and $\zeta_\beta=g(s_\beta)$ arbitrary on C. This implies that $H(s_\alpha, s_\beta)\neq H$ and $H(s_\beta, s_\alpha+l(C)) \neq H$.

Thus it only remains for us to prove that $H(s_\alpha, s_\beta)\neq(0)$ and $H(s_\beta, s_\alpha+l(C)) \neq (0)$. By Theorem 4, $H(s_\alpha, s_\beta)\neq H$ and $H(s_\beta, s_\alpha+l(C))\neq H$ imply that $H(s_\beta, s_\alpha+l(C))\neq(0)$ and $H(s_\alpha, s_\beta)\neq(0)$ respectively.

4. Main results. In this section, we shall treat with the hyponormal operators only. It is known that a hyponormal operator satisfies the condition (A) (see [8: Theorem 1] and [6]).

The following two lemmas were proved by Stampfli in [7].

LEMMA 5. *Let T be hyponormal and let $z_0 \in \sigma_c(T)$. If $x \in$ Domain($(T-z_0I)^{-1}$) then $x \in$ Domain($(T^*-\bar{z}_0I)^{-1}$) and* $\|(T^*-\bar{z}_0I)^{-1}x\| \leqq \|(T-z_0I)^{-1}x\|$.

PROOF. We may assume, without loss of generality, that $z_0 = 0$. Let $x \in$ Domain(T^{-1}), then $\|T^*T^{-1}x\| \leqq \|TT^{-1}x\| = \|x\|$. Thus T^*T^{-1} may be extended to a bound linear operator on H. Let $x_1, x_2 \in$ Domain(T^{-1}) and set $T^{-1}x_i = y_i$ for $i=1,2$. Then $L(x_2) = (T^{*-1}x_1, x_2) = (x_1, T^{-1}x_2) = (Ty_1, T^{-1}x_2) = (y_1, T^*T^{-1}x_2)$ so $|L(x_2)| \leqq \|y_1\| \cdot \|x_2\|$. Hence $L(x_2)$ is a bounded linear functional on Domain(T^{-1}) and can be extended to all of H. By the Riesz's representation theorem, there exists a vector w on H such that $L(x_2) = (w, x_2)$ i.e. $w = T^{*-1}x_1 \in H$.

Now, $|(T^{*-1}x_1, x_2)| \leqq \|y_1\| \cdot \|x_2\| = \|T^{-1}x_1\| \cdot \|x_2\|$ thus, $\|T^{*-1}x_1\| \leqq \|T^{-1}x_1\|$ which completes the proof.

LEMMA 6. *If an operator T on H with $\sigma(T) = \sigma_c(T)$ is hyponormal and satisfies the condition* (B), *then for each $\zeta \in \rho(T:x)$, $(T^*-\bar{\zeta}I)^{-1}x$ exists and is weakly continuous on $\overline{\rho(T:x)}$ for fixed $x \in H$.*

PROOF. By Lemma 5, $x \in$ Domain($(T^*-\bar{\zeta}I)^{-1}$). Thus $(T^*-\bar{\zeta}I)^{-1}x$ is well-defined for $\zeta \in \rho(T:x)$. Let $\zeta_0 \in \rho(T:x)$ and let $R(\zeta:T,x)$ be the maximal single-valued analytic continuation of $(T-\zeta I)^{-1}x$. Then $R(\zeta:T,x)$ is analytic in $J = \{z : |z-\zeta_0| < \delta\}$ and continuous strongly on $\tilde{J} = \{z : |z-\zeta_0| \leqq \delta\}$ for some $\delta > 0$; and hence bounded on $\tilde{J}$ by the maximum modulus principle. Therefore $\|(T-\zeta I)^{-1}x\| \leqq M$ for all $\zeta \in J$ and for some $M > 0$. By Lemma 5, we have $\|(T^*-\bar{\zeta}I)^{-1}x\| \leqq M$ for all $\zeta \in J$ and hence, $\|(T^*-\bar{\zeta}I)^{-1}x - (T^*-\bar{\zeta}_0I)^{-1}x\| \leqq 2M$ for all $\zeta \in J$. Given $y \in H$ choose $v \in H$ such that $\|y-(T-\zeta_0I)v\| < \varepsilon$ which is possible since Range($(T-\zeta_0I)$) is dense in H. Then for each $\zeta \in J$, we have,

$$
\begin{aligned}
&|(\{(T^*-\bar{\zeta}I)^{-1} - (T^*-\bar{\zeta}_0I)^{-1}\}\, x, y)| \\
&\quad \leqq 2M \cdot \|y - (T-\zeta_0I)v\| + |(\{(T^*-\bar{\zeta}I)^{-1} - (T^*-\bar{\zeta}_0I)^{-1}\}x, (T-\zeta_0I)v)| \\
&\quad \leqq 2M \cdot \varepsilon + |\zeta-\zeta_0| \cdot |((T^*-\bar{\zeta}_0I)^{-1}(T^*-\bar{\zeta}I)^{-1}x, (T-\zeta_0I)v)| \\
&\quad \leqq 2M \cdot \varepsilon + |\zeta-\zeta_0| \cdot \|(T^*-\bar{\zeta}I)^{-1}x\| \cdot \|v\| \\
&\quad \leqq 2M \cdot \varepsilon + |\zeta-\zeta_0| \cdot M \cdot \|v\| \\
&\quad \leqq 3M \cdot \varepsilon \qquad \text{for } |\zeta-\zeta_0| \text{ sufficiently small.}
\end{aligned}
$$

By this and by the Painlevé's theorem, we have the following theorem.

THEOREM 6. *If a hyponormal operator T with $\sigma(T)=\sigma_c(T)$ satisfies the condition* (B), *then* $\sigma(T:x)\supset\overline{\sigma(T^*:x)}$.

PROOF. $(T^*-\bar{z}I)^{-1}x$ is analytic for $z\in\rho(T)$ and continuous weakly for $z\in\rho(T:x)$ by Lemma 6. Hence by the Painlevé's theorem, $(T^*-\bar{z}I)^{-1}x$ may be continuable analytically across the subarc of C, which implies that $\overline{\rho(T^*:x)}\supset\rho(T:x)$ i.e. $\sigma(T:x)\supset\overline{\sigma(T^*:x)}$.

This proof is similar to one by Stampfli in [7].

THEOREM 7. *If T is a hyponormal operator with $\sigma(T)=\sigma_c(T)$ and satisfies the condition* (B), *then for $H(s_\alpha, s_\beta)$ and $H(s_\beta, s_\alpha+l(C))$ being given in Theorem* 3,

$$H = H(s_\alpha, s_\beta) \oplus H(s_\beta, s_\alpha + l(C));$$

and $H(s_\alpha, s_\beta)$, $H(s_\beta, s_\alpha + l(C))$ reduce T.

PROOF. By Theorem 6, $H(s_\alpha, s_\beta)\subset H^*(s_\alpha, s_\beta)$ and $H(s_\beta, s_\alpha + l(C))\subset H^*(s_\beta, s_\alpha+l(C))$; and by Theorem 3, we have $H(s_\alpha, s_\beta)\perp H^*(s_\beta, s_\alpha + l(C))$ and $H(s_\beta, s_\alpha+l(C))\perp H^*(s_\alpha, s_\beta)$, in particular, $H(s_\alpha, s_\beta)\perp H(s_\beta, s_\alpha + l(C))$. And this implies that

$$H(s_\alpha, s_\beta) \subset H\ominus H^*(s_\beta, s_\alpha + l(C))$$

and

$$H(s_\beta, s_\alpha + l(C)) \subset H\ominus H^*(s_\alpha, s_\beta). \tag{1}$$

Conversely, suppose for any fixed non-zero vector

$$x\in H\ominus H^*(s_\beta, s_\alpha+l(C)),\quad \sigma(T:x)\cap\mathrm{arc}(g(s_\beta),\ g(s_\alpha+l(C))]\neq\emptyset .$$

Since $H\ominus H^*(s_\beta, s_\alpha+l(C))$ is invariant under T, $T|(H\ominus H^*(s_\beta, s_\alpha+l(C)))$ is hyponormal (see [5]). Hence, by Theorem 3, we have

$$\{x\in H\ominus H^*(s_\beta, s_\alpha+l(C)): \sigma(T|(H\ominus H^*(s_\beta, s_\alpha+l(C))):x)$$
$$\subset \mathrm{arc}(g(s_\beta), g(s_\alpha+l(C))]\}\neq(0)$$

because $\sigma(T|(H\ominus H^*(s_\beta, s_\alpha+l(C))))\cap\mathrm{arc}(g(s_\beta), g(s_\alpha+l(C))]\neq\emptyset$ by the hypothesis.

Therefore there exists a non-zero vector $x_0 \in H \ominus H^*(s_\beta, s_\alpha + l(C))$ such that $\sigma(T: x_0) \subset \mathrm{arc}(g(s_\beta), g(s_\alpha + l(C))]$. This implies that $x_0 \in H(s_\beta, s_\alpha + l(C)) \subset H^*(s_\beta, s_\alpha + l(C))$. This is a contradiction. Therefore

$$H \ominus H^*(s_\beta, s_\alpha + l(C)) \subset H(s_\alpha, s_\beta)$$

and also

$$H \ominus H^*(s_\alpha, s_\beta) \subset H(s_\beta, s_\alpha + l(C)). \tag{2}$$

By (1) and (2), we have $H \ominus H^*(s_\beta, s_\alpha + l(C)) = H(s_\alpha, s_\beta)$ and $H \ominus H^*(s_\alpha, s_\beta) = H(s_\beta, s_\alpha + l(C))$; hence,

$$\begin{aligned} H &= H(s_\alpha, s_\beta) \oplus H(s_\beta, s_\alpha + l(C)) \oplus (H^*(s_\beta, s_\alpha + l(C)) \ominus H(s_\beta, s_\alpha + l(C))) \\ &= H(s_\alpha, s_\beta) \oplus H(s_\beta, s_\alpha + l(C)) \oplus (H^*(s_\alpha, s_\beta) \ominus H(s_\alpha, s_\beta)) \end{aligned}$$

and

$$\begin{aligned} &H \ominus (H(s_\alpha, s_\beta) \oplus H(s_\beta, s_\alpha + l(C))) \\ &= (H^*(s_\beta, s_\alpha + l(C)) \ominus H(s_\beta, s_\alpha + l(C))) \cap (H^*(s_\alpha, s_\beta) \ominus H(s_\alpha, s_\beta)) \\ &\subset H^*(s_\beta, s_\alpha + l(C)) \cap H^*(s_\alpha, s_\beta) = (0). \end{aligned}$$

Therefore $H = H(s_\alpha, s_\beta) \oplus H(s_\beta, s_\alpha + l(C))$. It is clear that $H(s_\alpha, s_\beta)$ and $H(s_\beta, s_\alpha + l(C))$ reduce T by Theorem 3.

It is known that a hyponormal operator is normaloid (i.e. $\|T\| = \max\{|\lambda| : \lambda \in \sigma(T)\}$; see [5]). Therefore we have the following theorem.

THEOREM 8. *If an operator T on H with $\sigma(T) = \sigma_c(T)$ is hyponormal and satisfies the condition* (B), *then T is normal.*

PROOF. Let $\Delta: 0 = s_1 < s_2 < \cdots < s_{k+1} = l(C)$ be any partition of $l(C)$ such that $\max_{1 \leq j \leq k}(s_{j+1} - s_j) \leqq 2 \cdot l(C)/k$, and let $I_j = \mathrm{arc}(g(s_j), g(s_{j+1})]$, then we can construct, by Theorem 3, $H_j = \{x \in H : \sigma(T: x) \subset I_j\}$ and by Theorem 7, we have $H = \bigoplus_j H_j$ where each H_j reduces T and $\sigma(T|H_j) \subset I_j$. Clearly, $T|H_j$ is also hyponormal and hence for any $x = \bigoplus_j x_j \in H$, $x_j \in H_j$ and for any $\lambda_j \in I_j$ we have

$$\begin{aligned} \|Tx - \bigoplus_j \lambda_j x_j\|^2 &= \sum_{j=1}^{k} \|(Tx_j - \lambda_j x_j)\|^2 \\ &\leqq \sum_{j=1}^{k} \|T|H_j - \lambda_j I\|^2 \cdot \|x_j\|^2 \\ &\leqq (\max_{1 \leqq j \leqq k}[\max\{|\lambda| : \lambda \in \sigma(T|H_j - \lambda_j I)\}])^2 \cdot \sum_{j=1}^{k} \|x_j\|^2 \end{aligned}$$

$$\leqq \{\max_{1\leqq j\leqq k}(s_{j+1}-s_j)\}^2 \cdot \|x\|^2$$
$$\leqq 4\cdot l(C)^2/k^2 \cdot \|x\|^2 \to 0 \quad \text{as} \quad k\to\infty .$$

And also we have $\|T^*x - \bigoplus_j \overline{\lambda}_j x_j\| \to 0$. Therefore,

$$|\|Tx\| - \|T^*x\|| \leqq |\|Tx\| - \|\bigoplus_j \lambda_j x_j\|| + |\|\bigoplus_j \overline{\lambda}_j x_j\| - \|T^*x\||$$
$$\leqq \|Tx - \bigoplus_j \lambda_j x_j\| + \|T^*x - \bigoplus_j \overline{\lambda}_j x_j\| \to 0. \text{ i.e. } \|Tx\| = \|T^*x\| .$$

By Theorem 1 and Theorem 8, we have the following theorem.

THEOREM 9. *If a hyponormal operator T satisfies the condition* (B), *then T is normal.*

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