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TENSOR FIELDS AND CONNECTIONS ON CROSS-SECTIONS IN THE COTANGENT BUNDLE

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Introduction. E. M. Patterson and the present author [6] recently studied vertical and complete lifts of tensor fields and connections from a manifold M to its cotangent bundle ${}^cT(M)$. When a 1-form is given in an n-dimensional manifold M, the 1-form defines a cross-section in the cotangent bundle ${}^cT(M)$, which is an n-dimensional submanifold in the $2n$-dimensional cotangent bundle ${}^cT(M)$.

The main purpose of the present paper is to study the behaviour on the cross-section of the lifts of tensor fields and connections in a manifold M to its cotangent bundle ${}^cT(M)$.

In § 1 and § 2, we review the results obtained in [6] on vertical and complete lifts of tensor fields and connections from a manifold to its cotangent bundle ${}^cT(M)$. In § 3, we study the behaviour of the lifts of tensor fields and of Riemann extension of connections [1] on the cross-sections. We examine, in § 4, the behaviour of the lifts of almost complex structures on the cross-sections. We show in § 5 that the tensor discovered by Slebodzinski [3] appears in our present theory. Finally we study in §6 the behaviour of the complete lift of a connection on the cross-sections.

The manifold, functions, vector fields, 1-forms, tensor fields and connections appearing in the discussion will be supposed to be of the differentiability class C^∞.

The indices $A, B, C, D, \cdots$ run from 1 to $2n$, the indices $a, b, c, \cdots, h, i, j, \cdots$ from 1 to n and the indices $\bar{a}, \bar{b}, \bar{c}, \cdots, \bar{h}, \bar{i}, \bar{j}, \cdots$ from $n+1$ to $2n$. We use the notations $x^A = (x^h, x^{\bar{h}})$ and $x^{\bar{h}} = p_h$.

1. Vertical lifts of tensor fields. Let M be an n-dimensional differentiable manifold of class C^∞, ${}^cT(M)$ its cotangent bundle, and π the projection ${}^cT(M) \to M$. Let the manifold M be covered by a system of coordinate neighbourhoods $\{U; x^h\}$, where (x^h) is a local coordinate system defined in the neighbourhood U. Let (p_i) be the cartesian coordinate system in each cotangent space ${}^cT_P(M)$ at P of M with respect to the natural coframe dx^i

in M, P being an arbitrary point in U whose coordinates are (x^h). Then we can introduce local coordinates (x^h, p_i) in the open set $\pi^{-1}(U)$ of ${}^cT(U)$. We call them coordinates induced in $\pi^{-1}(U)$ from $\{U; x^h\}$ or simply *induced coordinates* in $\pi^{-1}(U)$. The projection π is represented by $(x^h, p_i) \to (x^h)$.

In ${}^cT(M)$, there exists a 1-form

$$p = p_i dx^i, \tag{1.1}$$

which we call the *basic 1-form* in ${}^cT(M)$. The exterior derivative of p is

$$dp = dp_i \wedge dx^i. \tag{1.2}$$

We call this the *basic 2-form* in ${}^cT(M)$. If we put

$$dp = \frac{1}{2} \varepsilon_{CB} dx^C \wedge dx^B, \tag{1.3}$$

we see that ε_{CB} given by

$$\varepsilon_{CB} = \begin{pmatrix} 0 & \delta_i^j \\ -\delta_j^i & 0 \end{pmatrix} \tag{1.4}$$

are components of a tensor field of type (0, 2) in ${}^cT(M)$. Consequently we can define a tensor field ε^{BA} of type (2, 0) by

$$\varepsilon_{BC} \varepsilon^{AB} = \delta_C^A \tag{1.5}$$

and find that ε^{BA} has components

$$\varepsilon^{BA} = \begin{pmatrix} 0 & -\delta_i^h \\ \delta_h^i & 0 \end{pmatrix}. \tag{1.6}$$

We now take a function f in M. The function $f \circ \pi$ in ${}^cT(M)$ induced from f in M is called the *vertical lift of* f and is denoted by

$$f^V = f \circ \pi. \tag{1.7}$$

A vector field X in M is, in a natural way, regarded as a function in ${}^cT(M)$. This function is called the *vertical lift of the vector field* X to ${}^cT(M)$ and is denoted by X^V. When X in M has local components X^h with respect to the natural frame ∂_h in M, X^V in ${}^cT(M)$ has local expression

$$X^V = p_i X^i. \tag{1.8}$$

When a 1-form $\omega = \omega_i dx^i$ is given in M, it is also regarded as a 1-form in ${}^cT(M)$. If we write $\tilde{\omega} = \tilde{\omega}_B dx^B$, then $\tilde{\omega}$ has components

$$\tilde{\omega}_B = (\omega_i, 0)$$

in ${}^cT(M)$. Thus we can define a vector field $\tilde{\omega}_B \varepsilon^{BA}$ in ${}^cT(M)$. We call this vector field in ${}^cT(M)$ the *vertical lift of a 1-form* in M to ${}^cT(M)$ and denote it by ω^V. The ω^V has the components

$$\omega^V = \begin{pmatrix} 0 \\ \omega_i \end{pmatrix}. \tag{1.9}$$

The vertical lift ω^V of a 1-form ω in M to ${}^cT(M)$ satisfies

$$\begin{cases} \omega^V f^V = 0, & \text{for any } f \in \mathfrak{T}_0^0(M), \\ \omega^V X^V = (\omega(X))^V, & \text{for any } X \in \mathfrak{T}_0^1(M), \end{cases} \tag{1.10}$$

which characterize ω^V, where $\mathfrak{T}_s^r(M)$ denotes the set of tensor fields of type (r, s) in M.

When we are given a tensor field F of type (1, 1) in M with local components F_i^h, we can easily see that $\tilde{F}_B dx^B = p_a F_i^a dx^i$ is a 1-form in ${}^cT(M)$. Thus we can define a vector field $\tilde{F}_B \varepsilon^{BA}$ in ${}^cT(M)$. We call this the *vertical lift of the tensor field* F of type (1, 1) in M to ${}^cT(M)$ and denote it by F^V. The F^V has the components

$$F^V = \begin{pmatrix} 0 \\ p_a F_i^a \end{pmatrix}. \tag{1.11}$$

The F^V satisfies

$$\begin{cases} F^V f^V = 0, & \text{for any } f \in \mathfrak{T}_0^0(M), \\ F^V X^V = (FX)^V, & \text{for any } X \in \mathfrak{T}_0^1(M), \end{cases} \tag{1.12}$$

which characterize F^V.

We also have

$$[F^V, G^V] = (FG - GF)^V, \quad \text{for any } F, G \in \mathfrak{T}_1^1(M). \tag{1.13}$$

Suppose that there is given a vector-valued 2-form N in M with local components $N_{ji}{}^h$. We can easily see that

$$\widetilde{N}_{CB}dx^C \wedge dx^B = p_a N_{ji}{}^a dx^j \wedge dx^i$$

is a 2-form in ${}^cT(M)$ and consequently that $\widetilde{N}_{CB}\varepsilon^{BA}$ is a tensor field of type (1, 1) in ${}^cT(M)$. We call this the *vertical lift of the vector-valued 2-form* N in M to ${}^cT(M)$, and denote it by N^V. The N^V has components

$$(1.14) \qquad N^V = \begin{pmatrix} 0 & 0 \\ p_a N_{ji}{}^a & 0 \end{pmatrix}.$$

The N^V satisfies

$$(1.15) \qquad \begin{cases} N^V \omega^V = 0, & \text{for any } \omega \in \mathfrak{T}_1^0(M), \\ N^V F^V = 0, & \text{for any } F \in \mathfrak{T}_1^1(M). \end{cases}$$

We can repeat the same argument and define the *vertical lift of a vector-valued r-form* in M to ${}^cT(M)$.

2. Complete lifts of tensor fields. Suppose that there is given a vector field X in M. From X we can construct a function $X^V = p_a X^a$ in M. The gradient $\widetilde{X}_B$ of X^V has components

$$\widetilde{X}_B = (p_a \partial_i X^a, X^i)$$

in ${}^cT(M)$. We can define a vector field $-\widetilde{X}_B\varepsilon^{BA}$ corresponding to this gradient in ${}^cT(M)$. We call this vector field the *complete lift* of X in M to ${}^cT(M)$ and denote it by X^C. The X^C has components

$$(2.1) \qquad X^C = \begin{pmatrix} X^h \\ -p_a \partial_i X^a \end{pmatrix}.$$

The complete lift of X in M to ${}^cT(M)$ has properties

$$(2.2) \qquad \begin{cases} X^C f^V = (Xf)^V, & \text{for any } f \in \mathfrak{T}_0^0(M), \\ X^C Y^V = [X, Y]^V, & \text{for any } Y \in \mathfrak{T}_0^1(M), \end{cases}$$

which characterize the complete lift X^C. The complete lift X^C of X in M to ${}^cT(M)$ has further properties:

(2.3) $$[X^C, \omega^V] = (\pounds_X \omega)^V, \quad \text{for any} \quad \omega \in \mathfrak{T}_1^0(M),$$

where $\pounds_X$ denotes the Lie derivative with respect to X,

(2.4) $$[X^C, F^V] = (\pounds_X F)^V, \quad \text{for any} \quad F \in \mathfrak{T}_1^1(M),$$

(2.5) $$N^V X^C = (N_X)^V,$$

for any vector-valued 2-form in M, where N_X is a tensor field of type (1, 1) such that $N_X Y = N(X, Y)$ for any $Y \in \mathfrak{T}_0^1(M)$, and

(2.6) $$[X^C, Y^C] = [X, Y]^C, \quad \text{for any } X, Y \in \mathfrak{T}_0^1(M).$$

We note here that (1.15) and (2.5) characterize the vertical lift N^V.

Now take a tensor field F of type (1, 1) in M with local components $F_i{}^h$. Then $p_a F_i{}^a dx^i$ is a 1-form in ${}^cT(M)$ and its exterior differential

$$d(p_a F_i{}^a dx^i) = p_a \partial_j F_i{}^a dx^j \wedge dx^i + F_i{}^a dp_a \wedge dx^i$$

gives, when it is written as $\frac{1}{2} \widetilde{F}_{CB} dx^C \wedge dx^B$, a tensor field of type (0, 2) whose components are

$$\widetilde{F}_{CB} = \begin{pmatrix} p_a(\partial_j F_i{}^a - \partial_i F_j{}^a) & F_i{}^j \\ -F_j{}^i & 0 \end{pmatrix}.$$

We define a tensor field of type (1, 1) by $\widetilde{F}_{CB} \varepsilon^{BA}$ and call this tensor field the *complete lift of* F in M to ${}^cT(M)$ and denote it by F^C. The F^C has components

(2.7) $$F^C = \begin{pmatrix} F_i{}^h & 0 \\ p_a(\partial_i F_h{}^a - \partial_h F_i{}^a) & F_h{}^i \end{pmatrix}.$$

The complete lift F^C has the properties

(2.8) $$\begin{aligned} F^C \omega^V &= (\omega F)^V, && \text{for any } \omega \in \mathfrak{T}_1^0(M), \\ F^C G^V &= (GF)^V, && \text{for any } G \in \mathfrak{T}_1^1(M), \\ F^C X^C &= (FX)^C + (\pounds_X F)^V, && \text{for any } X \in \mathfrak{T}_0^1(M) \end{aligned}$$

which characterize F^C, where ωF denotes a 1-form defined by

$$(\omega F)(X) = \omega(FX), \quad \text{for any } X \in \mathfrak{T}_0^1(M).$$

The complete lift F^C of $F \in \mathfrak{T}_1^1(M)$ has further properties

$$(2.9) \qquad \begin{aligned} F^C N^V &= (NF)^V, \quad \text{for any } N \in \mathfrak{T}_2^1(M), \\ N^V F^C &= (NF)^V, \quad \text{for any } N \in \mathfrak{T}_2^1(M), \end{aligned}$$

where NF is a tensor field of type (1, 2) defined by

$$(NF)(X, Y) = N_X(FY), \quad \text{for any } X, Y \in \mathfrak{T}_0^1(M), \text{ and}$$

$$(2.10) \qquad F^C G^C + G^C F^C = (FG+GF)^C + (N_{F,G})^V,$$

for any $F, G \in \mathfrak{T}_1^1(M)$, where $N_{F,G}$ is the Nijenhuis tensor formed with F and G:

$$(2.11) \qquad \begin{aligned} 2N_{F,G}(X, Y) = {} & [FX, GY] + [GX, FY] \\ & - F[GX, Y] - G[FX, Y] - F[X, GY] - G[X, FY] \\ & + (FG+GF)[X, Y] \end{aligned}$$

for any $X, Y \in \mathfrak{T}_0^1(M)$. From equation (2.10) we have, on putting $F=G$,

$$(2.12) \qquad (F^C)^2 = (F^2)^C + N^V,$$

where N is the Nijenhuis tensor formed from F:

$$(2.13) \qquad N(X, Y) = [FX, FY] - F[FX, Y] - F[X, FY] + F^2[X, Y].$$

From (2.12), we see that, when F defines a complex structure, that is, $F^2 = -1$ and $N = 0$, its complete lift F^C to ${}^cT(M)$ defines an almost complex structure in ${}^cT(M)$.

We can moreover prove that, F being an almost complex structure,

$$(2.14) \qquad F^C + \frac{1}{2}(NF)^V$$

is also an almost complex structure. (See, Satô, [2]).

Now take a vector-valued 2-form N in M with local components $N_{ji}{}^h$. Then $p_a N_{ji}{}^a dx^j \wedge dx^i$ is a 2-form in ${}^cT(M)$ and consequently its exterior differential

$$d(p_a N_{ji}{}^a dx^j \wedge dx^i) = p_a(\partial_k N_{ji}{}^a)\, dx^k \wedge dx^j \wedge dx^i + N_{ji}{}^a dp_a \wedge dx^j \wedge dx^i,$$

gives, when it is written as $\frac{1}{3}\tilde{N}_{DCB}dx^D\wedge dx^C\wedge dx^B$, a tensor field $\tilde{N}_{DCB}$ of type (0, 3) in ${}^cT(M)$, where

$$N_{kji} = p_a(\partial_k N_{ji}{}^a + \partial_j N_{ik}{}^a + \partial_i N_{kj}{}^a),$$
$$N_{ji\bar{h}} = N_{i\bar{h}j} = N_{\bar{h}ji} = N_{ji}{}^h,$$

all the other components being zero, from which we can define a tensor field of type (1, 2) $\tilde{N}_{CBE}\varepsilon^{EA}$ in ${}^cT(M)$. We call this *the complete lift of* N in M to ${}^cT(M)$ and denote it by N^C. The N^C has components

$$\tag{2.15}\begin{aligned}\tilde{N}_{ji}{}^h &= N_{ji}{}^h,\\ \tilde{N}_{ji}{}^{\bar{h}} &= -p_a(\partial_j N_{ih}{}^a+\partial_i N_{hj}{}^a+\partial_h N_{ji}{}^a),\\ \tilde{N}_{j\bar{i}}{}^{\bar{h}} &= N_{jh}{}^i,\\ \tilde{N}_{\bar{j}i}{}^{\bar{h}} &= N_{hi}{}^j,\end{aligned}$$

all the others being zero. The N^C satisfies

$$\tag{2.16} N^C(X^C, Y^C) = (N(X, Y))^C - ((\pounds_X N)_Y - (\pounds_Y N)_X + N_{[X,Y]})^V,$$

which characterizes N^C.

We can prove that the Nijenhuis tensor of the complete lift F^C of F is the complete lift N^C of the Nijenhuis tensor N formed with F.

We know that if F defines a complex structure in M, then F^C defines an almost complex structure in ${}^cT(M)$. Following (2.16) and the fact above, F^C actually defines a complex structure in ${}^cT(M)$.

Suppose now that F defines an almost complex structure in M. We know that $\bar{F}=F^C+\frac{1}{2}(NF)^V$ defines an almost complex structure in ${}^cT(M)$. We thus consider the Nijenhuis tensor $\bar{N}$ of $\bar{F}$. The Nijenhuis tensor $\bar{N}$ of $\bar{F}$ has components

$$\tag{2.17}\begin{aligned}\bar{N}_{ji}{}^h &= N_{ji}{}^h,\\ \bar{N}_{ji}{}^{\bar{h}} &= -p_a(\partial_j N_{ih}{}^a + \partial_i N_{hj}{}^a + \partial_h N_{ji}{}^a)\\ &\quad + \frac{1}{2}p_a\{F_j{}^t\partial_t(N_{is}{}^a F_h{}^s) - F_i{}^t\partial_t(N_{js}{}^a F_h{}^s)\\ &\quad - (\partial_j(N_{is}{}^a F_t{}^s) - \partial_i(N_{js}{}^a F_t{}^s))F_h{}^t\\ &\quad + (\partial_j F_t{}^a - \partial_t F_j{}^a)N_{is}{}^t F_h{}^s - (\partial_i F_t{}^a - \partial_t F_i{}^a)N_{js}{}^t F_h{}^s\end{aligned}$$

$$- (\partial_j F_h{}^t - \partial_h F_j{}^t) N_{is}{}^a F_t{}^s + (\partial_i F_h{}^t - \partial_h F_i{}^t) N_{js}{}^a F_t{}^s$$

$$- (\partial_j F_i{}^t - \partial_i F_j{}^t) N_{ts}{}^a F_h{}^s + \frac{1}{2}(N_{js}{}^a N_{ih}{}^s - N_{is}{}^a N_{jh}{}^s)\} ,$$

all the others being zero.

3. Lifts of vector fields on the cross-sections. Suppose that there is given a global 1-form W in M whose local expression is $W = W_i(x)\, dx^i$. Then the 1-form W defines a cross-section in ${}^cT(M)$, whose parametric representation is

$$x^h = x^h , \quad p_h = W_h(x) . \tag{3.1}$$

Thus the tangent vectors $B_i{}^A = \partial_i x^A$ to the cross-section have components

$$B_i{}^A = \begin{pmatrix} \delta_i^h \\ \partial_i W_h \end{pmatrix} . \tag{3.2}$$

On the other hand, the fibre being represented by

$$x^h = \text{const.}, \quad p_h = p_h , \tag{3.3}$$

the tangent vectors $C_{\bar{i}}{}^A = \partial_{\bar{i}} x^A$ to the fibre have components

$$C_{\bar{i}}{}^A = C^{iA} = \begin{pmatrix} 0 \\ \delta_h^i \end{pmatrix} . \tag{3.4}$$

The vectors $B_i{}^A$ and $C_{\bar{i}}{}^A$, being linearly independent, form a frame along the cross-section. We call this the *frame* (B, C) along the cross-section. The coframe $(B^h{}_A, C^{\bar{h}}{}_A)$ corresponding to this frame is given by

$$\begin{aligned} B^h{}_A &= (\delta_i^h, 0) \\ C^{\bar{h}}{}_A &= C_{hA} = (-\partial_i W_h, \delta_h^i) . \end{aligned} \tag{3.5}$$

We call this coframe the *coframe* (B, C) along the cross-section.

The basic 1-form $p = p_i\, dx^i$ has the expression $p = W_i\, dx^i$ and the basic 2-form the expression $dp = \frac{1}{2}(\partial_j W_i - \partial_i W_j)\, dx^j \wedge dx^i$ on the cross-section. The vertical lift ω^V of a 1-form $\omega = \omega_i\, dx^i$ has the expression

$$C_{\bar{i}}{}^{A}\omega^{\bar{i}} = C^{iA}\omega_i = \begin{pmatrix} 0 \\ \omega_i \end{pmatrix} \tag{3.6}$$

on the cross-section.

The complete lift X^C of a vector field X in M to ${}^cT(M)$, having components (2. 1) with respect to the natural frame, has components

$$\begin{pmatrix} X^h \\ -\pounds_X W_h \end{pmatrix}$$

with respect to the frame (B, C) along the cross-section. Thus We have

$$X^C:\ B_i{}^A X^i - C^{iA}(\pounds_X W_i), \tag{3.7}$$

from which

PROPOSITION 3.1. *The complete lift X^C of a vector field X in M to ${}^cT(M)$ is tangent to the cross-section determined by a 1-form W in M if and only if the Lie derivative of W with respect to X vanishes in M.*

Suppose now that an affine connection ∇ without torsion is given in M and denote by Γ_{ji}^h the components of the connection. Then

$$ds^2 = 2\,\delta p_i\, dx^i, \tag{3.8}$$

where

$$\delta p_i = dp_i - \Gamma_{ji}^h dx^j p_h, \tag{3.9}$$

defines a Riemannian metric in ${}^cT(M)$. We call this metric in ${}^cT(M)$ the Riemann extension of ∇ and denote it by ∇^R [1]. With respect to the Riemann extension ∇^R, the fibre given by $dx^h = 0$ is null and the horizontal distribution given by $\delta p_i = 0$ is also null.

The Riemann extension ∇^R has components

$$\nabla^R: \begin{pmatrix} -2\Gamma_{ji}^h p_h & \delta_i^j \\ \delta_j^i & 0 \end{pmatrix} \tag{3.10}$$

with respect to the natural frame and components

$$\nabla^R : \begin{pmatrix} \nabla_j W_i + \nabla_i W_j & \delta_i^j \\ \delta_j^i & 0 \end{pmatrix} \tag{3.11}$$

with respect to the frame (B, C) along the cross-section, from which we have

PROPOSITION 3.2. *If M has an affine connection ∇ without torsion and ${}^cT(M)$ has the Riemann extension ∇^R as its metric, then the cross-section determined by a 1-form W in M is null with respect to ∇^R if and only if*

$$\nabla_j W_i + \nabla_i W_j = 0 . \tag{3.12}$$

PROPOSITION 3.3. *When M has Riemannian metric g and the Levi-Civita connection ∇ of g and ${}^cT(M)$ has the Riemann extension ∇^R as its metric, the cross-section determined by a 1-form W in M is null with respect to ∇^R if and only if W is a Killing vector field in M.*

4. Lifts of almost complex structures on cross-sections. Suppose that the manifold M has a complex structure F. Then the cotangent bundle ${}^cT(M)$ has the complex structure F^C.

Now the F^C has the components (2.7) with respect to the natural frame and consequently has components

$$\begin{pmatrix} F_i^{\ h} & 0 \\ (\partial_i F_h^{\ a} - \partial_h F_i^{\ a}) W_a - F_i^{\ t} \partial_t W_h + F_h^{\ t} \partial_i W_t, & F_h^{\ i} \end{pmatrix} \tag{4.1}$$

with respect to the frame (B, C) along the cross-section determined by W. Thus we have

$$\begin{aligned} \widetilde{F}_B^A B_i^{\ B} &= F_i^{\ h} B_h^{\ A} + \{(\partial_i F_h^{\ a} - \partial_h F_i^{\ a}) W_a - F_i^{\ t} \partial_t W_h + F_h^{\ t} \partial_i W_t\} C^{hA} , \\ \widetilde{F}_B^A C_{\bar{i}}^{\ B} &= F_t^{\ i} C^{tA} . \end{aligned} \tag{4.2}$$

Thus the cross-section is analytic if and only if

$$P_{ih} = (\partial_i F_h^{\ a} - \partial_h F_i^{\ a}) W_a - F_i^{\ t} \partial_t W_h + F_h^{\ t} \partial_i W_t = 0 . \tag{4.3}$$

We can easily verify that P_{ih} are components of a tensor field of type (0, 2) in M. On the other hand, equation (4.3) is the condition for W_i to be covariant analytic. [5]. Thus we have

PROPOSITION 4.1. *Suppose that M has a complex structure F. Then the cross-section determined by a 1-form W in ${}^cT(M)$ with complex structure F^C is analytic if and only if W is covariant analytic in M.*

Now suppose that M has an almost complex structure F. Then the Nijenhuis tensor $\tilde{N}$ of the complete lift F^C of F has components (2.15) with respect to the natural frame in ${}^cT(M)$. Thus we have

$$\tilde{N}_{CB}{}^A B_j{}^C B_i{}^B = N_{ji}{}^h B_h{}^A - Q_{jih} C^{hA}, \tag{4.4}$$

where

$$Q_{jih} = (\partial_j N_{ih}{}^a + \partial_i N_{hj}{}^a + \partial_h N_{ji}{}^a) W_a + N_{ji}{}^t \partial_t W_h - N_{ih}{}^t \partial_j W_t - N_{hj}{}^t \partial_i W_t . \tag{4.5}$$

We can easily verify that Q_{jih} are components of a tensor field of type (0, 3) in M. From (4.4) we have

PROPOSITION 4.2. *In order that $\tilde{N}_{CB}{}^A B_j{}^C B_i{}^B$ be tangent to the cross-section determined by a 1-form W, it is necessary and sufficient that $Q_{jih}=0$ in M.*

We know that when M has an almost complex structure F, the cotangent bundle ${}^cT(M)$ has also an almost complex structure

$$\bar{F} = F^C + \frac{1}{2}(NF)^V .$$

The almost complex structure $\bar{F}$ has components

$$\bar{F} : \begin{pmatrix} F_i{}^h & 0 \\ (\partial_i F_h{}^a - \partial_h F_i{}^a + \frac{1}{2} N_{it}{}^a F_h{}^t) W_a & F_h{}^i \end{pmatrix}$$

with respect to the natural frame in ${}^cT(M)$ and components

$$\bar{F} : \begin{pmatrix} F_i{}^h & 0 \\ (\partial_i F_h{}^a - \partial_h F_i{}^a) W_a - F_i{}^t \partial_t W_h + F_h{}^t \partial_i W_t + \frac{1}{2} N_{it}{}^a F_h{}^t W_a) & F_h{}^i \end{pmatrix}$$

with respect to the frame (B, C) along the cross-section determined by the 1-form W. Thus we have

$$\begin{aligned} \bar{F}_B^A B_i^B &= F_i^h B_h^A \\ &+ \{(\partial_i F_h^a - \partial_h F_i^a) W_a - F_i^t \partial_t W_h + F_h^t \partial_i W_t + \frac{1}{2} N_{it}^a F_h^t W_a\} C^{hA}, \\ \bar{F}_B^A C^{iB} &= F_h^i C^{hA}. \end{aligned} \tag{4.6}$$

Thus the cross-section is almost analytic if and only if

$$(\partial_i F_h^a - \partial_h F_i^a) W_a - F_i^t \partial_t W_h + F_h^t \partial_i W_t + \frac{1}{2} N_{it}^a F_h^t W_a = 0.$$

But the last equation means that W_i is almost covariant analytic [2], [5]. Thus we have

PROPOSITION 4.3. *Suppose that M has an almost complex structure F. Then the cross-section determined by W in ${}^cT(M)$ with almost complex structure $F^C + \frac{1}{2}(NF)^V$ is almost analytic if and only if W is almost covariant analytic in M.*

We now consider the Nijenhuis tensor $\bar{N}$ of $\bar{F} = F^C + \frac{1}{2}(NF)^V$. The $\bar{N}$ has components (2.17), or equivalently, by virtue of the relation $N_{is}^a F_h^s = -N_{ih}^s F_s^a$,

$$\begin{aligned} \bar{N}_{ji}^h &= N_{ji}^h, \\ \bar{N}_{ji}^{\bar{h}} &= -p_a(\partial_j N_{ih}^a + \partial_i N_{hj}^a + \partial_h N_{ji}^a) \\ &\quad - \frac{1}{2} p_a [F_j^t \partial_t (N_{ih}^s F_s^a) - F_i^t \partial_t (N_{jh}^s F_s^a) \\ &\quad + \{\partial_j (N_{is}^a F_t^s) - \partial_i (N_{js}^a F_t^s)\} F_h^t \\ &\quad + (\partial_j F_t^a - \partial_t F_j^a) N_{ih}^s F_s^t - (\partial_i F_t^a - \partial_t F_i^a) N_{jh}^s F_s^t \\ &\quad - (\partial_j F_h^t - \partial_h F_j^t) N_{it}^s F_s^a + (\partial_i F_h^t - \partial_h F_i^t) N_{jt}^s F_s^a \\ &\quad - (\partial_j F_i^t - \partial_i F_j^t) N_{th}^s F_s^a \\ &\quad - \frac{1}{2} (N_{js}^a N_{ih}^s - N_{is}^a N_{jh}^s)], \end{aligned} \tag{4.8}$$

all the others being zero. Thus we have

$$\bar{N}_{CB}^A B_j^C B_i^B = N_{ji}^h B_h^A + R_{jih} C^{hA}, \tag{4.9}$$

where

$$
(4.10)\qquad \begin{aligned}
R_{jih} = &- N_{ji}{}^{t}\partial_t W_h - (\partial_j N_{ih}{}^{a} + \partial_i N_{hj}{}^{a} + \partial_h N_{ji}{}^{a}) W_a \\
&- \frac{1}{2}[F_j{}^{t}\partial_t(N_{ih}{}^{s}F_s{}^{a}) - F_i{}^{t}\partial_t(N_{jh}{}^{s}F_s{}^{a}) \\
&+ \{\partial_j(N_{is}{}^{a}F_t{}^{s}) - \partial_i(N_{js}{}^{a}F_t{}^{s})\}F_h{}^{t} \\
&+ (\partial_j F_t{}^{a} - \partial_t F_j{}^{a}) N_{ih}{}^{s}F_s{}^{t} - (\partial_i F_t{}^{a} - \partial_t F_i{}^{a}) N_{jh}{}^{s}F_s{}^{t} \\
&- (\partial_j F_h{}^{t} - \partial_h F_j{}^{t}) N_{it}{}^{s}F_s{}^{a} + (\partial_i F_h{}^{t} - \partial_h F_i{}^{t}) N_{jt}{}^{s}F_s{}^{a} \\
&- (\partial_j F_i{}^{t} - \partial_i F_j{}^{t}) N_{th}{}^{s}F_s{}^{a} - \frac{1}{2}(N_{js}{}^{a}N_{ih}{}^{s} - N_{is}{}^{a}N_{jh}{}^{s})] W_a .
\end{aligned}
$$

We can easily verify that R_{jih}, or rather R_{jih} minus last term containing $N_{ij}{}^{e}N_{hu}{}^{i}$'s are components of a tensor field of type (0, 2) in M. From (4.9) we have

PROPOSITION 4.4. *The vector $\bar{N}_{CB}{}^{A}B_j{}^{C}B_i{}^{B}$ is tangent to the cross-section determined by W_i if and only if $R_{jih} = 0$.*

5. The Slebodzinski tensor. From (4.10), we have

$$
(5.1)\qquad \begin{aligned}
R_{jih} + R_{ihj} + R_{hji} = &- N_{ji}{}^{t}\partial_t W_h - N_{ih}{}^{t}\partial_t W_j - N_{hj}{}^{t}\partial_t W_i \\
&- 3(\partial_j N_{ih}{}^{a} + \partial_i N_{hj}{}^{a} + \partial_h N_{ji}{}^{a}) W_a \\
&- [(F_j{}^{t}\partial_t N_{ih}{}^{s} + F_i{}^{t}\partial_t N_{hj}{}^{s} + F_h{}^{t}\partial_t N_{ji}{}^{s}) F_s{}^{a}W_a \\
&- \{N_{ji}{}^{s}(\partial_s F_h{}^{t}) + N_{ih}{}^{s}(\partial_s F_j{}^{t}) + N_{hj}{}^{s}(\partial_s F_i{}^{t})\} F_t{}^{a}W_a \\
&- (\partial_j N_{ih}{}^{a} + \partial_i N_{hj}{}^{a} + \partial_h N_{ji}{}^{a}) W_a \\
&- \{(\partial_j F_i{}^{t} - \partial_i F_j{}^{t}) N_{th}{}^{s} \\
&+ (\partial_i F_h{}^{t} - \partial_h F_i{}^{t}) N_{tj}{}^{s} \\
&+ (\partial_h F_j{}^{t} - \partial_h F_j{}^{t}) N_{ti}{}^{s}\} F_s{}^{a}W_a \\
&+ \frac{1}{2}(N_{jt}{}^{a}N_{ih}{}^{t} + N_{it}{}^{a}N_{hj}{}^{t} + N_{ht}{}^{a}N_{ji}{}^{t}) W_a]
\end{aligned}
$$

But, we have on the other hand

$$
(5.2)\qquad \begin{aligned}
&\partial_j N_{ih}{}^{a} + \partial_i N_{hj}{}^{a} + \partial_h N_{ji}{}^{a} \\
&= -\frac{1}{2}\{(\partial_j N_{ih}{}^{t} - \partial_i N_{jh}{}^{t}) + (\partial_i N_{hj}{}^{t} - \partial_h N_{ij}{}^{t})
\end{aligned}
$$

$$+ (\partial_h N_{ji}{}^t - \partial_j N_{hi}{}^t)\} F_t{}^s F_s{}^a$$

$$= -\frac{1}{2}\{\partial_j(N_{ih}{}^t F_t{}^s) - N_{ih}{}^t \partial_j F_t{}^s - \partial_i(N_{jh}{}^t F_t{}^s) + N_{jh}{}^t \partial_i F_t{}^s$$

$$+ \partial_i(N_{hj}{}^t F_t{}^s) - N_{hj}{}^t \partial_i F_t{}^s - \partial_h(N_{ij}{}^t F_t{}^s) + N_{ij}{}^t \partial_h F_t{}^s$$

$$+ \partial_h(N_{ji}{}^t F_t{}^s) - N_{ji}{}^t \partial_h F_t{}^s - \partial_j(N_{hi}{}^t F_t{}^s) + N_{hi}{}^t \partial_j F_t{}^s\} F_s{}^a$$

$$= \frac{1}{2}\{\partial_j(N_{it}{}^s F_h{}^t) + N_{ih}{}^t \partial_j F_t{}^s - \partial_i(N_{jt}{}^s F_h{}^t) - N_{jh}{}^t \partial_i F_t{}^s$$

$$+ \partial_i(N_{ht}{}^s F_j{}^t) + N_{hj}{}^t \partial_i F_t{}^s - \partial_h(N_{it}{}^s F_j{}^t) - N_{ij}{}^t \partial_h F_t{}^s$$

$$+ \partial_h(N_{jt}{}^s F_i{}^t) + N_{ji}{}^t \partial_h F_t{}^s - \partial_j(N_{ht}{}^s F_i{}^t) - N_{hi}{}^t \partial_j F_t{}^s\} F_s{}^a$$

$$= \frac{1}{2}\{F_j{}^t(\partial_i N_{ht}{}^s - \partial_h N_{it}{}^s) + F_i{}^t(\partial_h N_{jt}{}^s - \partial_j N_{ht}{}^s)$$

$$+ F_h{}^t(\partial_j N_{it}{}^s - \partial_i N_{jt}{}^s)$$

$$+ (\partial_j F_i{}^t - \partial_i F_j{}^t) N_{th}{}^s + (\partial_i F_h{}^t - \partial_h F_i{}^t) N_{tj}{}^s$$

$$+ (\partial_h F_j{}^t - \partial_j F_h{}^t) N_{ti}{}^s$$

$$+ 2(N_{ji}{}^t \partial_h F_t{}^s + N_{ih}{}^t \partial_j F_t{}^s + N_{hj}{}^t \partial_i F_t{}^s)\} F_s{}^a .$$

Thus, we have from (5.1)

(5.3) $$R_{jih} + R_{ihj} + R_{hji}$$

$$= -(N_{ji}{}^t \partial_t W_h + N_{ih}{}^t \partial_t W_j + N_{hj}{}^t \partial_t W_i) - (\partial_j N_{ih}{}^a + \partial_i N_{hj}{}^a + \partial_h N_{ji}{}^a) W_a$$

$$- [F_j{}^t \partial_t N_{ih}{}^s + F_i{}^t \partial_t N_{hj}{}^s + F_h{}^t \partial_t N_{ji}{}^s$$

$$+ \frac{1}{2}\{F_j{}^t(\partial_i N_{ht}{}^s - \partial_h N_{it}{}^s) + F_i{}^t(\partial_h N_{jt}{}^s - \partial_j N_{ht}{}^s) + F_h{}^t(\partial_j N_{it}{}^s - \partial_i N_{jt}{}^s)\}$$

$$+ N_{ji}{}^t(\partial_h F_t{}^s - \partial_t F_h{}^s) + N_{ih}{}^t(\partial_j F_t{}^s - \partial_t F_j{}^s) + N_{hj}{}^t(\partial_i F_t{}^s - \partial_t F_i{}^s)$$

$$- \frac{1}{2}\{(\partial_j F_i{}^t - \partial_i F_j{}^t) N_{th}{}^s + (\partial_i F_h{}^t - \partial_h F_i{}^t) N_{tj}{}^s$$

$$+ (\partial_h F_j{}^t - \partial_j F_h{}^t) N_{ti}{}^s\}] F_s{}^a W_a$$

$$- \frac{1}{2}[N_{jt}{}^a N_{ih}{}^t + N_{it}{}^a N_{hj}{}^t + N_{ht}{}^a N_{ji}{}^t] W_a ,$$

that is

(5.4) $$R_{jih} + R_{ihj} + R_{hji} + Q_{jih} + N_{ih}{}^t(\partial_j W_t - \partial_t W_j)$$

$$+ N_{hj}{}^t(\partial_i W_t - \partial_t W_i) + S_{jih}{}^s F_s{}^a W_a$$

$$+\frac{1}{2}(N_{jt}{}^{a}N_{ih}{}^{t}+N_{it}{}^{a}N_{hj}{}^{t}+N_{ht}{}^{a}N_{ji}{}^{t})W_{a}=0,$$

where

$$\begin{aligned}S_{jih}{}^{s} &= F_{j}{}^{t}\partial_{t}N_{ih}{}^{s}+F_{i}{}^{t}\partial_{t}N_{hj}{}^{s}+F_{h}{}^{t}\partial_{t}N_{ji}{}^{s}\\ &+\frac{1}{2}\{F_{j}{}^{t}(\partial_{i}N_{ht}{}^{s}-\partial_{h}N_{it}{}^{s})+F_{i}{}^{t}(\partial_{h}N_{jt}{}^{s}-\partial_{j}N_{ht}{}^{s})\\ &+F_{h}{}^{t}(\partial_{j}N_{it}{}^{s}-\partial_{i}N_{jt}{}^{s})\}\\ &+N_{ji}{}^{t}(\partial_{h}F_{t}{}^{s}-\partial_{t}F_{h}{}^{s})+N_{ih}{}^{t}(\partial_{j}F_{t}{}^{s}-\partial_{t}F_{j}{}^{s})+N_{hj}{}^{t}(\partial_{i}F_{t}{}^{s}-\partial_{t}F_{i}{}^{s})\\ &-\frac{1}{2}\{(\partial_{j}F_{i}{}^{t}-\partial_{i}F_{j}{}^{t})N_{th}{}^{s}+(\partial_{i}F_{h}{}^{t}-\partial_{h}F_{i}{}^{t})N_{tj}{}^{s}\\ &+(\partial_{h}F_{j}{}^{t}-\partial_{j}F_{h}{}^{t})N_{ti}{}^{s}\}\,.\end{aligned}\tag{5.5}$$

Equation (5.4) shows the tensor character of $S_{jih}{}^{s}$. This is the tensor first introduced by Slebodzinski. [3]. (The expression of Slebodzinski tensor in Math. Rev. 30 (1965), p. 652, 3438, should be read as 2[ ]+[ ]+2[ ]−[ ].) T. J. Willmore [4] showed that this tensor is identically zero.

6. Complete lift of a connection on cross-sections. Suppose that there is given a symmetric affine connection ∇ in M whose components are Γ_{ji}^{k}. Then (3.10) defines a Riemannian metric in ${}^{c}T(M)$ which is called the Riemann extension of ∇.

We construct the Levi-Civita connection ∇^{C} from this Riemann extension and call it *complete lift of the symmetric affine connection* ∇ to the cotangent bundle ${}^{c}T(M)$. The complete lift ∇^{C} has components $\widetilde{\Gamma}_{CB}^{A}$ given by

$$\begin{aligned}&\widetilde{\Gamma}_{ji}^{h}=\Gamma_{ji}^{h},\quad \widetilde{\Gamma}_{j\bar{i}}^{h}=0,\quad \widetilde{\Gamma}_{\bar{j}i}^{h}=0,\quad \widetilde{\Gamma}_{\bar{j}\bar{i}}^{h}=0.\\ &\widetilde{\Gamma}_{ji}^{\bar{h}}=p_{a}(\partial_{h}\Gamma_{ji}^{a}-\partial_{j}\Gamma_{ih}^{a}-\partial_{i}\Gamma_{jh}^{a}+2\Gamma_{ht}^{a}\Gamma_{ji}^{t}),\\ &\widetilde{\Gamma}_{j\bar{i}}^{\bar{h}}=-\Gamma_{jh}^{i},\quad \widetilde{\Gamma}_{\bar{j}i}^{\bar{h}}=-\Gamma_{hi}^{j},\quad \widetilde{\Gamma}_{\bar{j}\bar{i}}^{\bar{h}}=0,\end{aligned}\tag{6.1}$$

and the curvature tensor of the complete lift ∇^{C} components $\widetilde{R}_{DCB}{}^{A}$ given by

$$\begin{aligned}&\widetilde{R}_{kji}{}^{h}=R_{kji}{}^{h},\\ &\widetilde{R}_{kji}{}^{\bar{h}}=(\nabla_{h}R_{kji}{}^{a}-\nabla_{i}R_{kjh}{}^{a}\\ &\qquad+\Gamma_{ht}^{a}R_{kji}{}^{t}+\Gamma_{kt}^{a}R_{ihj}{}^{t}+\Gamma_{jt}^{a}R_{hik}{}^{t}+\Gamma_{it}^{a}R_{kjh}{}^{t})p_{a},\\ &\widetilde{R}_{kj\bar{i}}{}^{\bar{h}}=-R_{kjh}{}^{i},\quad \widetilde{R}_{k\bar{j}i}{}^{\bar{h}}=-R_{hik}{}^{j},\quad \widetilde{R}_{\bar{k}ji}{}^{\bar{h}}=-R_{hij}{}^{k},\end{aligned}\tag{6.2}$$

all the others being zero, where $R_{kji}{}^{h}$ are components of the curvature tensor of ∇.

Suppose now that there is given a global 1-form W in M. Then the W defines a cross-section in ${}^cT(M)$. The vectors (3. 2) are tangent to the cross-section and (3. 4) are n linearly independent vectors which are not tangent to the cross-section. We take the vectors C^{iA} as normals to the cross-section and define an affine connection induced on the cross-section. The components of the induced affine connection are given by

$$(\partial_j B_i{}^A + \tilde{\Gamma}^A_{CB} B_j{}^C B_i{}^B) B^h{}_A = \Gamma^h_{ji}\,. \tag{6. 3}$$

From this equation we see that the quantity

$$\partial_j B_i{}^A + \tilde{\Gamma}^A_{CB} B_j{}^C B_i{}^B - \Gamma^h_{ji} B_h{}^A \tag{6. 4}$$

is a linear combination of the vectors $C_{\bar{i}}{}^A$. To find the coefficients, we put $A=\bar{h}$ in (6. 4) and find

$$\begin{aligned}&\partial_j\partial_i W_h + W_a(\partial_h \Gamma^a_{ji} - \partial_j \Gamma^a_{ih} - \partial_i \Gamma^a_{jh} + 2\Gamma^a_{ht}\Gamma^t_{ji})\\ &\quad - \Gamma^a_{jh}\partial_i W_a - \Gamma^a_{hi}\partial_j W_a - \Gamma^a_{ji}\partial_a W_h\\ &= \nabla_j \nabla_i W_h + R_{hij}{}^a W_a\,.\end{aligned}$$

Thus representing (6. 4) by ${}'\nabla_j B_i^A$, we have

$${}'\nabla_j B_i^A = (\nabla_j \nabla_i W_h + R_{hij}{}^a W_a) C^{hA}, \tag{6. 5}$$

which is the equation of Gauss for the cross-section determined by W_i. Thus we have

PROPOSITION 6.1. *In order that the cross-section in ${}^cT(M)$ determined by a 1-form W in M with symmetric affine connection ∇ be totally geodesic, it is necessary and sufficient that W satisfies*

$$\nabla_j \nabla_i W_h + R_{hij}{}^a W_a = 0\,. \tag{6. 6}$$

On the other hand, since the components $\tilde{\Gamma}^A_{CB}$ are given by (6. 1) we can easily verify that

$$\partial_j C_{\bar{i}}{}^A + \tilde{\Gamma}^A_{CB} B_j{}^C C_{\bar{i}}{}^B - \Gamma^i_{jh} C_{\bar{h}}{}^A = 0\,,$$

that is

$$\partial_j C^{iA} + \tilde{\Gamma}^A_{CB} B_j{}^C C^{iB} - \Gamma^i_{jh} C^{hA} = 0\,.$$

Thus denoting by $'\nabla_j C^{iA}$ the left hand member of this equation, we get

$$(6.7) \qquad '\nabla_j C^{iA} = 0 .$$

This is the equation of Weingarten for the cross-section.

Applying the operator $'\nabla_k$ to (6.5), we find

$$'\nabla_k ' \nabla_j B_i^A = \nabla_k(\nabla_j \nabla_i W_h + R_{hij}{}^a W_a) C^{hA} ,$$

from which, remembering that

$$'\nabla_k ' \nabla_j B_i^A - '\nabla_j ' \nabla_k B_i^A = \widetilde{R}_{DCB}{}^A B_k{}^D B_j{}^C B_i{}^B - R_{kji}{}^h B_h{}^A ,$$

we find

$$(6.8) \qquad \widetilde{R}_{DCB}{}^A B_k{}^D B_j{}^C B_i{}^B - R_{kji}{}^h B_h{}^A = [(\nabla_k R_{hij}{}^a - \nabla_j R_{hik}{}^a) W_a$$
$$- R_{kji}{}^a \nabla_a W_h - R_{kjh}{}^a \nabla_i W_a + R_{hij}{}^a \nabla_k W_a - R_{hik}{}^a \nabla_j W_a] C^{hA} .$$

Thus we have

PROPOSITION 6.2. *In order that $\widetilde{R}_{DCB}{}^A B_k{}^D B_j{}^C B_i{}^B$ is tangent to the cross-section, it is necessary and sufficient that*

$$(6.9) \qquad (\nabla_k R_{hij}{}^a - \nabla_j R_{hik}{}^a) W_a$$
$$= R_{kji}{}^a \nabla_a W_h + R_{kjh}{}^a \nabla_i W_a - R_{hij}{}^a \nabla_k W_a + R_{hik}{}^a \nabla_j W_a .$$

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