Tôhoku Math. Journ.
Vol. 19, No. 2, 1967

ON BOREL-TYPE METHODS, II

DAVID BORWEIN AND BRUCE L. R. SHAWYER

(Received February 6, 1967)

1. Introduction. In this paper, we define, and investigate the properties of the strong Borel-type methods $[B', \alpha, \beta]_p$, $[B, \alpha, \beta]_p$, which, when the index $p=1$, reduce to the methods $[B', \alpha, \beta]$, $[B, \alpha, \beta]$ considered in [1]. We use * to designate generalization of theorems, lemmas and definitions of [1]: e.g. Theorem 3* is a generalization of Theorem 3 of [1].

Suppose that σ, a_n $(n = 0, 1, \cdots)$ are arbitrary complex numbers, that $\alpha > 0$, that β is real and that N is a positive integer greater than $-\beta/\alpha$. Whenever $q > 1$, q' denotes the number conjugate to q, so that

$$\frac{1}{q} + \frac{1}{q'} = 1 .$$

Let x be a real variable in the range $[0, \infty)$: in all limits and order relations involving x, it is to be understood that $x \to \infty$.

Let

$$s_n = \sum_{\nu=0}^{n} a_\nu , \quad s_{-1} = 0 , \quad \sigma_N = \sigma - s_{N-1} ,$$

and define Borel-type sums

$$a_{\alpha,\beta}(x) = \sum_{n=N}^{\infty} \frac{a_n x^{\alpha n+\beta-1}}{\Gamma(\alpha n+\beta)} ; \qquad s_{\alpha,\beta}(x) = \sum_{n=N}^{\infty} \frac{s_n x^{\alpha n+\beta-1}}{\Gamma(\alpha n+\beta)} .$$

It is known that the convergence of either series for all $x \geqq 0$ implies the convergence, for all $x \geqq 0$, of the other.

Borel-type means are defined by

$$A_{\alpha,\beta}(x) = \int_0^x e^{-t} a_{\alpha,\beta}(t)\, dt ; \qquad S_{\alpha,\beta}(x) = \alpha e^{-x} s_{\alpha,\beta}(x) .$$

Borel-type methods are defined as follows:

1. Summability:

(i) If $A_{\alpha,\beta}(x) \to \sigma_N$, we say that $s_n \to \sigma(B', \alpha, \beta)$,
(ii) If $S_{\alpha,\beta}(x) \to \sigma$, we say that $s_n \to \sigma(B, \alpha, \beta)$.

3*. Strong summability with index p:

(i) If

$$\int_0^x e^t \,|A_{\alpha,\beta-1}(t)-\sigma_N|^p \,dt = o(e^x),$$

we say that $s_n \to \sigma[B', \alpha, \beta]_p$.

(ii) If

$$\int_0^x e^t \,|S_{\alpha,\beta-1}(t)-\sigma|^p \,dt = o(e^x),$$

we say that $s_n \to \sigma[B, \alpha, \beta]_p$.

We assume henceforth that the series defining $a_{\alpha,\beta}(x)$, $s_{\alpha,\beta}(x)$ are convergent for all $x \geqq 0$, and, since the actual choice of N in the definitions is clearly immaterial, that $\alpha N+\beta \geqq 2$. The functions $a_{\alpha,\beta}(x)$, $a_{\alpha,\beta-1}(x)$, $s_{\alpha,\beta}(x)$ and $s_{\alpha,\beta-1}(x)$ are then all continuous for $x \geqq 0$. Further, we assume, without loss of generality, that $a_0 = a_1 = \cdots = a_{N-1} = 0$, so that $\sigma_N = \sigma$.

Given a function $f(x)$, we write for $\delta > 0$

$$f_\delta(x) = \{\Gamma(\delta)\}^{-1} \int_0^x (x-t)^{\delta-1} f(t)\,dt$$

whenever the integral exists in the Lebesgue sense.

2. Preliminary Results.

LEMMA A. *Suppose that $f(t)$ is a non-negative function, integrable L in every finite interval $(0, x)$, that $\alpha > 0$ and that $\alpha+\beta > 0$. Then*

$$\int_0^x f(t)\,dt = o(e^{\alpha x})$$

if and only if

$$\int_0^x e^{\beta t} f(t)\,dt = o(e^{(\alpha+\beta)x}).$$

This can readily be proved by integration by parts.

LEMMA B. *If $f(t)$ is non-negative and integrable L^p in every finite interval $(0, x)$, where $p>1$, then, for $0<\delta<1/p$ and $q=p/(1-\delta p)$, $f_\delta(t)$ is integrable L^q in every finite interval $(0, x)$ and*

$$\left(\int_0^x \{f_\delta(t)\}^q\,dt\right)^{1/q} \leqq K\left(\int_0^x \{f(t)\}^p\,dt\right)^{1/p}$$

where K is a constant independent of x.

For a proof, see [2], page 290, Theorem 393.

LEMMA 5*. *If $f(t)$ is integrable L^p in every finite interval $(0, x)$, where $p \geqq 1$, and*

$$\int_0^x |f(t)|^p\,dt = o(e^{px}),$$

then, for $q \geqq p$ and $\delta > \dfrac{1}{p}-\dfrac{1}{q}$, $f_\delta(x)$ is integrable L^q in every finite interval $(0, x)$, and

$$\int_0^x |f_\delta(t)|^q\,dt = o(e^{qx})\,.$$

PROOF. Let $0<\mu<1$, $\dfrac{1}{\lambda}=1-\dfrac{1}{p}+\dfrac{1}{q}$, so that $(\delta-1)\lambda>-1$. Using Hölder's inequality twice, we obtain that

$$|f_\delta(x)|^p \leqq \left(\int_0^x |f(u)|(x-u)^{\delta-1}\,du\right)^p$$

$$\leqq e^{p\mu x}\int_0^x |f(u)|^p\,(x-u)^{(\delta-1)\lambda p/q}\,e^{-p\mu u}\,du\left(\int_0^x (x-u)^{(\delta-1)\lambda}\,e^{-p\mu(x-u)/(p-1)}\,du\right)^{p-1}\ {}^{1)}$$

$$\leqq Ke^{p\mu x}\left(\int_0^x |f(u)|^p\,(x-u)^{(\delta-1)\lambda}\,e^{-p\mu u}\,du\right)^{p/q}\left(\int_0^x |f(u)|^p\,e^{-p\mu u}\,du\right)^{1-p/q}$$

1) The second integral does not appear when $p=1$.

where, since $(\delta-1)\lambda > -1$, K is finite and independent of x; whence in view of Lemma A with $\alpha = q$, $\beta = -\mu q$,

$$|f_\delta(x)|^q = o\left(e^{q\mu x} e^{(q-p)(1-\mu)x} \int_0^x |f(u)|^p (x-u)^{(\delta-1)\lambda} e^{-p\mu u}\, du\right).$$

Thus

$$\int_0^x |f_\delta(t)|^q e^{-(p-q)t}\, dt = o\left(\int_0^x |f(u)|^p e^{-p\mu u}\, du \int_u^x (t-u)^{(\delta-1)\lambda} e^{p\mu t}\, dt\right)$$

$$= o\left(\int_0^x |f(u)|^p\, du\right) = o(e^{px})$$

since $(\delta-1)\lambda > -1$, and so, in view of Lemma A with $\alpha = p$, $\beta = q-p$,

$$\int_0^x |f_\delta(t)|^q\, dt = o(e^{qx}).$$

This completes the proof of Lemma 5*.

3. Theorems. This section is divided into two parts. The first contains theorems concerning relations between methods of the same type: that is between "B" methods or between "B'" methods. The second contains theorems giving interrelations between "B" and "B'" methods.

3.1. To each "B" theorem stated in this section, there corresponds an exactly analogous "B'" theorem which can be proved by replacing "B" by "B'", "σ" by "σ_N" and "$S_{\alpha,\beta}(x)$" by "$A_{\alpha,\beta}(x)$" respectively in the appropriate proof outlined below.

THEOREM 3*. *If* $s_n \to \sigma[B, \alpha, \beta]_q$ *then* $s_n \to \sigma(B, \alpha, \beta-\delta)$ *where* $q > 1$ *and* $\delta < (q-1)/q$.

PROOF. Assume without loss of generality, that $\sigma = 0$. Let $0 < \theta < 1$. Using Lemma A with $\alpha = q$, $\beta = -\theta q$, we obtain that

$$
\begin{aligned}
|\Gamma(1-\delta)s_{\alpha,\beta-\delta}(x)| &= \left|\int_0^x (x-t)^{-\delta}\, s_{\alpha,\beta-1}(t)\, dt\right| \\
&\leqq \left\{\int_0^x e^{-\theta q t}\, |s_{\alpha,\beta-1}(t)|^q\, dt\right\}^{1/q} \left\{\int_0^x e^{\theta q' t}(x-t)^{-\delta q'}\, dt\right\}^{1/q'} \\
&= o\left(e^{(1-\theta)x}\left\{\int_0^x e^{\theta q'(x-u)}\, u^{-\delta q'}\, du\right\}^{1/q'}\right) \\
&= o(e^x)
\end{aligned}
$$

(since $\delta q' < 1$, $\int_0^\infty e^{-\theta q' u}\, u^{-\delta q'}\, du < \infty$), and so it follows that $s_n \to 0\,(B, \alpha, \beta-\delta)$. This completes the proof of Theorem 3*.

THEOREM 5*. *If* $s_n \to \sigma(B, \alpha, \beta)$ *then* $s_n \to \sigma[B, \alpha, \beta+1]_q$ *where* $q > 0$.

This follows immediately from the definitions.

THEOREM 9*. *If* $s_n \to \sigma[B, \alpha, \beta]_p$ *then* $s_n \to \sigma[B, \alpha, \beta+\delta]_q$ *provided*

i) $p > q > 0, \quad \delta = 0,$

or ii) $q \geqq p \geqq 1, \quad \delta > \frac{1}{p} - \frac{1}{q},$

or iii) $q > p > 1, \quad \delta = \frac{1}{p} - \frac{1}{q}.$

PROOF. Using Hölder's inequality, we obtain, for $p > q > 0$, that

$$
\begin{aligned}
\int_0^x e^t\, |S_{\alpha,\beta-1}(t) - \sigma|^q\, dt &\leqq \left\{\int_0^x e^t\, |S_{\alpha,\beta-1}(t)-\sigma|^p\, dt\right\}^{q/p} \left\{\int_0^x e^t\, dt\right\}^{1-q/p} \\
&= o(e^x),
\end{aligned}
$$

from which case (i) follows.

Case (ii) can readily be proved by means of Lemmas A and 5*, and case (iii) by means of Lemmas A and B. The final theorem in this section exhibits an exact relation between the strong and ordinary methods; it can be proved in a similar way to Theorem 11 of [1] by using Minkowski's inequality instead of the triangle inequality.

THEOREM 11*. *For* $q > 1$, $s_n \to \sigma[B, \alpha, \beta]_q$ *if and only if* $s_n \to \sigma(B, \alpha, \beta)$

and

$$\int_0^x e^t |S'_{\alpha,\beta}(t)|^q dt = o(e^x).$$

3. 2.

THEOREM 15*. *For* $q > 1$, $s_n \to \sigma[B, \alpha, \beta]_q$ *if and only if* $s_n \to \sigma[B', \alpha, \beta]_q$ *and* $a_n \to 0[B, \alpha, \beta]_q$.

THEOREM 18*. *For* $q > 1$, $s_n \to \sigma[B', \alpha, \beta]_q$ *if and only if* $s_n \to \sigma[B, \alpha, \beta+1]_q$.

Proofs of these theorems can be constructed from the proofs of Theorems 15 and 18 of [1], by using

i) Theorems 3*, 11* instead of Theorems 3, 11 of [1],
ii) Lemma A to give equivalent statements about means and sums,

e.g.
$$\int_0^x e^t |S'_{\alpha,\beta}(t)|^q dt = o(e^x)$$

if and only if

$$\int_0^x |s_{\alpha,\beta}(t) - s_{\alpha,\beta-1}(t)|^q dt = o(e^{qx}),$$

iii) Lemma 5* with $p = q$ instead of Lemma 5 of [1],
iv) Minkowski's inequality instead of the triangle inequality,
v) (applicable only to the proof of Theorem 18*), Theorem 15* instead of Theorem 15 of [1].

REFERENCES

[1] D. BORWEIN AND B. L. R. SHAWYER, On Borel-type Methods, Tôhoku Math. Journ., 18 (1966), 283–298.
[2] G. H. HARDY, J. E. LITTLEWOOD AND G. PÓLYA, Inequalities, Cambridge University Press, (1934).

DEPARTMENT OF MATHEMATICS,
THE UNIVERSITY OF WESTERN ONTARIO,
LONDON, ONTARIO, CANADA.