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A CRITICAL EXAMINATION OF THE THEORY OF CURVES IN THREE DIMENSIONAL DIFFERENTIAL GEOMETRY

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CONTENTS

1. 1. Introduction. The theory of curves in the differential geometry of 3-dimensional Euclidean space is not properly treated in books on elementary differential geometry. Firstly, the relationship between the two parts of the fundamental theorem is not clearly explained [Nomizu, 7]. Secondly, a number of theorems on special curves are often loosely or incorrectly stated or proved; for example,

(i) the condition for a curve to be a plane curve [Gifford, 3],
(ii) the condition for a curve to be a helix,
(iii) the condition for a curve to lie on a sphere [Wong, 10].

All these defects can be traced to one common cause :— The difficulty of studying the properties of a curve when one or more functions intrinsically associated with the curve vanish at some of its points.

The purpose of this paper is two-fold: To give a more detailed study of the Frenet curves as defined by K. Nomizu [7] and to present what we believe to be a unified and correct treatment of the special curves mentioned in the last paragraph. The main tools used are a lemma on the nature of subsets of a topological space on which one or more continuous functions vanish (§ 1.3), and a method to construct on the real line R a C^∞ function whose set of zeros is an arbitrarily given closed subset of R (§ 1.4).

It is easy to see that the techniques we develop in this paper can also be used to refine some of the results in the theory of curves in affine and projective differential geometries, and in fact, in the differential geometry of any kind of one-parameter family of geometric figures.

§ 1.2—§ 1.5 are preliminary in nature. In § 2.1—§ 2.3, we consider the Frenet curves and determine how far such a curve determines its curvature and pseudo-torsion (Theorem 2.1); a direct consequence of our result is that for a closed Frenet curve, the total pseudo-torsion, modulo 2π, is an intrinsic quantity of the curve (Theorem 2.4). In § 3.1—§ 5.2, we study in some detail the conditions usually given for a curve to be a plane curve, a helix or a spherical curve. We prove that in general a curve satisfying any of these conditions contains a dense subset which is the union of a countable number of arcs of different types. In each case, we shall show by an example that such curves actually exist. In § 6.1, we prove that C^∞ regular helices, as well as C^∞ regular plane curves and spherical curves, are Frenet curves, and in § 6.2, we obtain necessary and sufficient conditions for a Frenet curve to be a plane curve, helix or a spherical curve in terms of its curvature and pseudo-torsion.

1.2. Open and closed subsets of the real line or of a proper interval. We state here without proof some known properties of the real line R and proper intervals on it which are used or helpful in this paper (see, for example, Hobson [5], p. 116). (By a proper interval we mean an interval with non-empty interior.) We assume that R has the usual topology.

(a) Any open set of R is the union of a countable family of disjoint open intervals.
(b) Any closed set of R consists of (i) the end-points of a countable

family of disjoint intervals, (ii) the limit points of the set of such end-points and (iii) a countable number of proper (closed) intervals.

(c) Any open subset of a proper interval L, being the intersection of an open subset of R with L, is the union of a countable number of disjoint intervals which are all open in L and all of which except at most two are open intervals of R.

1.3. A lemma on certain subsets of a proper interval. In this section we prove the main lemma of the paper. Although it is in effect a theorem on coverings of a topological space by the closures of open sets, we shall formulate it in terms of the sets of zeros of some continuous functions, for convenience of application. The motivation of this lemma will be clear later when it is used to study the global structure of some types of curves which are characterized by the vanishing or non-vanishing of functions intrinsically associated with the curves.

Throughout this paper, if f is a function defined on a set, we mean by $f = 0$ that f is everywhere zero, and by $f \neq 0$ that f is nowhere zero. If L is a proper interval which is closed at one end or at both ends, we mean by a C^k function on L a function which can be extended to a C^k function on an open interval containing L.

LEMMA 1.1. *Let $f_1, \cdots, f_n$ be continuous (real-valued) functions of which f_1 is defined on a proper interval L of the real line, and f_i $(2 \leqq i \leqq n)$ is defined on the set $G_{i-1} \equiv \{s \in L : f_1(s) \neq 0, \cdots, f_{i-1}(s) \neq 0\}$. Then there exist $n+1$ open sets $B_1, B_2, \cdots, B_n, G_n$ of L with the following properties:*

$$f_1 = 0 \quad on \quad B_1,$$

$$f_1 \neq 0 \quad and \quad f_2 = 0 \quad on \quad B_2,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$f_1 \neq 0,\ f_2 \neq 0, \cdots, f_{n-1} \neq 0, \quad and \quad f_n = 0 \quad on \quad B_n,$$

$$f_1 \neq 0,\ f_2 \neq 0, \cdots, \quad and \quad f_n \neq 0 \quad on \quad G_n;$$

and

$$\overline{L} = \overline{B}_1 \cup \overline{B}_2 \cup \cdots \cup \overline{B}_n \cup \overline{G}_n,$$

where the closure operation is taken in R. Thus the component intervals of $B_1, B_2, \cdots, B_n, G_n$, taken together, form a countable family of disjoint proper intervals each of which is open in L, and the union of these component intervals is a dense subset of L.

This lemma is easily seen to be a particular case of the following

LEMMA 1.2. *Let $G_1, \cdots, G_n$ be open sets of a topological space X such that $G_1 \supset G_2 \supset \cdots \supset G_n$. Let $f_1, \cdots, f_n$ be continuous real-valued functions defined on G_0 ($\equiv X$), $G_1, \cdots, G_{n-1}$, respectively, such that*

$$G_i = \{x \in G_{i-1} : f_i(x) \neq 0\} \qquad \text{for} \quad i = 1, 2, \cdots, n.$$

Furthermore, let

$$A_i = \{x \in G_{i-1} : f_i(x) = 0\} \qquad \text{for} \quad i = 1, 2, \cdots, n.$$

Then

$$X = \overline{A_1^0} \cup \overline{A_2^0} \cup \cdots \cup \overline{A_n^0} \cup \overline{G_n},$$

where A_i^0 denotes the interior of A_i.

We first prove lemma 1.2. If $n=1$, we have

$$G_1 = X \backslash A_1,$$

where the set on the right consists of those elements of X which do not belong to A_1. Therefore

$$X = A_1^0 \cup \overline{G}_1. \tag{1.1}$$

If $n > 1$, let us apply the above result for A_1, G_1 and X to A_{r+1}, G_{r+1} and G_r for any $r : 1 \leqq r < n$. Then we have

$$G_r = A_{r+1}^0 \cup \overline{G}_{r+1} \text{ (relative to } G_r).$$

Therefore,

$$G_r = A_{r+1}^0 \cup (\overline{G}_{r+1} \cap G_r) \text{ (relative to } X).$$

Hence we have

$$\overline{G}_r \subset \overline{A_{r+1}^0 \cup G_{r+1}} = \overline{A_{r+1}^0} \cup \overline{G}_{r+1} \tag{1.2}$$

which holds for any $r : 1 \leqq r < n$.

Induction using (1.1) and (1.2) then completes the proof of Lemma 1.2. Lemma 1.1 follows immediately from Lemma 1.2 and the properties of open subsets of a proper interval given in §1.2.

1. 4. Two C^∞ real functions. We know that the set of zeros of a continuous function defined on the real line R is a closed subset of R and that the zeros of an analytic function defined on R and not identically zero are (countably many) isolated points. Here we shall construct an example to show that

Given any closed subset F of R, there exists a C^∞, non-negative, function whose set of zeros is precisely the given subset F.

This result is already known if the function is required only to be continuous (see Kelley [6], p. 134, J.). To construct our function, we first note that the function

$$\psi(s) = \begin{cases} \exp\left(\dfrac{1}{s-b} - \dfrac{1}{s-a}\right) & \text{if } a < s < b \\ 0 & \text{otherwise} \end{cases}$$

is a C^∞ function the set of whose zeros is $R\backslash(a, b)$ and that the derivatives of ψ of all orders vanish at the points a, b.

Let F be an arbitrarily given closed subset of R. Then by § 1.2, $R\backslash F = \bigcup_{j=1}^{\infty} G_j$, where G_j $(j=1, 2, \cdots)$ are disjoint open intervals, some of which may be empty sets.

We consider first the case where each $G_j \equiv (a_j, b_j)$ is finite. For each $j \geqq 1$ let

$$\psi_j(s) = \begin{cases} \exp\left(\dfrac{1}{s-b_j} - \dfrac{1}{s-a_j}\right) & \text{if } s \in G_j \\ 0 & \text{otherwise.} \end{cases}$$

Then for each j there exists a positive number M_j having the property $|\psi_j^{(k)}(s)| \leqq M_j$ for all k such that $0 \leqq k \leqq j$ and for all $s \in R$.

We assert that the function $f(s)$ defined on R by

$$f(s) = \sum_{j=1}^{\infty} 2^{-j} f_j(s), \quad \text{where} \quad f_j(s) = \frac{1}{M_j} \psi_j(s),$$

has the desired properties. Obviously, the set of zeros of f is precisely the given closed set F. We now prove that f is of class C^∞. In fact, $|f_j^{(k)}(s)| \leqq 1$ for any k and j such that $0 \leqq k \leqq j$ and for all $s \in R$. Therefore, for each fixed $k \geqq 0$, the series $\sum_{j=1}^{\infty} 2^{-j} f_j^{(k)}(s)$ is a uniformly convergent series of continuous functions. Therefore (from analysis), for each $k \geqq 1$, the series $\sum_{j=1}^{\infty} 2^{-j} f_j^{(k)}(s)$ represents a continuous function, which in our case is precisely the derivative of the continuous function $\sum_{j=1}^{\infty} 2^{-j} f_j^{(k-1)}(s)$. It follows from this by induction that $f \in C^\infty$, with $f^{(k)}(s) = \sum_{j=1}^{\infty} 2^{-j} f_j^{(k)}(s)$. This completes the proof of our assertion in the case where each G_j is finite.

If $R \backslash F = (a_0, \infty) \cup \bigcup_{j=1}^{\infty} (a_j, b_j)$, where the intervals are disjoint and each (a_j, b_j) is finite, we need only put

$$f(s) = f_0(s) + \sum_{j=1}^{\infty} 2^{-j} f_j(s),$$

where f_j $(j \geqq 1)$ are as defined above and

$$f_0(s) = \begin{cases} \exp\left(-\dfrac{1}{s-a_0}\right) & \text{if } s > a_0 \\ 0 & \text{otherwise.} \end{cases}$$

The remaining cases

$$R \backslash F = (-\infty, b_0) \cup \bigcup_{j=1}^{\infty} (a_j, b_j)$$

or

$$R \backslash F = (-\infty, b_0) \cup (a_0, \infty) \cup \bigcup_{j=1}^{\infty} (a_j, b_j)$$

can be treated in a similar way. This completes the construction of our function.

We note the C^∞ function f constructed above has the property that *not only f but also its derivatives of all orders vanish at each point of the closed subset F.*

We now construct for later use another C^∞ function with certain properties.

Let L be a proper interval which is open on the right, and $\{s_j\}$, $j=1,2,\cdots$, a strictly monotonic increasing sequence of points in L with no limit point in L. We require a C^∞ function ϕ defined on L and having the following properties:

$$\phi(s)\begin{cases} =0 & \text{if } s\in\{L\cap(-\infty,s_1]\}\cup[s_4,s_5]\cup\cdots\cup[s_{4j},s_{4j+1}]\cup\cdots \\ >0 \text{ and } <1 & \text{if } s\in(s_1,s_2)\cup(s_3,s_4)\cup\cdots\cup(s_{2i-1},s_{2i})\cup\cdots \\ =1 & \text{if } s\in[s_2,s_3]\cup[s_6,s_7]\cup\cdots\cup[s_{4j-2},s_{4j-1}]\cup\cdots \end{cases} \tag{1.3}$$

For each $i\geqq 1$ let $\psi_i(t)$ be the C^∞ function defined on R by

$$\psi_i(t)=\begin{cases} \exp\left(\dfrac{1}{t-s_{2i}}-\dfrac{1}{t-s_{2i-1}}\right) & \text{if } t\in(s_{2i-1},s_{2i}) \\ 0 & \text{o herwise.} \end{cases}$$

Let

$$\phi_i(s)=\frac{\displaystyle\int_{s_{2i-1}}^{s}\psi_i(t)\,dt}{\displaystyle\int_{s_{2i-1}}^{s_{2i}}\psi_i(t)\,dt}, \quad s\in R. \tag{1.4}$$

Then ϕ_i is C^∞ on R, and

$$\phi_i(s)\begin{cases} =0 & \text{if } s\leqq s_{2i-1} \\ >0 \text{ and } <1 & \text{if } s\in(s_{2i-1},s_{2i}) \\ =1 & \text{if } s\geqq s_{2i}. \end{cases}$$

Now let $\{c_i\}$, $i=1,2,\cdots$, be any sequence of real numbers and

$$\tilde{\phi}(s)=\sum_{i=1}^{\infty}c_i[\phi_i(s)-\phi_{i+1}(s)], \quad s\in L, \tag{1.5}$$

where $\phi_i(s)$ are as defined by (1.4). Then $\tilde{\phi}$ is C^∞ on L, takes the constant value 0 on $L\cap(-\infty,s_1]$ and the constant value c_i on the interval $[s_{2i},s_{2i+1}]$ for $i=1,2,\cdots$, and is strictly monotonic on each of the intervals (s_{2i-1},s_{2i}). Thus taking

$$c_i=\begin{cases} 1 \text{ if } i \text{ is odd}, \\ 0 \text{ if } i \text{ is even}, \end{cases}$$

we obtain a C^∞ function $\phi(s)$ which has the required properties (1. 3).

1. 5. Some auxiliary results. The following are some properties of 3-dimensional vector functions which will be used in the sequel.

LEMMA 1.3. (a) *Let* $\mathbf{v}(t)$ *be a* C^1 *vector function defined on a proper interval I such that* $\mathbf{v}\neq 0$ (*nowhere zero*). *Then the condition*

$$\mathbf{v}\times\frac{d\mathbf{v}}{dt}=0 \text{ on } I$$

is equivalent to the existence of a constant unit vector **a** *and a positive scalar function* v *such that* $\mathbf{v}=v\mathbf{a}$.

(b) *Let* $\mathbf{v}(t)$ *be a* C^2 *vector function defined on I such that* $\mathbf{v}\times\dfrac{d\mathbf{v}}{dt}\neq 0$. *Then the condition*

$$\left|\mathbf{v},\frac{d\mathbf{v}}{dt},\frac{d^2\mathbf{v}}{dt^2}\right|=0$$

is equivalent to the existence of a constant unit vector **b** *such that*

$$\mathbf{b}\cdot\mathbf{v}=0,\qquad \mathbf{b}\cdot\frac{d\mathbf{v}}{dt}=0,\qquad \mathbf{b}\cdot\frac{d^2\mathbf{v}}{dt^2}=0.$$

The proof of this lemma which we shall omit is easy and follows the familiar line.

2.1. Regular curves and Frenet curves. We denote by E^3 a 3-dimensional Euclidean space, and by $\mathbf{x}$ the position vector in E^3. A parametrized curve $\mathbf{x}=\mathbf{x}(t)$, where t runs through a proper interval, is said to be *regular* if its tangent vector $d\mathbf{x}/dt$ is continuous and nowhere zero. (Thus, a regular curve can be parametrized by its arc length.) A parametrized curve $\mathbf{x}=\mathbf{x}(t)$ is said to be C^k (where k is a positive integer or ∞) if the vector function $\mathbf{x}(t)$ is C^k.

We shall hereafter denote by L a proper interval. Let

$$\Gamma:\mathbf{x}=\mathbf{x}(s),\quad s\in L, \tag{2. 1}$$

be a regular curve in E^3 parametrized by its arc length s. Then the tangent vector $\mathbf{x}'$ is a unit vector. (Here and in what follows, a dash denotes differentiation with respect to the arc length s.)

The regular curve Γ defined by (2. 1) has a topology induced from the usual topology of the interval L by the parametrization (2. 1). This is the topology on Γ which will be used throughout this paper.

We shall see that in the problems we study, the assumption that a regular curve is C^∞ imposes no more restrictions on the curve than the assumption that it is C^k for suitably large k. Therefore, for convenience, we shall consider only C^∞ regular curves.

Following Nomizu [7], we shall define a C^∞ *Frenet curve* as a C^∞ regular curve $\Gamma : \mathbf{x}(s)$, $s \in L$, for which there exist three C^∞ vector functions $\mathbf{e}_1(s)$, $\mathbf{e}_2(s)$, $\mathbf{e}_3(s)$ and two C^∞ scalar functions $k_1(s)$, $k_2(s)$ satisfying the following conditions:

(i) $\mathbf{e}_1(s) = \mathbf{x}'(s)$,

(ii) At each point $s \in L$, $\mathbf{e}_1(s)$, $\mathbf{e}_2(s)$, $\mathbf{e}_3(s)$ form a right-handed orthonormal frame of E^3,

(iii) The Frenet equations

$$
\begin{cases}
\mathbf{e}_1 = k_1\mathbf{e}_2 , \\
\mathbf{e}_2' = -k_1\mathbf{e}_1 + k_2\mathbf{e}_3 , \\
\mathbf{e}_3' = -k_2\mathbf{e}_2
\end{cases}
\tag{2. 2}
$$

hold on L.

For a Frenet curve Γ, the function k_1 is uniquely determined (except for a sign) by the first Frenet equation, and we call $k_1(s)$ the *curvature* of Γ at the point s. On any arc of Γ on which $k_1 \neq 0$ (nowhere zero), the unit vector $\mathbf{e}_2$ is uniquely determined except for sign, and consequently k_2 is uniquely determined. In this case, $k_2(s)$ is called the *torsion* of Γ at the point s. If $k_1 = 0$ (everywhere zero) on an arc of Γ, then $\mathbf{e}_2$ is not uniquely determined by (2. 2) and consequently nor is k_2. In general, we call k_2 a *pseudo-torsion* of Γ.

A C^∞ regular curve with $\mathbf{x}'' \neq 0$ is a Frenet curve for which $\mathbf{e}_1 = \mathbf{t}$, $\mathbf{e}_2 = \varepsilon\mathbf{n}$, $\mathbf{e}_3 = \varepsilon\mathbf{b}$, $k_1 = \varepsilon\kappa$ and $k_2 = \tau$, where $\varepsilon = \pm 1$, $\mathbf{t}$ the unit tangent vector, $\mathbf{n}$ the unit principal normal vector, $\mathbf{b}$ the unit binormal vector, κ (>0) the curvature and τ the torsion, as defined in standard textbooks. We call such a curve a *special Frenet curve.*

It should be noted that not all C^∞ regular curves are Frenet curves, as was shown by Nomizu [7] with an example. The problem of finding a necessary and sufficient condition for a C^∞ regular curve to be a Frenet curve has been studied by several authors, including Hartman and Wintner [4], Nomizu [7], and our colleague Yim-Ming Wong [11], but the problem in its general form has not been solved.

2.2. Indeterminateness of k_1 and k_2 of a Frenet curve. A lemma. It is well known that given any two C^∞ functions k_1, k_2 defined on L, there always exists a C^∞ Frenet curve, uniquely determined except for its position in E^3, with k_1 as curvature and k_2 as pseudo-torsion. On the other hand, two different pairs of C^∞ functions k_1, k_2 and $\widetilde{k}_1, \widetilde{k}_2$ may determine the same Frenet curve. We shall now find the condition for this to be the case. For this purpose, we need the following lemma, which, besides being interesting and important in its own right, will also be used in a decisive way in our later work (§ 2.3, § 3.2, and § 6.2).

LEMMA 2.1. *If f,g are C^∞ functions on a proper interval L such that*

$$f(s)^2 + g(s)^2 = 1 \quad \textit{for all} \quad s \in L, \tag{2.3}$$

then there exists a C^∞ function θ on L such that

$$\cos\theta(s) = f(s), \quad \sin\theta(s) = g(s). \tag{2.4}$$

Furthermore, if s_0 is any point on L and θ_0 is the unique constant satisfying the conditions

$$\cos\theta_0 = f(s_0), \quad \sin\theta_0 = g(s_0), \quad 0 \leqq \theta_0 < 2\pi,$$

then the function θ is given explicitly by

$$\theta(s) = \int_{s_0}^{s} (fg' - gf')\, ds + \theta_0. \tag{2.5}$$

PROOF. Let C be the unit circle in the plane with rectangular coordinates (x, y) and origin O. Then the point $P(s):(f(s), g(s))$ moves continuously on C as s moves continuously on L. Let A be the point on C such that $\angle AOP(s_0) = \theta_0$, where s_0 and θ_0 are as defined in the lemma.

If s_1 is any point on L, we can reach it by allowing s to increase or decrease continuously from s_0. When s, starting from s_0, moves continuously towards s_1 and finally reaches it, the point $P(s)$, starting from $P(s_0)$, moves continuously along C until it finally stops at the position $P(s_1)$. (The point $P(s)$ may go back and forth or around C one or more times before finally stopping at the position $P(s_1)$.) We put $\theta(s_1)$ equal to the sum of θ_0 and the 'algebraic' angle described by the radius vector $OP(s)$. This defines a single-valued continuous function θ on L such that $\theta(s_0) = \theta_0$. (For another proof of this fact, see Chern [1], Chapter 1, § 3.2.)

We assert that θ is the function satisfying the required conditions. Obviously, θ satisfies (2.4). We now prove that θ is given by (2.5) so that it is C^∞. Since θ is continuous, we have, on account of (2.4), that

$$f'(s) = -\sin\theta(s)\lim_{\Delta s\to 0}\frac{\theta(s+\Delta s)-\theta(s)}{\Delta s},$$

$$g'(s) = \cos\theta(s)\lim_{\Delta s\to 0}\frac{\theta(s+\Delta s)-\theta(s)}{\Delta s}.$$

Since $\sin\theta$ and $\cos\theta$ cannot be both zero at the same point, the above two equations show that θ' exists and is continuous.

Let us now put

$$\phi(s) \equiv \int_{s_0}^{s}(fg'-gf')\,ds+\theta_0 .$$

Then it follows from this and (2.4) that

$$\theta' = fg' - gf' = \phi' .$$

This, together with $\theta(s_0) = \theta_0 = \phi(s_0)$, proves that $\theta = \phi$.

2.3. Indeterminateness of k_1 and k_2 of a Frenet curve, continued.

THEOREM 2.1. *Two pairs of C^∞ functions k_1, k_2 and $\widetilde{k}_1, \widetilde{k}_2$ on L determine the same Frenet curve (up to a Euclidean motion) iff there exists a C^∞ function θ on L such that*

$$k_1\sin\theta = 0,\quad k_1\cos\theta = \widetilde{k}_1,\quad k_2+\theta' = \widetilde{k}_2 . \tag{2.6}$$

PROOF. *Necessity.* Assume that k_1, k_2 and $\widetilde{k}_1, \widetilde{k}_2$ determine the same Frenet curve $\Gamma: \mathbf{x}(s)$, $s\in L$. Then there exist along Γ two C^∞ families of Frenet frames $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ and $\{\widetilde{\mathbf{e}}_1, \widetilde{\mathbf{e}}_2, \widetilde{\mathbf{e}}_3\}$ such that

$$\begin{cases}\mathbf{x}' = \mathbf{e}_1 ,\\ \mathbf{e}_1' = \qquad + k_1\mathbf{e}_2 ,\\ \mathbf{e}_2' = -k_1\mathbf{e}_1 \qquad + k_2\mathbf{e}_3 ,\\ \mathbf{e}_3 = \qquad - k_2\mathbf{e}_2 ;\end{cases} \tag{2.7}$$

$$(2.\tilde{7})\qquad \begin{cases} \mathbf{x}' = \tilde{\mathbf{e}}_1\,, \\ \tilde{\mathbf{e}}_1' = \qquad + \tilde{k}_1\tilde{\mathbf{e}}_2\,, \\ \tilde{\mathbf{e}}_2' = -\tilde{k}_1\tilde{\mathbf{e}}_1 \qquad + \tilde{k}_2\tilde{\mathbf{e}}_3\,, \\ \tilde{\mathbf{e}}_3' = \qquad - \tilde{k}_2\tilde{\mathbf{e}}_2\,. \end{cases}$$

It follows from $(2.7)_1$ and $(2.\tilde{7})_1$ that

$$(2.8)\qquad \tilde{\mathbf{e}}_1 = \mathbf{e}_1\,.$$

On account of this and Lemma 2.1, there exists a C^∞ function θ on L such that

$$(2.9)\qquad \begin{cases} \tilde{\mathbf{e}}_2 = \mathbf{e}_2\cos\theta + \mathbf{e}_3\sin\theta\,, \\ \tilde{\mathbf{e}}_3 = -\mathbf{e}_2\sin\theta + \mathbf{e}_3\cos\theta\,, \end{cases}$$

because $f\equiv\tilde{\mathbf{e}}_2\cdot\mathbf{e}_2$ and $g\equiv\tilde{\mathbf{e}}_2\cdot\mathbf{e}_3$ are C^∞ functions satisfying condition (2.3) of Lemma 2.1.

Now differentiating $(2.9)_1$ and making use of (2.7), (2.8) and (2.9), we get

$$(2.10)\qquad \begin{aligned} \tilde{\mathbf{e}}_2' &= (-\mathbf{e}_2\sin\theta+\mathbf{e}_3\cos\theta)\,\theta' + (-k_1\mathbf{e}_1+k_2\mathbf{e}_3)\cos\theta - k_2\mathbf{e}_2\sin\theta \\ &= \tilde{\mathbf{e}}_3(\theta'+k_2) - \tilde{\mathbf{e}}_1 k_1\cos\theta\,. \end{aligned}$$

Comparison of this with $(2.\tilde{7})_3$ gives $(2.6)_2$ and $(2.6)_3$.

Similarly, differentiating $(2.9)_2$ and then making use of (2.7), (2.8) and (2.9), we get

$$(2.11)\qquad \begin{aligned} \tilde{\mathbf{e}}_3' &= (-\mathbf{e}_2\cos\theta-\mathbf{e}_3\sin\theta)\,\theta' - (-k_1\mathbf{e}_1+k_2\mathbf{e}_3)\sin\theta - k_2\mathbf{e}_2\cos\theta \\ &= -(\theta'+k_2)\,\tilde{\mathbf{e}}_2 + k_1\sin\theta\,\tilde{\mathbf{e}}_1\,. \end{aligned}$$

Comparison of this with $(2.\tilde{7})_4$ gives $(2.6)_1$ and $(2.6)_3$. Hence the necessity of conditions (2.6) is completely proved.

Sufficiency. Let $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be the family of Frenet frames of a Frenet curve Γ determined by k_1, k_2. Then (2.7) hold. Using this $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ and the C^∞ function θ on L whose existence is assumed, we define by (2.8) and (2.9) a C^∞ family of right-handed orthonormal frames $(\tilde{\mathbf{e}}_1, \tilde{\mathbf{e}}_2, \tilde{\mathbf{e}}_3\}$ on Γ. We now prove that $(2.\tilde{7})$ hold; when this is done, it follows from $(2.\tilde{7})_1$ that

$\Gamma: \mathbf{x}(s)$, $s \in L$, is also a Frenet curve determined by the functions $\tilde{k}_1$, $\tilde{k}_2$. To prove $(2.\tilde{7})$, we first differentiate (2.9) and then make use of (2.7), (2.8) and (2.9). The result is (2.10) and (2.11) which, because of (2.6), reduce to $(2.\tilde{7})_3$ and $(2.\tilde{7})_4$. Next, differentiating (2.8) and then making use of (2.9), we get

$$\tilde{\mathbf{e}}_1' = \mathbf{e}_1 = k_1\mathbf{e}_2 = k_1(\tilde{\mathbf{e}}_2 \cos\theta - \tilde{\mathbf{e}}_3 \sin\theta),$$

which, because of (2.6), reduces to $(2.\tilde{7})_2$. Lastly, $(2.\tilde{7})_1$ follows from $(2.7)_1$ and (2.8). Thus, the sufficiency of conditions (2.6) is completely proved.

THEOREM 2.2. *Let $\Gamma: \mathbf{x}(s)$, $s \in L$, be any C^∞ Frenet curve with curvature k_1 and pseudo-torsion k_2, and let $G = \{s \in L : k_1(s) \neq 0\}$. Then there exist C^∞ functions $\tilde{k}_1$ and $\tilde{k}_2$ not equal to $\pm k_1$ and k_2, respectively, and determining the same Frenet curve iff $\overline{G} \not\supset L$.*

PROOF. By Theorem 2.1, $\tilde{k}_1$ and $\tilde{k}_2$ determine the same Frenet curve as k_1 and k_2 iff there exists a C^∞ function θ on L such that

$$(2.6) \qquad k_1 \sin\theta = 0, \quad k_1 \cos\theta = \tilde{k}_1, \quad k_2 + \theta' = \tilde{k}_2 .$$

Condition $(2.6)_1$ requires that $\sin\theta = 0$ on G, and consequently, by continuity, also on $\overline{G} \cap L$. Therefore, $\theta(\overline{G} \cap L) \subset \{0, \pm\pi, \pm 2\pi, \cdots\}$. If $\overline{G} \supset L$, then $\theta(L) = \theta(\overline{G} \cap L)$, so that $\cos\theta = \pm 1$ and $\theta' = 0$ on L. This proves the necessity of the condition.

Assume now that $\overline{G} \not\supset L$. Then the set $L \backslash \overline{G}$ is non-empty, and consequently, contains some open interval, say (a, b). Let ϕ be a C^∞ function on the real line R, constructed as in §1.4, which vanishes exactly on $R\backslash(a, b)$. Then it is easy to see that the restriction of ϕ to L is a function θ satisfying the conditions $k_1 \sin\theta = 0$ and $\theta' \not\equiv 0$ on L. This completes the proof of Theorem 2.2.

2.4. Some remarks and consequences. It is obvious from (2.2) that, for a Frenet curve, $k_1 = 0$ is a necessary and sufficient condition for it to be a line-segment, and $k_2 = 0$ is a sufficient but not a necessary condition (in view of Theorem 2.2) for it to be a plane curve (unless $\overline{G} \supset L$).

Nomizu [7] defined a normal curve as a regular curve with the property that, for every $s_0 \in L$, there exists an integer $m = m(s_0)$ such that $\mathbf{e}_1^{(m)}(s_0) \neq 0$. He proved that a normal curve of class C^∞ is a Frenet curve (but of course not all Frenet curves are normal curves). He also proved that, for a normal curve, k_1 is uniquely determined except for a sign and k_2 is uniquely determined. This result is a consequence of our Theorem 2.2 because, for a normal curve, the set $L\backslash G=\{s\in L: k_1(s)=0\}$ consists of isolated points so that the condition $\overline{G}\supset L$ is satisfied.

Obviously, if $\Gamma: \mathbf{x}(s)$, $s\in L$, is a regular curve, then any scalar function of the form $F(\mathbf{x}', \mathbf{x}'', \cdots)$ is an intrinsic quantity of the curve. If Γ is a Frenet curve, such a scalar function can be expressed in terms of the curvature k_1 and the pseudo-torsion k_2 and their derivatives, resulting in an expression which should be invariant under the transformation (2. 6). Conversely, it is probably true (but we have not been able to prove it) that a function of k_1, k_2 and their derivatives which is invariant under the transformation (2. 6) can be expressed as a scalar function of $\mathbf{x}', \mathbf{x}'', \cdots$.

We now prove

THEOREM 2.3. *If* $F(u_0, u_1, \cdots, u_p; v_0, v_1, \cdots, v_q)$ *is any* C^∞ *scalar function on a suitable open subset of the Euclidean* $(p+q+2)$*-space such that* $F(0, 0, \cdots, 0; v_0, v_1, \cdots, v_q)=0$ *and* $F(-u_0, -u_1, \cdots, -u_p; v_0, v_1, \cdots, v_q) = F(u_0, u_1, \cdots, u_p; v_0, v_1, \cdots, v_q)$, *then for any Frenet curve,* $F(k_1, k_1', \cdots, k_1^{(p)}; k_2, k_2', \cdots, k_2^{(q)})$ *is an intrinsic quantity of the curve.*

PROOF. Let $G=\{s\in L: k_1(s)\neq 0\}$. Then since $\sin\theta=0$ on G by $(2. 6)_1$, we have $\theta(G)\subset\{0, \pm\pi, \pm 2\pi, \cdots\}$. Therefore, $\cos\theta=\varepsilon=\pm 1$ and $\theta'=0$, and consequently also $k_1^{(r)}=\varepsilon\tilde{k}_1^{(r)}$ and $k_2^{(r)}=\tilde{k}_2^{(r)}$ $(r\geqq 0)$, all hold on G, and by continuity also on $\overline{G}\cap L$. Hence

$$(2. 12)\qquad F(\tilde{k}_1, \tilde{k}_1', \cdots, \tilde{k}_1^{(p)}; \tilde{k}_2, \tilde{k}_2', \cdots, \tilde{k}_2^{(q)}) = F(k_1, k_1', \cdots, k_1^{(p)}; k_2, k_2', \cdots, k_2^{(q)})$$

holds on $\overline{G}\cap L$. On the other hand, on $L\backslash\overline{G}$, $k_1^{(r)}=0$ $(r\geqq 0)$ and consequently, $\tilde{k}_1^{(r)}=0$ $(r\geqq 0)$ by $(2. 6)_2$. Therefore (2. 12) also holds on $L\backslash\overline{G}$.

2. 5. Total pseudo-torsion of a closed Frenet curve. For closed regular curves, W. Fenchel's theorem on the total curvature is well-known [2]. For regular curves Γ whose torsion τ is everywhere defined, there is the following interesting theorem of W. Scherrer's on total torsion [9]:

If Γ *is any regular closed curve on a sphere, then its total torsion* $\int_\Gamma \tau ds=0$. *Conversely, if on a surface* S, *the total torsion of every closed curve on it is zero, then* S *is a sphere or a portion of it.*

More recently, G. Saban [8] showed that Scherrer's theorem remains true if the total torsion is replaced by the integral $\int_\Gamma \kappa^n \tau \, ds$, where κ is the curvature, τ the torsion, and n any positive or negative integer.

Of course, if the torsion of a regular curve is not everywhere defined, we cannot speak of the total torsion, nor of any integral involving the torsion. However, in the case of Frenet curves, although the pseudo-torsion is not an intrinsic quantity of the curve, we have the following

THEOREM 2.4. (a) *For any closed Frenet curve* Γ, *the total pseudo-torsion* $\int_\Gamma k_2 \, ds \pmod{2\pi}$ *is an intrinsic quantity of the curve.*

(b) *More generally, if* Γ *is any Frenet curve, and the curvature of* Γ *is not zero at the points* $\mathbf{x}(s_1)$, $\mathbf{x}(s_2)$, *where* $s_1 < s_2$, *then the integral* $\int_{s_1}^{s_2} k_2 \, ds$ $\pmod{\pi}$ *has an intrinsic meaning.*

PROOF. If Γ is a closed Frenet curve, it follows from $(2.6)_3$ that

$$\int_\Gamma (\tilde{k}_2 - k_2)\, ds = \int_\Gamma \theta' \, ds = \theta \Big|_\Gamma \equiv 0 \pmod{2\pi}.$$

Therefore,

$$\int_\Gamma \tilde{k}_2 \, ds \equiv \int_\Gamma k_2 \, ds \pmod{2\pi},$$

which proves (a).

To prove (b), we need only observe that, on account of $(2.6)_1$, $\theta(s_1)$ and $\theta(s_2)$ are integral multiples of π (one or both of which may be zero).

3.1. Plane curves and plane arcs.

DEFINITION. A C^∞ regular curve is called a *plane curve* if it lies on a plane. A plane curve is called a *plane arc* if it contains no line-segments.

It is easy to see that a C^∞ regular curve $\mathbf{x}(s)$, $s \in L$, is a line-segment iff $\mathbf{x}'' = \mathbf{0}$. For a Frenet curve, this condition can be replaced by $k_1 = 0$.

A correct statement of the conditions for a regular curve to be a plane curve usually given in textbooks is as follows:

LEMMA 3.1. *For a regular curve* $\Gamma : \mathbf{x}(s)$, $s \in L$, *with* $\mathbf{x}'' \neq \mathbf{0}$ (*then* Γ *is* *a special Frenet curve with* $\kappa > 0$), *the following conditions are equivalent*:

(i) Γ *is a plane arc*,

(ii) $\tau = 0$,

(iii) $|\mathbf{x}', \mathbf{x}'', \mathbf{x}'''| = 0$.

The proof of this follows familiar lines and will not be given here. Now, omitting the condition $\mathbf{x}'' \neq \mathbf{0}$, we prove

THEOREM 3.1. *A* C^∞ *regular curve* $\Gamma : \mathbf{x}(s)$, $s \in L$, *satisfies condition*

$$(3.1) \qquad |\mathbf{x}', \mathbf{x}'', \mathbf{x}'''| = 0$$

iff it has a dense subset which is the union of a countable number of line-segments and plane arcs.

PROOF. *Sufficiency.* This follows immediately from the fact that any line-segment or plane arc satisfies (3.1).

Necessity. We assume that condition (3.1) holds. Consider the continuous function $f = |\mathbf{x} \times \mathbf{x}''|$ $(= |\mathbf{x}''|)$ on L. Let I be any proper interval of L, and Γ_1 the arc of Γ corresponding to $\bar{I} \cap L$.

If $f = 0$ on I, then by Lemma 1.3 (a), there exists a constant unit vector $\mathbf{a}$ and a scalar C^∞ function v of s such that $\mathbf{x}' = v\mathbf{a}$. Therefore, $\mathbf{x}(s) = u(s)\mathbf{a} + \mathbf{c}$, where $u(s)$ is some scalar C^∞ function of s and $\mathbf{c}$ is some constant vector. Hence Γ_1 is a line-segment. Conversely, it is obvious that if Γ_1 is a line-segment, then $f = 0$ on I.

If $f \neq 0$ on I, then since (3.1) holds on I, there exists, by Lemma 1.3 (b), a constant unit vector $\mathbf{b}$ such that $\mathbf{b} \cdot \mathbf{x}' = 0$. Therefore, $\mathbf{b} \cdot \mathbf{x} =$ constant, and consequently, Γ_1 is a plane curve. But since $f \neq 0$ on I, Γ_1 cannot contain any line-segments. Hence Γ_1 is a plane arc.

Now by Lemma 1.1 there exist two open sets B, G of L with the properties that

$$f = 0 \quad \text{on} \quad B, \quad f \neq 0 \quad \text{on} \quad G$$

and

$$(3.2) \qquad \bar{L} = \bar{B} \cup \bar{G}.$$

We have shown above that the arcs of Γ corresponding to the intervals of B are line-segments, and those corresponding to the intervals of G are plane arcs.

Because of (3. 2), the union of these line-segments and plane arcs is a dense subset of the curve Γ. This completes the proof of our theorem.

Since a plane curve $\mathbf{x}(s)$, $s \in L$, must satisfy condition (3. 1), we have

COROLLARY 3.1. *A plane curve has a dense subset which is the union of a countable number of line-segments and plane arcs.*

3. 2. An example. We now give an example to show that regular curves of the type described in Theorem 3.1 actually exist and are not necessarily plane curves.

EXAMPLE 3.1. Let L be any proper interval and $[a, b]$ a closed interval lying in the interior of L. Let $\{s_j\}$, $j=0, 1, 2, \cdots$, be any strictly monotonic increasing sequence of points on L such that $a=s_0$ and b is the limit point of $\{s_j\}$. Let

$$I_0 = L \cap (-\infty, a], \qquad I_\infty = L \cap [b, \infty),$$

$$A = \bigcup_{j=0}^{\infty} (s_{2j}, s_{2j+1}), \qquad B = \bigcup_{j=0}^{\infty} [s_{2j}, s_{2j+1}].$$

Then $L \backslash A$ and $\overline{B}$ $(=B \cup \{b\})$ are closed subsets of L, and there exist C^∞ functions $k_1(s)$ and $k_2(s)$, $s \in L$, constructed as in §1.4, which vanish precisely on $L \backslash A$ and $\overline{B}$ respectively. The Frenet curve Γ determined by k_1 and k_2 satisfies (3. 1) and consists of a countably infinite number of line-segments (corresponding to the intervals $I_0, I_\infty, [s_1, s_2], \cdots, [s_{2j-1}, s_{2j}], \cdots$ *of* $L \backslash A$) and a countably infinite number of plane arcs (corresponding to the intervals $[s_0, s_1]$, $\cdots, [s_{2j}, s_{2j+1}], \cdots$ *of* $\overline{B}$).

We now show that if the function $k_2(s)$ is suitably chosen, we can ensure that no two plane arcs of Γ lie on parallel planes.

We first construct a suitable function $k_2(s)$. For each $j \geqq 1$, let $\psi_j(s)$ be a non-negative C^∞ function on R vanishing precisely on $R \backslash (s_{2j-1}, s_{2j})$ such that (see §1.4)

$$|\psi_j^{(k)}(s)| \leqq 1 \quad \text{for} \quad 0 \leqq k \leqq j, \ s \in R.$$

Let $\{c_j\}$, $j=1, 2, \cdots$, be a monotonic decreasing sequence of positive numbers such that, if we denote $\int_{s_{2j-1}}^{s_{2j}} c_j \psi_j(t)\, dt$ by δ_j, then

$$0 < \delta_{j+1} \leqq \delta_j < 1 \quad \text{for each } j = 1, 2, \cdots \tag{3. 3}$$

Also let $\psi_0(s)$ be any C^∞ function on R vanishing precisely on $[a, b]$. Now define

$$k_2(s) = \psi_0(s) + \sum_{j=1}^{\infty} 2^{-j} c_j \psi_j(s), \quad s \in L. \tag{3.4}$$

Using an argument similar to that used in § 1.4, we easily prove that $k_2(s)$ is a C^∞ function on L vanishing precisely on $\bar{B}$.

Let $\Gamma : \mathbf{x}(s)$, $s \in L$, be the Frenet curve determined by this function $k_2(s)$ and any C^∞ function $k_1(s)$ on L which vanishes precisely on $L \backslash A$. We now show that no two plane arcs of Γ lie on parallel planes. First fix any $j \geqq 1$ and consider the interval $I_j = [s_{2j-1}, s_{2j}]$. On I_j, we have $k_1 = 0$. Therefore the arc Γ_j of Γ corresponding to I_j is a line-segment on which $\mathbf{e}_1 =$ constant. Let $\{\mathbf{e}_1, \mathbf{a}_2, \mathbf{a}_3\}$ be a constant right-handed orthonormal frame defined on Γ_j. Then by Lemma 2.1, there exists a C^∞ function $\theta(s)$ on I_j such that

$$\begin{cases} \mathbf{e}_2(s) = \cos\theta(s)\,\mathbf{a}_2 + \sin\theta(s)\,\mathbf{a}_3\,, \\ \mathbf{e}_3(s) = -\sin\theta(s)\,\mathbf{a}_2 + \cos\theta(s)\,\mathbf{a}_3 \end{cases}$$

hold on I_j. Since $\mathbf{e}_3' = -k_2(s)\,\mathbf{e}_2(s)$, it follows from these that $\theta'(s) = k_2(s)$. Therefore, using (3.4), we have that

$$\theta(s_{2j}) - \theta(s_{2j-1}) = \int_{s_{2j-1}}^{s_{2j}} k_2(s)\, ds = 2^{-j}\delta_j\,;$$

in other words, the angle between the vectors $\mathbf{e}_3(s_{2j-1})$ and $\mathbf{e}_3(s_{2j})$ is

$$\Delta\theta_j = 2^{-j}\delta_j\,. \tag{3.5}$$

We note that on the other hand, $\mathbf{e}_3(s)$ is constant on each plane arc of Γ.

Let us now take any two plane arcs of Γ, say those corresponding to the intervals $[s_{2p}, s_{2p+1}]$ and $[s_{2q}, s_{2q+1},]$ where $0 \leqq p < q$. From (3.3) and (3.5) it follows that

$$\sum_{j=p+1}^{q} \Delta\theta_j < \delta_{p+1} \sum_{j=p+1}^{\infty} 2^{-j} = 2^{-p}\delta_{p+1} < \frac{\pi}{2}\,. \tag{3.6}$$

If $\Delta\theta_{p,q}$ denotes the angle between the vectors $\mathbf{e}_3(s_{2p+1})$ and $\mathbf{e}_3(s_{2q})$, we have, by a well-known result in spherical trigonometry, that

$$\Delta\theta_{p+1} - \sum_{j=p+2}^{q} \Delta\theta_j \leqq \Delta\theta_{p,q} \leqq \sum_{j=p+1}^{q} \Delta\theta_j\,.$$

Therefore, by (3. 5) and (3. 6)

$$2^{-(p+1)}\,\delta_{p+1} - 2^{-(p+1)}\,\delta_{p+2} < \Delta\theta_{p,q} < 2^{-p}\,\delta_{p+1} < \frac{\pi}{2}\,.$$

Using (3. 3), we have

$$0 < \Delta\theta_{p,q} < \frac{\pi}{2}\,,$$

which proves that $\mathbf{e}_3(s_{2p+1})$ and $\mathbf{e}_3(s_{2q})$ cannot be parallel. But these vectors are normal vectors to the planes on which the plane arcs of Γ corresponding to $[s_{2p}, s_{2p+1}]$ and $[s_{2q}, s_{2q+1}]$ lie. Hence these plane arcs do not lie on parallel planes.

REMARK. In this example we have shown that a C^∞ regular curve defined on a compact set such as $[a, b]$ and satisfying condition (3. 1) can actually contain an infinite number of plane arcs all lying on different planes. If we merely want to illustrate this fact for a noncompact set such as $[a, b)$, the construction is much simpler, for then we need not use a uniformly convergent series to define the function $k_2(s)$.

4. 1. Helices and helical arcs.

DEFINITION. A C^∞ regular curve is called a *helix* if there exists a fixed direction, called the directrix, with which the tangents make a constant angle. A helix which is not a plane curve and contains no line-segments is called a *helical arc.*

Let Γ be a helix and α the constant angle mentioned in the definition. Then it is easy to see that Γ is a line-segment if $\alpha=0$, and Γ is a plane curve if $\alpha=\pi/2$.

A correct statement and proof of the conditions for a curve to be a helix usually given in books on elementary differential geometry is as follows:

LEMMA 4.1. *For a* C^∞ *regular curve* $\Gamma: \mathbf{x}(s)$, $s \in L$, *with* $\mathbf{x}'' \neq 0$ (*then* Γ *is a special Frenet curve with* $\kappa > 0$), *the following conditions are equivalent*:

(i) Γ *is a plane arc or a helical arc,*
(ii) τ/κ *is constant,*
(iii) $|\mathbf{x}'', \mathbf{x}''', \mathbf{x}^{(\mathrm{iv})}| = 0\,.$

PROOF. Assume that (i) is true. There exists a constant unit vector $\mathbf{p}$

making a constant angle α $(0<\alpha\leqq\pi/2)$ with the unit tangent vectors of Γ, i.e.

$$\mathbf{t}\cdot\mathbf{p}=\cos\alpha\,. \tag{4.1}$$

Differentiation of this gives $\kappa\mathbf{n}\cdot\mathbf{p}=0$, so that

$$\mathbf{n}\cdot\mathbf{p}=0\,. \tag{4.2}$$

It follows from (4.1) and (4.2) that

$$\mathbf{p}=\cos\alpha\mathbf{t}\pm\sin\alpha\mathbf{b}\,.$$

Differentiation of this leads to

$$\frac{\tau}{\kappa}=\pm\cot\alpha=\text{finite constant}\,.$$

This proves that (i) implies (ii).

To prove that (ii) implies (i), let α be the unique constant such that $0<\alpha\leqq\frac{\pi}{2}$ and $\frac{\tau}{\kappa}=\pm\cot\alpha$. Then the vector

$$\mathbf{p}\equiv\cos\alpha\mathbf{t}\pm\sin\alpha\mathbf{b}$$

is a constant vector, and $\mathbf{t}\cdot\mathbf{p}=\cos\alpha$. Therefore Γ is a plane arc or a helical arc (according as $\alpha=\frac{\pi}{2}$ or $\neq\frac{\pi}{2}$).

To show that (ii) is equivalent to (iii), we need only observe that for a special Frenet curve, we have

$$|\mathbf{x}\ ,\mathbf{x}\ \ ,\mathbf{x}^{(\mathrm{iV})}|=\kappa^5\frac{d}{ds}\left(\frac{\tau}{\kappa}\right).$$

We now prove

THEOREM 4.1. *A helix either is a plane curve or has a dense subset which is the union of a countable number of helical arcs and line-segments.*

PROOF. Let $\Gamma:\mathbf{x}(s)$, $s\in L$ be a helix, and $\mathbf{p}$ a constant unit vector along its directrix which makes a constant angle α $(0\leqq\alpha\leqq\pi/2)$, with the unit tangent vectors of Γ.

We have observed that if $\alpha=0$, then Γ is a line-segment; if $\alpha=\pi/2$, then Γ is a plane curve.

We now consider the case that $0<\alpha<\pi/2$. By Lemma 1.1, L has a dense subset which is the union of two countable families $\mathfrak{B}$, $\mathfrak{G}$ of disjoint proper intervals such that $|\mathbf{x}''|=0$ on each interval of $\mathfrak{B}$ and $|\mathbf{x}''|\neq 0$ on each interval of $\mathfrak{G}$. The arc of Γ corresponding to an interval of $\mathfrak{B}$ is a line-segment. Now let Γ_1 be the arc of Γ corresponding to an interval I of $\mathfrak{G}$. Then $|\mathbf{x}''|\neq 0$ on I, and we see from the proof of Lemma 4.1 that $\tau/\kappa=\pm\cot\alpha$ on Γ_1. But $0<\alpha<\pi/2$. Therefore $\tau\neq 0$. Hence Γ_1 is a helical arc.

The proof of the theorem can now be completed by using Lemma 1.1 as in the proof of Theorem 3.1.

Concerning condition (iii) in Theorem 4.1, we shall now prove

THEOREM 4.2. *A C^∞ regular curve $\Gamma:\mathbf{x}(s)$, $s\in L$, satisfies*

$$|\mathbf{x}'', \mathbf{x}''', \mathbf{x}^{(iv)}| = 0 \tag{4.3}$$

iff it has a dense subset which is the union of a countable number of helical arcs, plane arcs and line-segments.

PROOF. *Sufficiency.* This follows from the fact that any helical arc, plane arc or line-segment satisfies condition (4.3).

Necessity. We assume that condition (4.3) is satisfied. Consider the continuous functions $f_1=|\mathbf{x}''|$ and $f_2=|\mathbf{x}''\times\mathbf{x}'''|$ on L. Let I be any proper interval of L, and Γ_1 the arc of Γ corresponding to $\bar{I}\cap L$.

If $f_1=0$ on I, then the arc Γ_1 is a line-segment.

If $f_1\neq 0$ and $f_2=0$ on I, then by Lemma 1.3 (a), there exists a constant unit vector $\mathbf{a}$ and a (C^∞) scalar function $v(s)$ such that $\mathbf{x}''=v\mathbf{a}$. Therefore

$$\mathbf{x}(s) = u(s)\,\mathbf{a} + s\mathbf{b} + \mathbf{c}\,,$$

where $u(s)$ is some (C^∞) scalar functions of s, and $\mathbf{b}$, $\mathbf{c}$ are some constant vectors. Hence the arc Γ_1, which cannot contain any line-segments because $\mathbf{x}''\neq 0$, is a plane arc.

If $f_2\neq 0$ (and consequently also $f_1\neq 0$) on I, then since condition (4.3) is satisfied, we know by Lemma 1.3 (b) that there exists a constant unit vector $\mathbf{b}$ such that $\mathbf{b}\cdot\mathbf{x}''=0$. Therefore, $\mathbf{b}\cdot\mathbf{e}_1=\mathbf{b}\cdot\mathbf{x}'=$ constant, and consequently Γ_1 is a helix. Moreover, because $|\mathbf{x}''\times\mathbf{x}'''|\neq 0$ on I, Γ_1 does not lie on a plane, nor does it contain any line-segments. Hence Γ_1 is a helical arc.

The proof of the theorem can now be completed by Lemma 1.1.

REMARK. If the curve Γ in Theorem 4.2 is assumed to be a Frenet curve, the proof is simpler. In this case, we have

$$|\mathbf{x}'', \mathbf{x}''', \mathbf{x}^{(\mathrm{iv})}| = k_1^3(k_1 k_2' - k_1' k_2).$$

Since on any interval $I \subset L$ on which $k_1 \neq 0$, the condition $k_1 k_2' - k_1' k_2 = 0$ is equivalent to $k_2/k_1 =$ constant, we can apply Lemma 1.1, with $n = 2$, $f_1 = k_1$ and $f_2 = k_2$, to obtain the desired results.

4.2. An example. We now give an example to show that the curves described in Theorem 4.2 actually exist.

EXAMPLE 4.1. Let L and $\{s_j\}$, I_0, I_∞ be as defined in Example 3.1. For each $j \geqq 0$, let ψ_j be a C^∞ function on R vanishing precisely on $R \backslash (s_j, s_{j+1})$ and satisfying the condition

$$|\psi_j^{(k)}(s)| \leqq 1 \quad \text{if} \quad 0 \leqq k \leqq j.$$

Let $k_1(s)$ be defined on L by

$$k_1(s) = \sum_{j=0}^{\infty} 2^{-j}[\psi_{3j}(s) + \psi_{3j+1}(s)], \quad s \in L.$$

Then $k_1(s)$ is C^∞ on L and vanishes precisely on the intervals I_0, I_∞, $[s_2, s_3]$, $[s_5, s_6], \cdots$ and at the points $s_1, s_4, \cdots$.

Now, let $\{c_j\}, j = 0, 1, 2, \cdots$, be an arbitrary bounded sequence of constants. Then the function

$$k_2(s) = \sum_{j=0}^{\infty} 2^{-j} c_j \psi_{3j+1}(s), \quad s \in L,$$

is C^∞ on L and vanishes precisely on the complement of the union of the intervals (s_1, s_2), $(s_4, s_5), \cdots$. Moreover, on each interval (s_{3j+1}, s_{3j+2}) $(j \geqq 0)$, we have $k_2(s) = c_j k_1(s)$.

The Frenet curve determined by k_1, k_2 then satisfies the condition $|\mathbf{x}'', \mathbf{x}''', \mathbf{x}^{(\mathrm{iv})}| = 0$, and consists of a countably infinite number of helical arcs (corresponding to the intervals $[s_1, s_2], \cdots, [s_{3j+1}, s_{3j+2}], \cdots$), a countably infinite number of plane arcs (corresponding to the intervals $[s_0, s_1], \cdots$, $[s_{3j}, s_{3j+1}], \cdots$) and a countably infinite number of line-segments (corresponding to the intervals I_0, I_∞, $[s_2, s_3], \cdots, [s_{3j-1}, s_{3j}], \cdots$).

4.3. The condition $k_2 : k_1$=constant. From the proof of Lemma 4.1, we have the following

THEOREM 4.3 (a). *Let $\Gamma : \mathbf{x}(s)$, $s \in L$, be a helix which is not a plane curve (in particular not a line-segment). Then on each component interval of the open set $G=\{s \in L : \mathbf{x}''(s) \neq 0\}$ of L, the relation $\tau = \varepsilon\lambda\kappa$ holds, where λ $(=\cot\alpha)$ is a non-zero constant, and $\varepsilon = \pm 1$ whose value may be different for different component intervals of G.*

The following converse to this theorem can be easily proved.

THEOREM 4.3 (b). *If a C^∞ regular curve Γ: $\mathbf{x}(s)$, $s \in L$, satisfies the condition that on each component interval of the open subset $G=\{s \in L : \mathbf{x}''(s) \neq 0\}$ of L, the relation $\tau = \lambda\kappa$ holds, where λ is a non-zero constant which may be different for different component intervals of G, then Γ has a dense subset which is the union of a countable number of helical arcs and line-segments.*

The curve in Theorem 4.3 (b), however, is generally not a helix even when the constant λ is the same for all the component intervals of G. We illustrate this by

EXAMPLE 4.2. Take $L=(-2, 2)$, and let $k_1(s)$ and $f(s) \geqq 0$ be C^∞ functions on L vanishing precisely on $[-1, 1]$ and $L \setminus (-1, 1)$, respectively. Let $M=\int_{-1}^{1} f(s)\,ds$ and

$$k_2(s) = \begin{cases} k_1(s) & \text{if } s \in L\setminus(-1,1) \\ \dfrac{\pi}{2M} f(s) & \text{if } s \in (-1,1). \end{cases}$$

Then k_2 is C^∞ on L, and the Frenet curve Γ determined by k_1 and k_2 satisfies the condition that $k_2 = k_1$ on each interval component of $G=\{s \in L : \mathbf{x}''(s) \neq 0\}$.

But Γ is not a helix. In fact, consideration of the rotation of the vector $\mathbf{e}_2(s)$ in the interval $[-1, 1]$ (see Example 3.1) establishes that $\mathbf{e}_2(1) = \mathbf{e}_3(-1)$. Now let us assume that $\mathbf{e}_1(s)$, $s \in L$, makes a constant angle α with a constant unit vector $\mathbf{p}$. Then, since $\cot\alpha = \lambda = 1$ on the section $s \in (-2, -1)$, we have $\mathbf{e}_3(-1) \cdot \mathbf{p} = \sin\frac{\pi}{4}$, by the formulas given in the proof of Lemma 4.1. Therefore $\mathbf{e}_2(1) \cdot \mathbf{p} = \sin\frac{\pi}{4} \neq 0$. On the other hand, we must have $\mathbf{e}_2(1) \cdot \mathbf{p} = 0$ at the end point $s=1$ of the section $s \in (1, 2)$. Therefore $\mathbf{p}$ does not exist, and Γ is not a helix.

In the example just given, we have $k_1=0$ and $k_2 \neq 0$ on $(-1, 1)$, so that although the condition $k_2=\lambda k_1$ is satisfied on $G=L\backslash(-1, 1)$ where $k_1 \neq 0$, it is not satisfied on $(-1, 1)$ where $k_1=0$. We shall consider in § 6.1 the case of a Frenet curve for which there exists a Frenet frame such that $k_2 = \lambda k_1$ (λ constant) holds all along the curve.

5. 1. Spherical curves and spherical arcs.

DEFINITION. A *circular arc* is an arc of a circle. A *spherical curve* is a C^∞ regular curve lying on a sphere. A *spherical arc* is a spherical curve which contains no circular arcs.

The condition usually given for a curve to be a spherical curve is that

$$(5.1) \qquad [\tau^{-1}(\kappa^{-1})']' + \tau\kappa^{-1} = 0 .$$

Here it is implicitly assumed that $\kappa \neq 0$, $\tau \neq 0$, so that the spherical curve satisfying the condition (5. 1) is actually a spherical arc.

It is known [Wong, 10] that the following is a necessary and sufficient condition for a C^4 regular curve $\mathbf{x}(s)$, $s \in L$, to be a spherical curve:

(i) k_1 is nowhere zero (so that the torsion k_2 of the curve is uniquely defined);

(ii) there exists a C^1-function f on L such that

$$f k_2 = (k_1^{-1})' , \quad f' + k_2 k_1^{-1} = 0 .$$

We note in this result that $k_1 \neq 0$ is a natural part of the condition and that we do not have to assume that $k_2 \neq 0$.

We now prove

THEOREM 5.1. *A spherical curve has a dense subset which is the union of a countable number of circular arcs and spherical arcs.*

PROOF. Let $\Gamma : \mathbf{x}(s)$, $s \in L$, be a spherical curve. By Lemma 1.1, the interval L has a dense subset which is the union of two countable families $\mathfrak{B}$ and $\mathfrak{G}$ of disjoint intervals open in L such that $k_2=0$ on each interval of $\mathfrak{B}$ and $k_2 \neq 0$ on each interval of $\mathfrak{G}$. Thus the arc of Γ corresponding to any component interval of $\mathfrak{B}$ is a circular arc, and that corresponding to any component interval of $\mathfrak{G}$ is a spherical arc. Hence our theorem is proved.

If $k_1 \neq 0$ and $k_2 \neq 0$, condition (5. 1) is equivalent to

$$(5.2) \qquad (-k_1 k_1'' + 2k_1'^2)\, k_2 + k_1 k_1' k_2' + k_1^2 k_2^3 = 0\,.$$

By Theorem 2.3, the expression on the left of equation (5. 2) is an intrinsic quantity of a Frenet curve. In fact, it can easily be verified that condition (5. 2) is equivalent to

$$(5.3) \qquad |\mathbf{x}', \mathbf{x}'', \mathbf{x}^{(iv)}| - |\mathbf{x}''|^2 |\mathbf{x}', \mathbf{x}'', \mathbf{x}'''| = 0\,,$$

or

$$(5.3') \qquad |\mathbf{x}', \mathbf{x}''', \mathbf{x}^{(iv)}| + (\mathbf{x}' \cdot \mathbf{x}''') |\mathbf{x}', \mathbf{x}'', \mathbf{x}'''| = 0\,.$$

These considerations enable us to formulate the usual condition for a C^∞ regular curve to be a spherical arc in the following new form:

A C^∞ regular curve $\Gamma: \mathbf{x}(s)$, $s \in L$, with $|\mathbf{x}', \mathbf{x}'', \mathbf{x}'''| \neq 0$, is a spherical arc iff $|\mathbf{x}', \mathbf{x}''', \mathbf{x}^{(iv)}| + (\mathbf{x}' \cdot \mathbf{x}''') |\mathbf{x}', \mathbf{x}'', \mathbf{x}'''| = 0$.

We now prove

THEOREM 5.2. *A C^∞ regular curve $\Gamma: \mathbf{x}(s)$, $s \in L$, satisfies condition* (5. 3) *or* (5. 3′) *iff it has a dense subset which is the union of a countable number of spherical arcs, plane arcs and line-segments.*

PROOF. *Sufficiency.* For a plane arc or line-segment, the vectors $\mathbf{x}', \mathbf{x}'$, $\mathbf{x}''', \mathbf{x}^{(iv)}$ all lie on the same plane, and so (5. 3) is satisfied.

For a spherical arc Γ_1 corresponding to an interval $I \subset L$, we have noted above that $k_1 \neq 0$ on I, so that Γ_1 is a Frenet curve. The set $J = \{s \in I : k_2(s) \neq 0\}$ is open in I and also dense in I; otherwise, Γ_1 will contain some circular arc. Since (5. 1) holds on each component interval of J, it follows that (5. 3) hold on J and hence on I $(= \bar{J} \cap I)$.

Necessity. If $\mathbf{x}'' \neq 0$ on an interval $I \subset L$, then the Frenet equations hold. If moreover, $k_2 \neq 0$ on I, then (5. 3) is equivalent to (5. 1); in this case, the arc of Γ corresponding to I is a spherical arc. The rest then follows on applying Lemma 1.1 with $n=2$ and $f_1 = |\mathbf{x}''|$, $f_2 = k_2$.

COROLLARY 5.1. *A C^∞ Frenet curve satisfies condition* (5. 2) *iff it has a dense subset which is the union of a countable number of spherical arcs, plane arcs and line-segments.*

5. 2. An example. We now give an example to show that curves of the type described in Theorem 5.2 actually exist.

EXAMPLE 5.1. We first note that, if on an interval I, we have $k_1 \neq 0$, $k_1' \neq 0$ and $k_2 \neq 0$, then on this interval I, condition (5.2) is equivalent to (5.1) and to

$$\rho^2 + \rho'^2/k_2^2 = R^2,$$

where $\rho = k_1^{-1}$, and R is a constant; that is, to

$$k_2 = \pm\rho'(R^2 - \rho^2)^{-1/2}. \tag{5.4}$$

Now let L be any bounded proper interval open on the right, and $\{s_j\}$, $j = 1, 2, \cdots$, a strictly monotonic increasing sequence of points on L with no limit point in L.

Let f be a non-negative C^∞ function defined on R, constructed as in § 1.4, which vanishes precisely at the points $s_1, s_2, \cdots$ and on $(-\infty, \alpha_0]$ and $[\beta_0, +\infty)$, where α_0, β_0 are two points on R such that $\bar{L} \subset (\alpha_0, \beta_0)$. Then since this f is positive elsewhere, the function h defined on L by

$$h(s) = \left[\int_{\alpha_0}^{s} f(t)\,dt\right]^{-1}$$

is C^∞ and always positive.

Let ϕ be a C^∞ function on L such that (see § 1.4)

$$\phi(s) \begin{cases} = 0 & \text{if } s \in \{L \cap (-\infty, s_1]\} \cup [s_4, s_5] \cup \cdots \cup [s_{4j}, s_{4j+1}] \cup \cdots \\ > 0 \text{ and } < 1 & \text{if } s \in (s_1, s_2) \cup (s_3, s_4) \cup \cdots \cup (s_{2i-1}, s_{2i}) \cup \cdots \\ = 1 & \text{if } s \in [s_2, s_3] \cup [s_6, s_7] \cup \cdots \cup [s_{4j-2}, s_{4j-1}] \cup \cdots \end{cases}$$

Then the function k_1 defined on L by $k_1(s) = \phi(s)\,h(s)$ is C^∞, has the same set of zeros as ϕ, and coincides with h on

$$[s_2, s_3] \cup [s_6, s_7] \cup \cdots \cup [s_{4j-2}, s_{4j-1}] \cup \cdots$$

Let us now take any sequence of constants R_j, $j = 1, 2, \cdots$, each greater than $\int_{\alpha_0}^{\beta_0} f(t)\,dt$, and define the function k_2 on L by

$$k_2(s) = \begin{cases} \rho'(R_j^2 - \rho^2)^{-1/2} & \text{if } s \in (s_{4j-2}, s_{4j-1}),\ j = 1, 2, \cdots \\ 0 & \text{elsewhere,} \end{cases}$$

where $\rho=k_1^{-1}$. This function k_2 on L is C^∞. For, on

$$(s_2, s_3)\cup(s_6, s_7)\cup \cdots \cup(s_{4j-2}, s_{4j-1})\cup \cdots,$$

we have

$$\rho = k_1^{-1} = h^{-1} = \int_{\alpha_0}^{s} f(t)\,dt,$$

so that $\rho' = f$. Therefore, $k_2 > 0$ in each of the intervals (s_{4j-2}, s_{4j-1}); moreover, at the points $s_2, s_3, s_6, s_7, \cdots$, the function ρ' $(=f)$ and consequently, also the function k_2 and their derivatives of all orders, are zero.

It follows from the above discussions that the Frenet curve determined by the C^∞ functions k_1 and k_2 just defined satisfies condition (5.2) and consists of a countably infinite number of

line-segments, corresponding to $\{L\cap(-\infty, s_1]\}$, $[s_4, s_5], \cdots$,
plane arcs, corresponding to $[s_1,s_2]$, $[s_3, s_4], \cdots$,
and spherical arcs, corresponding to $[s_2, s_3]$, $[s_6, s_7], \cdots$.

We note that the spheres on which the spherical arcs lie are of radii R_j $(j=1, 2, \cdots)$, which can be chosen to be all different.

6.1. (C^∞ regular) plane curves, helices, and spherical curves are Frenet curves. We have noticed (in § 5.1) that a C^∞ regular curve lying on a sphere is necessarily a special Frenet curve. We now prove

THEOREM 6.1. *A C^∞ regular curve which is a helix (and in particular, a plane curve) is necessarily a Frenet curve.*

In fact, we can prove

THEOREM 6.2. *A C^∞ regular curve is a helix whose tangents make a constant angle α $(0<\alpha\leqq\pi/2)$ with some fixed direction iff it has a C^∞ Frenet frame with respect to which $k_2 = k_1 \cot\alpha$ holds.*

PROOF. We first prove Theorem 6.1. Let $\Gamma : \mathbf{x}(s)$, $s\in L$, be a C^∞ regular curve which is a helix whose tangent vectors make a constant angle α $(0<\alpha\leqq\pi/2)$ with a constant unit vector $\mathbf{p}$. Then

$$\mathbf{e}_1\cdot\mathbf{p} = \cos\alpha, \tag{6.1}$$

where $\mathbf{e}_1 = \mathbf{x}'$. Define

$$\mathbf{e}_3 = \frac{1}{\sin\alpha}(\mathbf{p} - \cos\alpha\,\mathbf{e}_1), \quad \mathbf{e}_2 = \mathbf{e}_3 \times \mathbf{e}_1 . \tag{6.2}$$

It follows easily from (6. 1) and (6. 2) that $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is a C^∞ family of right-handed orthonormal frames along Γ.

It remains to verify the Frenet equations. Using (6. 1) and (6. 2) we have

$$\mathbf{e}_1' \cdot \mathbf{e}_3 = \frac{1}{\sin\alpha}\mathbf{e}_1' \cdot (\mathbf{p} - \cos\alpha\,\mathbf{e}_1) = 0 .$$

Therefore, $\mathbf{e}_1' = k_1\mathbf{e}_2$ for some C^∞ scalar function k_1. The verification of the other two Frenet equations can now be completed by the usual argumeat.

We have thus proved that Γ is a Frenet curve. Let us now prove the remaining parts of Theorem 6.2. For our particular choice of $\mathbf{e}_3$ given in (6. 2), we have

$$-\mathbf{e}_3' \cdot \mathbf{e}_2 = \cot\alpha\,\mathbf{e}_1' \cdot \mathbf{e}_2 , \quad \text{i.e.} \quad k_2 = k_1 \cot\alpha .$$

Conversely, let Γ be a C^∞ Frenet curve for which there exists a C^∞ Frenet frame such that $k_2 = k_1 \cot\alpha$ holds for some constant α $(0 < \alpha \leqq \pi/2)$. Define the unit vector

$$\mathbf{p} = \cos\alpha\,\mathbf{e}_1 + \sin\alpha\,\mathbf{e}_3 .$$

Then $\mathbf{p} \cdot \mathbf{e}_1 = \cos\alpha$ and

$$\mathbf{p}' = k_1 \cos\alpha\,\mathbf{e}_2 - k_2 \sin\alpha\,\mathbf{e}_2 = 0 ,$$

so that $\mathbf{p}$ is a constant vector. Hence Γ is a helix.

6. 2. Necessary and sufficient conditions for a Frenet curve to be a plane curve, helix or spherical curve. In §3.1-§5.2, we have dealt with the conditions usually given for a regular curve to be a plane curve, a helix or a spherical curve, and we have shown by examples that these conditions are necessary but not sufficient conditions. The results in Theorem 6.1 and Lemma 2.1 enable us to obtain necessary and sufficient conditions for a C^∞ regular curve to be a plane curve, a helix or a spherical curve. From the discussion in §6.1, it is seen that there is no loss of generality in stating our results for Frenet curves.

THEOREM 6.3. *A C^∞ Frenet curve $\Gamma : \mathbf{x}(s), s \in L$, is a helix which makes an angle α, $(0 < \alpha \leqq \pi/2)$ with its directrix iff there exists a C^∞ function*

$\phi(s)$ on L such that

$$\begin{cases} k_1 \sin\phi = 0\,, \\ k_1 \cos\phi \cot\alpha = k_2 + \phi'\,. \end{cases} \tag{6.3}$$

PROOF. *Necessity.* If the Frenet curve Γ makes a constant angle α, $0<\alpha\leqq\pi/2$, with a constant unit vector $\mathbf{p}$, then

$$\mathbf{e}_1\cdot\mathbf{p} = \cos\alpha\,.$$

Since $\sin\alpha\neq0$, it follows from Lemma 2.1 that there exists a C^∞ function $\phi(s)$ on L such that

$$\mathbf{p} = \cos\alpha\,\mathbf{e}_1 + \sin\alpha(-\sin\phi\,\mathbf{e}_2 + \cos\phi\,\mathbf{e}_3)\,. \tag{6.4}$$

Differentiation of this gives

$$\sin\alpha(k_1\sin\phi)\,\mathbf{e}_1 + [\cos\alpha\,k_1 - \sin\alpha(k_2+\phi')\cos\phi]\,\mathbf{e}_2 - \sin\alpha(k_2+\phi')\sin\phi\,\mathbf{e}_3 = 0\,.$$

From this it follows that

$$\begin{cases} k_1\sin\phi = 0\,, \\ k_1\cot\alpha = (k_2+\phi')\cos\phi\,, \\ \quad 0 = (k_2+\phi')\sin\phi\,, \end{cases}$$

which can easily be seen to be equivalent to (6.3).

Sufficiency. Let Γ be a Frenet curve satisfying condition (6.3) for some constant α, $0<\alpha\leqq\pi/2$, and some C^∞ function $\phi(s)$. Then the vector $\mathbf{p}$ defined by (6.4) is a constant unit vector which makes constant angle α with the tangent vectors of Γ.

REMARK. It is easy to see that this theorem is also a direct consequence of Theorems 2.1 and 6.2.

Putting $\alpha=\pi/2$, we have

COROLLARY 6.1. *A C^∞ Frenet curve $\Gamma:\mathbf{x}(s)$, $s\in L$, is a plane curve iff there exists a C^∞ function $\phi(s)$ on L such that*

$$(6.3') \qquad \begin{cases} k_1 \sin\phi = 0\,, \\ k_2 + \phi' = 0\,. \end{cases}$$

THEOREM 6.4. *A C^∞ Frenet curve $\Gamma : \mathbf{x}(s)$, $s \in L$, lies on a sphere of radius R iff there exists a C^∞ function $\phi(s)$ on L such that*

$$(6.5) \qquad \begin{cases} k_1 \sin\phi = \dfrac{1}{R}\,, \\ k_2 + \phi' = 0\,. \end{cases}$$

PROOF. *Necessity.* Let Γ lie on a sphere with centre at $\mathbf{c}$ and radius equal to R. Then $|\mathbf{x}-\mathbf{c}|^2 = R^2$ and $(\mathbf{x}-\mathbf{c})\cdot\mathbf{e}_1 = 0$. Therefore, by Lemma 2.1, there exists a C^∞ function $\phi(s)$ on L such that

$$(6.6) \qquad \mathbf{x} - \mathbf{c} = -R\sin\phi\,\mathbf{e}_2 + R\cos\phi\,\mathbf{e}_3\,.$$

Differentiation of this gives

$$\mathbf{e}_1 = k_1 R \sin\phi\,\mathbf{e}_1 - R(k_2+\phi')\cos\phi\,\mathbf{e}_2 - R(k_2+\phi')\sin\phi\,\mathbf{e}_3\,,$$

from which (6.5) follows.

Sufficiency. Let Γ be a Frenet curve satisfying condition (6.5) for some non-zero constant R and some C^∞ function $\phi(s)$ on L. Then the vector $\mathbf{c}$ defined by (6.6) is constant. Therefore, Γ lies on the sphere with centre $\mathbf{c}$ and radius R.

6.3. Some remarks.

REMARK 1. To construct a C^∞ regular plane curve, helix or spherical curve which contains a countably infinite number of arcs of different kinds, we need only construct first a suitable C^∞ function ϕ similar to the function $\widetilde{\phi}$ in § 1.4, then two C^∞ functions k_1 and k_2 which together with ϕ satisfy conditions (6.3'), (6.3) or (6.5) respectively, and finally, the Frenet curve determind by k_1 and k_2.

REMARK 2. If $k_1 \neq 0$ on a curve Γ, conditions (6.3) and (6.3') reduce to the usual conditions $k_2 = \pm k_1 \cot\alpha$ and $k_2 = 0$, respectively. On the other hand, conditions (6.5) are equivalent to

$$fk_2 = (k_1^{-1})' \quad \text{and} \quad f + k_2 k_1^{-1} = 0\,,$$

where

$$f = -R\cos\phi, \quad R^2 = f^2 + (k_1^{-1})^2 .$$

Thus we get back the results quoted at the beginning of §5.1 from Wong [10].

We also notice that condition (6.5) becomes (6.3′) when $R \to \infty$, as can be expected.

ADDED IN PROOF. Lemma 2.1 can be found in B. O'Neill, Elementary Differential Geometry, Academic Press, 1966, p. 50.

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