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ON THE EXISTENCE OF *O*-CURVES*)

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Introduction. In early this century, Perron [5] (also, cf. [1]) has shown the followings: There exists a k-parameter family of solutions, which tend to zero as time increases infinitely, of the system

$$(E) \qquad \dot{x} = Ax + f(t, x),$$

under the conditions

(c_1) k characteristic roots of the constant matrix A have negative real parts

and

(c_2) $f(t, x)$ is continuous and of the class $o(\|x\|)$ for $\|x\|$ small and t large, that is, for any $\varepsilon > 0$ there exist $T \geqq 0$ and $\delta > 0$ such that

$$\|f(t, x)\| \leqq \varepsilon\|x\|$$

if $t \geqq T$ and $\|x\| \leqq \delta$, where $\|x\|$ denotes a Euclidean norm.

Moreover, if the condition

(c_3) for given $\varepsilon>0$, there exist $\delta > 0$ and $T \geqq 0$ such that

$$\|f(t, x) - f(t, y)\| \leqq \varepsilon\|x-y\|$$

if $t \geqq T$, $\|x\| \leqq \delta$, $\|y\| \leqq \delta$

is satisfied, then for any given solution $x^*(t)$ of the system

$$\dot{x} = Ax$$

remaining near the trivial solution, we can find a solution $x(t)$ of (E) satisfying

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$$\|x(t)-x^*(t)\| \to 0 \text{ as } t \to \infty .$$

In this article, a solution which tends to zero as $t \to \infty$ will be called an O-curve, and we shall discuss a similar problem to the above, that is, the existence of O-curves of the system (E) by replacing the conditions (c_2) and (c_3) by the condition

(c_4) there exists a continuous function $\lambda(t, \alpha)$ such that

$$\|f(t,x)\| \leqq \lambda(t,\alpha), \text{ if } \|x\| \leqq \alpha$$

and that

$$(L) \qquad \int_0^\infty \int_{t_1}^\infty \cdots \int_{t_{p-1}}^\infty \lambda(t_p, \alpha)\, dt_p dt_{p-1} \cdots dt_1 < \infty$$

for any $\alpha \geqq 0$,

where p is some positive integer which will be given in theorems. Here, we shall apply similar arguments to those used in the previous paper [3].

As was stated in [1], under the conditions (c_1), (c_2) and (c_3), such k-parameter family of O-curves consists a k-manifold, and moreover, it is obvious that if A has a characteristic root with the positive real part, then the trivial solution of (E) is unstable under the condition (c_2). However, in our case, even if A has a characteristic root with the positive real part, and even if $f(t, 0) = 0$ for all $t \geqq 0$, the trivial solution of (E) is not necessarily unstable (see Example).

We can discuss the same problem for more general systems by assuming the existence of Liapunov functions. However, for simplicity, we shall only consider a linear system and its perturbed system.

1. We shall consider a linear system of differential equations with constant coefficients

$$(1) \qquad \frac{dx}{dt} = Ax,$$

where x is an n-vector and A is a real (n, n)-matrix.

First of all, we shall prove the following lemma.

LEMMA 1. *There exists a real non-singular matrix $P(t)$ such that both of $P(t)$ and $P(t)^{-1}$ are continuous and bounded on $[0, \infty)$ and that by the transformation $x=P(t)y$, the system* (1) *is transformed into a system*

$$\frac{dy}{dt} = A'y,$$

where $A' = \operatorname{diag}(A_1, \cdots, A_m)$, A_j *is an* (k_j, k_j)*-matrix of the form*

$$(2) \qquad A_j = \begin{pmatrix} \alpha_j & 0 & 0 & \cdots\cdots & 0 \\ 1 & \alpha_j & 0 & \cdots\cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots\cdots & 0 & 1 & \alpha_j \end{pmatrix} \quad (j = 1, \cdots, m)$$

(*if* $k_j = 1$, $A_j = \alpha_j$), $k_1 + \cdots + k_m = n$, *and* α_j *is the real part of a characteristic root of* A.

PROOF. By well-known normalization, there exists a real non-singular constant matrix P_1 such that

$$P_1^{-1} A P_1 = \operatorname{diag}(A_1, \cdots, A_s, A'_{s+1}, \cdots, A'_r).$$

Here, A_j is an (k_j, k_j)-matrix of the form (2), where α_j is a real characteristic root of A $(j=1, \cdots, s)$, and A'_j is an (k'_j, k'_j)-matrix of the form

$$A'_j = \begin{pmatrix} S_j & O & \cdots\cdots & O \\ E & S_j & O \cdots\cdots & O \\ \cdots & \cdots & \cdots & \cdots \\ O & O \cdots & O\ E & S_j \end{pmatrix} \quad (j = s+1, \cdots, r),$$

where k'_j is even (if $k'_j = 2$, $A'_j = S_j$), E and O are the unit and the zero, respectively, (2, 2)-matrix,

$$S_j = \begin{pmatrix} \alpha_j & \beta_j \\ -\beta_j & \alpha_j \end{pmatrix}$$

and $\alpha_j \pm i\beta_j$ $(\beta_j \neq 0)$ are characteristic roots of A. Let $Q'_j(t)$ be defined by

$$Q'_j(t) = \begin{pmatrix} \cos(\beta_j t) & \sin(\beta_j t) \\ -\sin(\beta_j t) & \cos(\beta_j t) \end{pmatrix}$$

and let $Q_j(t)$ be the (k'_j, k'_j)-matrix of the form

$$Q_j(t) = \mathrm{diag}(Q_j'(t), \cdots, Q_j'(t)) \quad (j = s+1, \cdots, r).$$

Putting

$$P_2(t) = \mathrm{diag}(E_1, \cdots, E_s, Q_{s+1}(t), \cdots, Q_r(t)),$$

where E_j is the unite (k_j, k_j)-matrix, the transformation $x = P_1P_2(t)z$ transforms the syetem (1) into a system

$$\frac{dz}{dt} = Cz$$

where $C = \mathrm{diag}(A_1, \cdots, A_s, A_{s+1}'', \cdots, A_r'')$ and A_j'' is the (k_j', k_j')-matrix of the form

$$A_j'' = \begin{pmatrix} \alpha_j E & O & \cdots\cdots & O \\ E & \alpha_j E & \cdots\cdots & O \\ \cdots & \cdots & \cdots & \cdots \\ O & \cdots\cdots & E & \alpha_j E \end{pmatrix} \quad (j = s+1, \cdots, r).$$

Finally, we can find a non-singular matrix R_j such that

$$R_j^{-1} A_j'' R_j = \mathrm{diag}(A_j^*, A_j^*), \quad j = s+1, \cdots, r,$$

where A_j^* is of the form (2) and the order of A_j^* is $k_j'/2$. Let $P_3 = \mathrm{diag}(E_1, \cdots, E_s, R_{s+1}, \cdots, R_r)$. Then, the matrix $P(t) = P_1P_2(t)P_3$ is the required one.

By Lemma 1, we can find a non-singular real (n, n)-matrix $P(t)$ such that both of $P(t)$ and $P(t)^{-1}$ are continuous and bounded on $[0, \infty)$ and that by the transformation

$$x = P(t)y \tag{3}$$

the system (1) is transformed into a system

$$\frac{dy}{dt} = By, \tag{4}$$

where B is a constant real (n, n)-matrix of a form

$$B=\operatorname{diag}(B_1,\cdots,B_l,C,D).$$

Here, B_ν is an (n_ν,n_ν)-matrix of the form

$$B_\nu=\begin{pmatrix}0&0&\cdots\cdots&0\\1&0&\cdots\cdots&0\\&\cdots\cdots\cdots&&\\0&0&\cdots\cdots1&0\end{pmatrix}\quad(\nu=1,\cdots,l)$$

(if $n_\nu=1$, $B_\nu=0$), C is a (q,q)-matrix and D is a (k,k)-matrix, $n_1+\cdots+n_l+q+k=n$. All characteristic roots of C have positive real parts and those of D have negative real parts. Let y be decomposed in the following way:

$$y=(y_1,\cdots,y_l,z,u)$$

and

$$y_\nu=(y_\nu^1,\cdots,y_\nu^{n_\nu})\quad(\nu=1,\cdots,l),$$

where y_ν is an n_ν-vector $(\nu=1,\cdots,l)$, z is a q-vector, u is a k-vector and y_ν^j is a scalar $(j=1,\cdots,n_\nu;\ \nu=1,\cdots,l)$. Now, assuming that $n_1\geqq n_2\geqq\cdots\geqq n_l$, we shall put

$$(5)\qquad\begin{cases}u=u\\v=(y_1^1,y_2^1,\cdots,y_l^1,z)\\w_j=(y_1^{j+1},\cdots,y_{m_j}^{j+1})\quad(j=1,\cdots,n_1-1),\end{cases}$$

where m_j is the largest integer ν such that $l\geqq\nu\geqq1$ and $n_\nu\geqq j+1$. Morover, we put $m_0=l+q$ and $w_0=v$. Obviously, this transformation is non-singular and tranforms the system (4) into a system

$$(6)\qquad\frac{du}{dt}=A^*u,\quad\frac{dv}{dt}=B^*v,\quad\frac{dw_j}{dt}=C_j^*w_{j-1}\quad(j=1,\cdots,n_1-1).$$

Here, $A^*=D$, B^* is the (m_0,m_0)-matrix diag (O^*,C) and C_j^* is the (m_j,m_{j-1})-matrix (E_j,O_j) $(j=1,\cdots,n_1-1)$, where O^* is the zero (l,l)-matrix, E_j is the unit (m_j,m_j)-matrix and O_j is the zero $(m_j,m_{j-1}-m_j)$-matrix (it must be noted that $m_0\geqq l\geqq m_1\geqq\cdots\geqq m_{n_1-1}$). In the case where $l=0$ (or $n_1=1$), the system (6) becomes

$$\frac{du}{dt}=A^*u,\quad\frac{dv}{dt}=B^*v$$

where $A^* = D$ and $B^* = C$ (or $B^* = \mathrm{diag}(O^*, C)$) and $v = z$ (or $v = (y_1^1, \cdots, y_l^1, z)$).

2. We consider a perturbed system

$$\frac{dx}{dt} = Ax + f(t, x) \tag{7}$$

of the system (1). It is the purpose of this article to prove the following theorem.

THEOREM 1. *We shall denote by p the maximum degrees of the elementary divisors of characteristic roots of A with zero real parts if such a root exists, and otherwise we shall put $p=1$. Suppose that $f(t, x)$ is continuous on $[0, \infty) \times R^n$ (R^n represents the Euclidean n-space) and satisfies the condition (c_4).*

Then, there exists at least one O-curve of (7).

Furthermore, if A satisfies the condition (c_1), then there exists a k-parameter family of O-curves of (7).

In the proof of this theorem, we shall apply the following lemma (for the proof, see [4]).

LEMMA 2. *Suppose that $f(t, x, y)$ and $g(t, x, y)$ are continuous and bounded on $[a, b] \times R^n \times R^m$. Then, for given $(x_0, y_0) \in R^n \times R^m$ there exists a solution $(x(t), y(t))$ of the system*

$$\frac{dx}{dt} = f(t, x, y), \qquad \frac{dy}{dt} = g(t, x, y)$$

such that $x(a) = x_0$ and $y(b) = y_0$.

3. Now, we shall prove Theorem 1.

As was stated in **1**, the system (1) can be transformed into a system of the form (6) by transformations of the forms (3) and (5). By these transformations, the system (7) will be transformed into a system

$$\begin{cases} \dfrac{du}{dt} = A^*u + F(t, u, v, w) \\ \dfrac{dv}{dt} = B^*v + G(t, u, v, w) \\ \dfrac{dw_j}{dt} = C_j^* w_{j-1} + H_j(t, u, v, w) \quad (j=1, \cdots, p-1), \end{cases} \tag{8}$$

where $w=(w_1, \cdots, w_{p-1})$. In the above, if $p=1$, the system (8) is considered to be

$$\begin{cases} \dfrac{du}{dt} = A^*u + F(t, u, v) \\ \dfrac{dv}{dt} = B^*v + G(t, u, v)\,. \end{cases}$$

By the properties of these transformations, the condition (c_4) implies that $F(t, u, v, w)$, $G(t, u, v, w)$ and $H_j(t, u, v, w)$ $(j=1, \cdots, p-1)$ are continuous on the domain

$$D = [0, \infty) \times R^k \times R^m \times R^{n-k-m}\,, \quad m = m_0\,,$$

and that there exists a continous function $\lambda^*(t, \alpha)$ satisfying

$$\int_0^\infty \int_{t_1}^\infty \cdots \int_{t_{p-1}}^\infty \lambda^*(t_p, \alpha)\, dt_p dt_{p-1} \cdots dt_1 < \infty \quad \text{(for any } \alpha \geqq 0)$$

and

$$\| F(t, u, v, w)\|,\ \|G(t, u, v, w)\|,\ \|H_j(t, u, v, w)\| \leqq \lambda^*(t, \alpha)$$

$(j=1, \cdots, p-1)$ for any (t, u, v, w) such that $t \in [0, \infty)$ and $\max\{\|u\|, \|v\|, \|w\|\} \leqq \alpha$.

Since all characteristic roots of A^* have negative real parts, we can find a Liapunov function $V(t, u)$ which is continuous in (t, u) on $[0, \infty) \times R^k$ and which satisfies the conditions

i) $\quad \| u \| \leqq V(t, u) \leqq K\|u\|\,,$

ii) $\quad |V(t, u) - V(t, u')| \leqq K\|u-u'\|\,,$

iii) $\quad \dot{V}(t, u) = \overline{\lim_{\delta \to +0}} \dfrac{1}{\delta} \{V(t+\delta, u + \delta A^*u) - V(t, u)\} \leqq - cV(t, u),$

where K and c are positive constants. On the other hand, the real part of each characteristic root of B^* is positive or zero and in the latter case, the corresponding elementary divisor is linear, and hence there exists a continuous Liapunov function $W(t, v)$ defined on $[0, \infty) \times R^m$ and satisfying the similar conditions to (i) and (ii) with the constant K and the condition

iv) $\quad \dot{W}(t, v) = \underline{\lim_{\delta \to +0}} \dfrac{1}{\delta} \{W(t+\delta, v + \delta B^*v) - W(t, v)\} \geqq 0\,.$

For any given $\alpha > 0$ we can find a constant $T = T(\alpha) \geqq 0$ such that

$$\Lambda_1(T) < \alpha, \quad \sum_{j=2}^{p} \left\{ K\Lambda_j(T) + \sum_{\nu=1}^{j-1} \Lambda_\nu(T) \right\}^2 < 4K^2\alpha^2,$$

where

$$\Lambda_j(t) = \int_t^\infty \int_{t_1}^\infty \cdots \int_{t_{j-1}}^\infty \lambda^*(t_j, 2K\alpha)\, dt_j\, dt_{j-1} \cdots dt_1 .$$

Let us define $F^*(t, u, v, w)$, $G^*(t, u, v, w)$ and $H_j^*(t, u, v, w)$ by replacing (t, u, v, w) in $F(t, u, v, w)$, $G(t, u, v, w)$ and $H_j(t, u, v, w)$, respectively, by $(t, \varphi(\|u\|)\, u, \varphi(\|v\|)\, v, \varphi(\|w\|)\, w)$, where

$$\varphi(r) = \begin{cases} \min\left\{1, \dfrac{2K\alpha}{r}\right\} & (r \neq 0) \\ 1 & (r = 0). \end{cases}$$

Obviously, $F^*(t, u, v, w)$, $G^*(t, u, v, w)$ and $H_j^*(t, u, v, w)$ $(j=1, \cdots, p-1)$ are continuous, and the norms of those are bounded by $\lambda^*(t, 2K\alpha)$ on D.

If we set

$$f(t, u, v, w) = \varphi(\|u\|)\, A^*u + F^*(t, u, v, w)$$

$$g(t, u, v, w) = \varphi(\|v\|)\, B^*v + G^*(t, u, v, w)$$

and

$$h_j(t, u, v, w) = \varphi(\|w\|) C_j^* w_{j-1} + H_j^*(t, u, v, w) \quad (j=1, \cdots, p-1),$$

these functions are continuous and bounded on D if t is restrained in a finite interval. Therefore, by applying Lemma 2, for any given constants a, b $(b > a \geqq 0)$ and $u_0 \in R^k$, we can find a solution $(u(t), v(t), w(t))$ of the system

$$\frac{du}{dt} = f(t, u, v, w), \qquad \frac{dv}{dt} = g(t, u, v, w), \qquad \frac{dw_j}{dt} = h_j(t, u, v, w)$$
$$(j = 1, \cdots, p-1)$$

such that

$$u(a) = u_0, \quad v(b) = 0 \quad \text{and} \quad w(b) = 0 .$$

If we can see that

$$(9) \qquad \|u(t)\|, \|v(t)\|, \|w(t)\| \leqq 2K\alpha \quad \text{on} \quad [a, b],$$

then $(u(t), v(t), w(t))$ is obviously a solution of the system (8) on $[a, b]$. Assuming that $\|u_0\| < \alpha$ and $a \geqq T(\alpha)$, (9) will be easily shown: So long as $\|u(t)\|$, $\|v(t)\| \leqq 2K\alpha$, we have

$$(10) \qquad \frac{d}{dt} V(t, u(t)) \leqq -cV(t, u(t)) + K\lambda^*(t, 2K\alpha),$$

$$\frac{d}{dt} W(t, v(t)) \geqq -K\lambda^*(t, 2K\alpha)$$

for almost everywhere, and hence, these inequalities and the assumptions

$$\|u(a)\| < \alpha, \; a \geqq T(\alpha), \; v(b) = 0$$

imply that $\|u(t)\|, \|v(t)\| \leqq 2K\alpha$ on $[a, b]$ and that

$$(11) \qquad \|v(t)\| \leqq K \int_t^b \lambda^*(s, 2K\alpha)\, ds \leqq K\Lambda_1(t).$$

Further, from the inequalities (11) and

$$\|w_j(t)\| \leqq \int_t^b \|w_{j-1}(s)\| ds + \int_t^b \lambda^*(s, 2K\alpha)\, ds \quad (j=1, \cdots, p-1),$$

it follows that

$$(12) \qquad \|w_j(t)\| \leqq K\Lambda_{j+1}(t) + \sum_{\nu=1}^{j} \Lambda_\nu(t) \quad (j = 1, \cdots, p-1),$$

and hence we have $\|w(t)\| \leqq 2K\alpha$ on $[a, b]$.

For given a and u_0 such that $a \geqq T(\alpha)$, $u_0 \in R^k$ and $\|u_0\| < \alpha$, let $(u(t\,; s), v(t\,; s), w(t\,; s))$ be a continuous function defined on $[a, \infty)$ such that it is a solution of the system (8) on $[a, s]$ satisfying the conditions

$$u(a\,; s) = u_0, \; v(s\,; s) = 0 \text{ and } w(s\,; s) = 0$$

and that

$$u(t\,; s) = u(s\,; s), \; v(s\,; s) = 0 \quad \text{and} \quad w(t\,; s) = 0 \quad \text{for all } t \geqq s$$

for any given $s \geqq a$. Since we have

$$\|u(t\,;s)\|,\ \|v(t\,;s)\|,\ \|w(t\,;s)\| \leqq 2K\alpha \quad \text{on} \quad [a,\infty)$$

by (9), the family $\{(u(t\,;s), v(t\,;s), w(t\,;s))\,; s \geqq a\}$ are uniformly bounded and equicontinuous on any bounded subinterval of $[a,\infty)$. From this, it follows that there exists a divergent sequence $\{s_\mu\}$, for which the sequence $\{(u(t\,;s_\mu),\ v(t\,;s_\mu),\ w(t\,;s_\mu))\}$ converges a solution $(u(t), v(t), w(t))$ of the system (8). Obviously, it satisfies the conditions $u(a)=u_0$ and

$$\|u(t)\|,\ \|v(t)\|,\ \|w(t)\| \leqq 2K\alpha \quad \text{on} \quad [a,\infty)\,.$$

Moreover, since each $(v(t\,;s), w(t\,;s))$ satisfies the conditions (11) and (12) on $[a,\infty)$, so does $(v(t), w(t))$. Hence, we obtain

$$v(t)\to 0\,,\ w(t)\to 0 \quad \text{as} \quad t\to\infty\,,$$

because $\Lambda_j(t)\to 0$ as $t\to\infty$ $(j=1,\cdots,p-1)$. Referring the inequality (10), we can easily see that

$$u(t)\to 0 \quad \text{as} \quad t\to\infty\,.$$

Thus, $(u(t), v(t), w(t))$ is an O-curve of the system (8). Hence, by the properties of transformations of the forms (3) and (5), we completely prove Theorem 1.

4. Let $x^*(t)$ be any bounded solution of the system (1) (or (7)), and consider the transformation

$$x = x^*(t) + y$$

in the system (7) (or (1), respectively). Then, we have

$$\frac{dy}{dt} = Ay + g(t,y)\,,$$

where $g(t,y)=f(t,x^*(t)+y)$ (or $-f(t,x^*(t))$). If $f(t,x)$ in (7) satisfies the condition (c_4), then $g(t,y)$ also satisfies the condition (c_4), where $\lambda(t,\alpha)$ will be replaced by $\lambda(t,\alpha+M)$ (or $\lambda(t,M)$, respectively) and M is a bound of $x^*(t)$.

Thus, the following theorem is an immediate consequence of Theorem 1.

THEOREM 2, *Under the same assumptions in Theorem* 1, *for any bounded solution* $x^*(t)$ *of one of the system* (1) *and* (7) *there exists a solution* $x(t)$ *of the other system such that*

$$\|x(t) - x^*(t)\| \to 0 \quad as \quad t \to \infty .$$

Furthermore, if A satisfies the condition (c_1), then there exists a k-parameter family of such solutions.

5. The results above can be extended for functional differential equations.

For a fixed constant $r > 0$, let C^n be the space of all continuous R^n-valued functions defined on $[-r, 0]$, and let

$$\|\varphi\|_r = \sup\{\|\varphi(\theta)\| \; ; \; \theta \in [-r, 0]\}$$

for a $\varphi \in C^n$. For any continuous R^n-valued function $x(t)$, we shall denote by x_t the function of C^n such that

$$x_t(\theta) = x(t + \theta), \; \theta \in [-r, 0] ,$$

and by $\dot{x}(t)$ the right-hand derivative of $x(t)$.

In the system

$$\dot{x}(t) = f(x_t) + X(t, x_t) , \tag{13}$$

we assume that $f(\varphi)$ is a continuous linear function of C^n into R^n and that $X(t, \varphi)$ is a completely continuous R^n-valued function defined on $[0, \infty) \times C^n$. Let $\mathfrak{A}$ be a linear operator such that

$$[\mathfrak{A}\varphi](\theta) = \begin{cases} \dot{\varphi}(\theta) & -r \leqq \theta < 0 \\ f(\varphi) & \theta = 0 \end{cases}$$

for $\varphi \in C^n$, for which the right-hand derivative $\dot{\varphi}(\theta)$ exists for each $\theta \in [-r, 0)$.

By referring Hale's result in [2], we can prove the following theorem by using the arguments similar to those in the proof of Theorem 1. The proof is omitted (refer [3]).

THEOREM 3. *Let p be the maximum dimensions of the eigen spaces of the spectra of $\mathfrak{A}$ with zero real parts if such a spectrum exists, and let $p=1$ otherwise.*

Suppose that there exists a continuous function $\lambda(t, \alpha)$ satisfying the inequality (L) in the condition (c_4) and

$$\|X(t, \varphi)\| \leqq \lambda(t, \alpha), \quad if \quad \|\varphi\|_r \leqq \alpha$$

for any $t \in [0, \infty)$ *and* $\alpha \geqq 0$ *and that k spectra of* $\mathfrak{A}$ *have negative real parts.*

Then, there exists a k-parameter family of O-curves of the system (13). *Moreover, for any given bounded solution* $x^*(t)$ *of the system* (13) *or*

$$\dot{x}(t) = f(x_t)\,, \tag{14}$$

we can find a k-parameter family of solutions $x(t)$ *of the system* (14) *or* (13) *satisfying*

$$\|x(t) - x^*(t)\| \to 0 \quad as \quad t \to \infty\,.$$

6. In the case where $f(t, 0) = 0$ and A has at least one characteristic root with positive real part, the condition (c_4) is not sufficient for the instability of the trivial solution of the system (7).

EXAMPLE. We consider a equation

$$\frac{dx}{dt} = ax + f(t, x)\,, \tag{15}$$

where

$$f(t, x) = -(a+b+1)\,e^{-bt} \sin[e^{bt}x]$$

and a, b are positive constants. Obviously, $f(t, 0) = 0$ and $f(t, x)$ satisfies the condition (c_4) and the Lipschitz condition

$$|f(t, x) - f(t, y)| \leqq (a+b+1)|x - y|\,.$$

However, the trivial solution of the system (15) is asymptotically stable. In fact, let $y = e^{bt}x$. Then, we have

$$\frac{dy}{dt} = (a+b)y - (a+b+1)\sin y = g(y)\,. \tag{16}$$

Since $g(y) = -y + O(y^3)$, the trivial solution of the system (16) is asymptotically stable, and hence so is that of the system (15).

However, we shall obtain the following theorem.

THEOREM 4. *Under the assumptions* (c_4) *with* $p = 1$ *and*

$$(c_5) \qquad f(t, 0) = 0 \quad for\ all \quad t \geqq 0\,,$$

if there exists a characteristic root of A with the positive real part, then the trivial solution of the system (7) *is not uniform-stable.*

PROOF. Let a be the real part of a characteristic root of A with the positive real part. Then, by Lemma 1 there exists a non-singular matrix $P(t)$ such that $P(t)$ and $P(t)^{-1}$ are continuous and bounded on $[0, \infty)$ and that the transformation $x = P(t)\begin{pmatrix} y \\ z \end{pmatrix}$, where y is a scalar and z is an $(n-1)$-vector, transforms the system (7) into a system

$$\frac{dy}{dt} = ay + g(t, y, z), \quad \frac{dz}{dt} = A_0\begin{pmatrix} y \\ z \end{pmatrix} + h(t, y, z). \tag{17}$$

Clearly, there exists a continuous function $\lambda^*(t, \alpha)$ such that

$$\int^{\infty} \lambda^*(t, \alpha)\, dt < \infty \quad (\text{for all} \quad \alpha \geqq 0)$$

and that if $|y| \leqq \alpha$ and $\|z\| \leqq \alpha$, then $|g(t, y, z)| \leqq \lambda^*(t, \alpha)$ and $\|h(t, y, z)\| \leqq \lambda^*(t, \alpha)$ for all $t \geqq 0$.

Now, suppose that the trivial solution of the system (7) is uniform-stable. Then, the trivial solution of the system (17) also is uniform-stable. Hence, for any $\varepsilon > 0$, there exists a $\delta(\varepsilon) > 0$ such that for any $t_0 \geqq 0$, if $|y_0| < \delta$ and $\|z_0\| < \delta$, then $|y(t; y_0, z_0, t_0)| < \varepsilon$ and $\|z(t; y_0, z_0, t_0)\| < \varepsilon$ for all $t \geqq t_0$, where $(y(t; y_0, z_0, t_0), z(t; y_0, z_0, t_0))$ is a solution of the system (17) through (t_0, y_0, z_0). Since

$$y(t; y_0, z_0, t_0) = e^{a(t-t_0)}\{y_0 + \int_{t_0}^{t} e^{a(t_0-s)}\, g(s, y(s; y_0, z_0, t_0), z(s; y_0, z_0, t_0))\, ds\},$$

we have

$$|y(t; y_0, z_0, t_0)| \geqq e^{a(t-t_0)}\{|y_0| - \int_{t_0}^{t} |g(s, y(s; y_0, z_0, t_0), z(s; y_0, z_0, t_0))|\, ds\}.$$

For a given $y_0, y_0 \neq 0$, we can find a $T(|y_0|) \geqq 0$ such that

$$\int_{T(|y_0|)}^{\infty} \lambda^*(t, \varepsilon) dt < \frac{|y_0|}{2}.$$

Hence, if $0 < |y_0| < \delta(\varepsilon)$, $\|z_0\| < \delta(\varepsilon)$ and $t_0 \geqq T(|y_0|)$, then we have

$$|y(t; y_0, z_0, t_0)| \geqq \frac{|y_0|}{2} e^{a(t-t_0)}.$$

Hence, there arises a contradiction. Thus, the trivial solution of the system (7) is not uniform-stable.

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