Tôhoku Math. Journ.
Vol. 19, No. 2, 1967

ON A DECOMPOSITION OF C-HARMONIC FORMS IN A COMPACT SASAKIAN SPACE

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(Received January 16, 1967)

0. Introduction. Let M be a compact regular Sasakian space, $\pi: M \to B$ the fibering of M. Recently S. Tanno [10] discussed relations between the Betti numbers of M and B by making use of the exact sequence of Gysin. On the other hand it is well known that any harmonic p-form ($p \leqq m+1$) in a compact Kählerian space M^{2m} is written in terms of effective harmonic forms and the fundamental 2-form of M^{2m}. The work by Tanno suggests that an analogous theorem is expected in a compact Sasakian space.

In this paper, first we fix our notations in §1 and introduce a notion of a C-harmonic form in a compact Sasakian space in §4. The decomposition theorem for C-harmonic form will be given in the last section. We shall give only outline of proofs by the following two reasons: (1) the discussions in §2 and §5 are similar to that of an almost Hermitian space and a Kählerian space, (2) the results in §4 are based on straightforward computations though they are rather complicated and it is expected to have a reformulation by Y. Ogawa in a forthcoming paper [4].

1. Preliminaries.[1] Consider an n dimensional Riemannian space M^n and let $\{x^\lambda\}$, $\lambda = 1, \cdots, n$, be its local coordinates. Denoting the positive definite Riemannian metric by $g_{\lambda\mu}$, the Riemannian curvature tensor and the Ricci tensor are given by

$$R_{\lambda\mu\nu}{}^{\omega} = \partial_\lambda \left\{ {\omega \atop \mu\nu} \right\} - \partial_\mu \left\{ {\omega \atop \lambda\nu} \right\} + \left\{ {\omega \atop \lambda\alpha} \right\} \left\{ {\alpha \atop \mu\nu} \right\} - \left\{ {\omega \atop \mu\alpha} \right\} \left\{ {\alpha \atop \lambda\nu} \right\},$$

$$R_{\mu\nu} = R_{\varepsilon\mu\nu}{}^{\varepsilon},$$

where $\left\{ {\nu \atop \lambda\mu} \right\}$ means the Christoffel symbol and $\partial_\lambda = \partial/\partial x^\lambda$.

Components of a skew-symmetric tensor $u_{\lambda_1 \cdots \lambda_p}$ are considered as coefficients of a differential form:

1) As to notations we follow S. Tachibana [8].

$$u = \frac{1}{p!} u_{\lambda_1 \cdots \lambda_p} dx^{\lambda_1} \wedge \cdots \wedge dx^{\lambda_p},$$

so we shall represent this fact by

$$u: (u)_{\lambda_1 \cdots \lambda_p} = u_{\lambda_1 \cdots \lambda_p}.$$

The exterior differential du and codifferential δu are given by the following formulas:

$$(du)_{\mu\lambda_1 \cdots \lambda_p} = \nabla_\mu u_{\lambda_1 \cdots \lambda_p} - \sum \nabla_{\lambda_i} u_{\lambda_1 \cdots \lambda_{i-1} \mu \lambda_{i+1} \cdots \lambda_p},$$ [2]

or

$$(du)_{\lambda_1 \cdots \lambda_{p+1}} = \sum (-1)^{i+1} \nabla_{\lambda_i} u_{\lambda_1 \cdots \hat{\lambda}_i \cdots \lambda_{p+1}}, \quad p \geqq 1,$$

$$(du)_\lambda = \nabla_\lambda u, \quad p = 0,$$

where $\widehat{\lambda_i}$ means that λ_i is omitted,

$$(\delta u)_{\lambda_2 \cdots \lambda_p} = -\nabla^\alpha u_{\alpha\lambda_2 \cdots \lambda_p}, \quad p \geqq 1,$$ [3]

$$\delta u = 0, \quad p = 0.$$

About the Laplacian operator $\triangle = d\delta + \delta d$, we know the following formulas:

$$(\Delta u)_{\lambda_1 \cdots \lambda_p} = -\nabla^\alpha \nabla_\alpha u_{\lambda_1 \cdots \lambda_p} + \sum R_{\lambda_i}{}^\sigma u_{\lambda_1 \cdots \lambda_{i-1} \sigma \cdots \lambda_p} + \sum_{j<i} R_{\lambda_j \lambda_i}{}^{\rho\sigma} u_{\lambda_1 \cdots \lambda_{j-1} \rho \cdots \lambda_{i-1} \sigma \cdots \lambda_p},$$
$$p \geqq 2,$$

$$(\Delta u)_\lambda = -\nabla^\alpha \nabla_\alpha u_\lambda + R_\lambda{}^\alpha u_\alpha, \quad p = 1,$$

$$\Delta u = -\nabla^\alpha \nabla_\alpha u, \quad p = 0.$$

A p-form u is called to be harmonic, if it satisfies $du = 0$ and $\delta u = 0$. Thus $\Delta u = 0$ holds good for a harmonic form u.

The inner product of p-forms u and v is given by

$$<u, v> = \frac{1}{p!} u_{\lambda_1 \cdots \lambda_p} v^{\lambda_1 \cdots \lambda_p},$$

where $v^{\lambda_1 \cdots \lambda_p}$ are contravariant components of v.

2) ∇ means the operator of covariant derivation.
3) We remark that δu has the opposite sign of that in [8].

Especially the norm $|u|$ of u is given by

$$|u|^2 = <u, u>, \qquad |u| \geqq 0.$$

Let $\eta = \eta_\lambda dx^\lambda$ be a 1-form and we identify η with the vector field $\eta^\lambda = g^{\lambda\alpha}\eta_\alpha$. The operator $i(\eta)$ is defined by

$$(i(\eta)u)_{\lambda_2\cdots\lambda_p} = \eta^\alpha u_{\alpha\lambda_2\cdots\lambda_p}, \qquad p \geqq 1,$$
$$i(\eta)u = 0, \qquad p = 0$$

Let $\varphi = (1/2)\varphi_{\lambda\mu}dx^\lambda \wedge dx^\mu$ be a 2-form and we define an operator $i(\varphi)$ by

$$(i(\varphi)u)_{\lambda_3\cdots\lambda_p} = (1/2)\varphi^{\alpha\beta}u_{\alpha\beta\lambda_3\cdots\lambda_p}, \qquad p \geqq 2,$$
$$i(\varphi)u = 0, \qquad p = 0, 1.$$

The exterior product of η or φ and a p-form u are given explicitly by the following formulas:

$$(\eta\wedge u)_{\alpha\lambda_1\cdots\lambda_p} = \eta_\alpha u_{\lambda_1\cdots\lambda_p} - \sum \eta_{\lambda_j} u_{\lambda_1\cdots\lambda_{j-1}\alpha\cdots\lambda_p},$$

or

$$(\eta\wedge u)_{\lambda_1\cdots\lambda_{p+1}} = \sum(-)^{i+1}\eta_{\lambda_i}u_{\lambda_1\cdots\hat{\lambda}_i\cdots\lambda_{p+1}}, \qquad p \geqq 1$$
$$(\eta\wedge u)_\lambda = u\eta_\lambda, \qquad p = 0,$$
$$(\varphi\wedge u)_{\alpha\beta\lambda_1\cdots\lambda_p} = \varphi_{\alpha\beta}u_{\lambda_1\cdots\lambda_p} - \sum \varphi_{\alpha\lambda_i}u_{\lambda_1\cdots\lambda_{i-1}\beta\cdots\lambda_p}$$
$$- \sum \varphi_{\lambda_j\beta}u_{\lambda_1\cdots\lambda_{j-1}\beta\cdots\lambda_p} + \sum_{j<i} \varphi_{\lambda_j\lambda_i}u_{\lambda_1\cdots\lambda_{j-1}\alpha\cdots\lambda_{i-1}\beta\cdots\lambda_p},$$

or

$$(\varphi\wedge u)_{\lambda_1\cdots\lambda_{p+2}} = \sum_{j<i}(-1)^{i+j+1}\varphi_{\lambda_j\lambda_i}u_{\lambda_1\cdots\hat{\lambda}_j\cdots\hat{\lambda}_i\cdots\lambda_{p+2}}, \qquad p \geqq 1,$$
$$(\varphi\wedge u)_{\lambda\mu} = u\varphi_{\lambda\mu}, \qquad p = 0.$$

Now suppose that M^n is compact orientable. Then the global inner product of p-forms u and v is defined by

$$(u, v) = \int_M <u, v> dV,$$

where dV maens the volume element of M^n. We shall denote the global norm of u by $\|u\|$, i.e., $\|u\|^2=(u,u)$, $\|u\| \geqq 0$.

Let u, v, w, φ and η be any p, $p-1$, $p-2$, 2 and 1 form respectively, then the following integral formulas are well known:

$$(du, v) = (u, \delta v)$$

$$(i(\eta)\, u, v) = (u, \eta \wedge v)\,, \qquad (i(\varphi)\, u, w) = (u, \varphi \wedge w)\,,$$

$$(\Delta u, u) = \|du\|^2 + \|\delta u\|^2\,. \tag{1.1}$$

Here we state the following lemmas which are useful for the later discussions.

LEMMA 1.1. *For a skew-symmetric tensor* $u^{\lambda\mu\nu}$ *we have*

$$R_{\lambda\mu\nu\omega} u^{\lambda\mu\nu} = 0\,.$$

LEMMA 1.2. *For a skew-symmetric tensor* $u^{\lambda\mu}$ *we have*

$$R_{\lambda\mu\alpha\beta} u^{\alpha\beta} = -2R_{\lambda\alpha\beta\mu} u^{\alpha\beta}\,.$$

2. Almost contact metric space. An n dimensional Riemannian space is called an almost contact metric space, if it admits a 1-form $\eta = \eta_\lambda dx^\lambda$ and a 2-form $\varphi = (1/2)\varphi_{\lambda\mu} dx^\lambda \wedge dx^\mu$ satisfying

$$|\eta| = 1 : \qquad \eta_\lambda \eta^\lambda = 1\,, \tag{2.1}$$

$$i(\eta)\,\varphi = 0 : \qquad \eta^\alpha \varphi_{\alpha\lambda} = 0\,, \tag{2.2}$$

$$\varphi_\alpha{}^\lambda \varphi_\mu{}^\alpha = -\delta_\mu^\lambda + \eta_\mu \eta^\lambda\,, \tag{2.3}$$

where we have put

$$\varphi_\alpha{}^\lambda = g^{\lambda\mu} \varphi_{\alpha\mu}$$

It is known that an almost contact metric space is orientable and its dimension n is necessarily odd: $n=2m+1$.

In this section we shall concern ourselves with an $n(=2m+1)$ dimensional almost contact metric space M^n.

We introduce an operator L by

$$Lu = \varphi \wedge u$$

for any form u.

It is evident that if a p-form u satisfies $i(\eta)u=0$ then we have $i(\eta)Lu=0$ and $i(\eta)i(2\varphi)u=0$.

First we can get

LEMMA 2.1.[4] *If a p-form u_p satisfies $i(\eta)u_p=0$, then we have*

$$i(2\varphi)L^k u_p = L^k i(2\varphi)u_p + k(n+1-2p-2k)L^{k-1}u_p,$$

where k is any non-negative integer and $L^{-1}\equiv 0$.

We shall call a p-form u to be effective if $i(\eta)u=0$ and $i(2\varphi)u=0$ hold good. A 0-form is always effective. From Lemma 2.1 we can get

LEMMA 2.2. *For an effective p-form u_p we have*

$$i(2\varphi)^k L^{k+s}u_p = (s+k)(s+k-1)\cdots(s+1) \times (n+1-2p-2s-2)\cdots(n+1-2p-2s-2k)L^s u_p,$$

where k is any positive integer and s non-negative integer.

Especially we have

LEMMA 2.3. *For an effective p-form u_p we have*

$$i(2\varphi)^k L^k u_p = k!(n+1-2p-2)\cdots(n+1-2p-2k)u_p,$$

where k is any positive integer.

From this lemma for a large k we get

THEOREM 2.1. *In a $2m+1$ dimensional almost contact metric space, there does not exist an effective p-form other than 0 for $p>m$.*

By virtue of Lemma 2.2 and the mathematical induction, we obtain the following

LEMMA 2.4. *If ϕ_{p-2k}, $k=0,1,\cdots,r$, are effective $(p-2k)$-forms and satisfy*

4) Proofs of lemmas in this section are analogous to that of an almost Hermitian space, see, for example, S. I. Goldberg, [2], p. 179–180.

$$\sum L^k \phi_{p-2k} = 0, \qquad r = \left[\frac{p}{2}\right],^{5)}$$

then we have $\phi_{p-2k} = 0$ *for* $p \leqq m+1$.

From these lemmas we have the following theorem which corresponds to Hodge-Lepage theorem in an almost Hermitian space.

THEOREM 2.2. *In a* $2m+1$ *dimensional almost contact metric space, if a p-form* u_p ($p \leqq m+1$) *satisfies* $i(\eta)u_p = 0$, *then it is written uniquely in the form*

$$u_p = \sum_{k=0}^{r} L^k \phi_{p-2k}, \qquad r = \left[\frac{p}{2}\right],$$

where ϕ_{p-2k} *are effective* $(p-2k)$*-forms.*

PROOF. The cases $p=0$ and $p=1$ are trivial. Assuming its validity for p such that $2 \leqq p \leqq m' < m$, we shall prove that for $p+2$. Let u_p be a p-form such that

$$i(\eta)u_p = 0, \qquad p \leqq m',$$

then there exists a p-form v_p uniquely such that

$$(2.1) \qquad i(2\varphi)Lv_p = u_p, \qquad i(\eta)v_p = 0.$$

In fact, by the assumption of the induction there exist uniquely effective forms ψ_{p-2k} such that

$$u_p = \sum L^k \psi_{p-2k}.$$

By Lemma 2.1 we know that

$$v_p = \sum L^k \phi_{p-2k}$$

is the unique solution of (2.1), where

$$\phi_{p-2k} = \frac{1}{2(k+1)(m-p+k)} \psi_{p-2k}.$$

5) [a] means the integer part of a.

Now let u_{p+2} be a $(p+2)$-form such that $i(\eta)u_{p+2} = 0$ and put

$$i(2\varphi)\,u_{p+2} = u_p\,,$$

then we have that $i(\eta)\,u_p = 0$. For this u_p we consider the v_p of (2. 1) and put

$$\phi_{p+2} = u_{p+2} - Lv_p\,.$$

Then ϕ_{p+2} is effective and we have the form

$$u_{p+2} = \phi_{p+2} + \sum L^{k+1}\,\phi_{p-2k}\,.$$

The uniqueness follows from Lemma 2.4. Q.E.D.

Let $A^p(M)$ be the vector space of p-forms on M^n satisfying $i(\eta)u_p = 0$. Then we can get the following two theorems.

THEOREM 2.3. *$i(2\varphi)L$ is an automorphism of $A^p(M)$ for $p \leqq m-1$.*

THEOREM 2.4. *$L: A^{p-2}(M) \to A^p(M)$ is an into isomorphism for $2 \leqq p \leqq m+1$.*

The following lemmas are necessary for the discussion in the later sections.

LEMMA 2.5. If u satisfies $i(\eta)u = 0$, *then we have* $|\eta\wedge u| = |u|$.

As a special case of Lemma 2.1 we have

LEMMA 2.6. *For any $(p-2)$-form v such that $i(\eta)v = 0$, we have*

$$i(2\varphi)\,Lv = L\,i(2\varphi)\,v + (n-2p+3)\,v\,.$$

Now we introduce an operator Φ by

$$\overset{*}{u} = \Phi u \;:\; \begin{cases} \overset{*}{u}_{\lambda_1\cdots\lambda_p} = \sum \varphi_{\lambda_i}{}^{\alpha}\, u_{\lambda_1\cdots\lambda_{i-1}\alpha\cdots\lambda_p}\,, & p \geqq 1\,, \\ \quad = 0\,, \qquad p = 0\,, \end{cases}$$

then $\overset{*}{u}$ is again a p-form for a p-form u.

LEMMA 2.7. *For any p-form u such that $i(\eta)u = 0$, we have*

$$i(2\varphi)\Phi u = \Phi\, i(2\varphi)u\,.$$

3. Identities in a Sasakian space. An n dimensional Sasakian space is a Riemannian space which admits a unit Killing vector field η^λ such that

$$\nabla_\lambda \nabla_\mu \eta_\nu = \eta_\mu g_{\lambda\nu} - \eta_\nu g_{\lambda\mu}\,. \tag{3.1}$$

In the following we shall consider an n dimensional Sasakian space M^n.

If we put $\varphi_\mu{}^\nu = \nabla_\mu \eta^\nu$, then $\varphi_{\mu\nu} = \varphi_\mu{}^\alpha g_{\alpha\nu}$, η_λ and $g_{\lambda\mu}$ give an almost contact metric structure to M^n and hence M^n is orientable and n is odd: $n = 2m+1$. As (3.1) becomes

$$\nabla_\lambda \varphi_{\mu\nu} = \eta_\mu g_{\lambda\nu} - \eta_\nu g_{\lambda\mu}, \tag{3.2}$$

we can get

$$\nabla^\lambda \varphi_{\lambda\nu} = -(n-1)\eta_\nu\,, \quad \nabla^\lambda \nabla_\lambda \varphi_{\mu\nu} = -2\varphi_{\mu\nu}\,.$$

Applying the Ricci's identity to η_λ we have

$$\nabla_\nu \nabla_\mu \eta_\lambda - \nabla_\mu \nabla_\nu \eta_\lambda = -R_{\nu\mu\lambda}{}^\alpha \eta_\alpha,$$

from which it follows that

$$R_{\nu\mu\lambda}{}^\varepsilon \eta_\varepsilon = \eta_\nu g_{\mu\lambda} - \eta_\mu g_{\nu\lambda}\,,$$

$$R_\nu{}^\varepsilon \eta_\varepsilon = (n-1)\eta_\nu\,.$$

Next, applying the Ricci's identity to $\varphi_\lambda{}^\alpha$ we have

$$\nabla_\rho \nabla_\sigma \varphi_\lambda{}^\alpha - \nabla_\sigma \nabla_\rho \varphi_\lambda{}^\alpha = R_{\rho\sigma\varepsilon}{}^\alpha \varphi_\lambda{}^\varepsilon - R_{\rho\sigma\lambda}{}^\varepsilon \varphi_\varepsilon{}^\alpha\,,$$

from which we can get the following formulas:

$$R_{\rho\sigma\varepsilon}{}^\alpha \varphi_\lambda{}^\varepsilon - R_{\rho\sigma\lambda}{}^\varepsilon \varphi_\varepsilon{}^\alpha = \varphi_{\rho\lambda}\delta_\sigma^\alpha - \varphi_\rho{}^\alpha g_{\sigma\lambda} - \varphi_{\sigma\lambda}\delta_\rho^\alpha + \varphi_\sigma{}^\alpha g_{\rho\lambda}\,,$$

$$\varphi_\lambda{}^\varepsilon R_{\varepsilon\mu\rho\sigma} = -R_{\rho\sigma\lambda\varepsilon}\varphi_\mu{}^\varepsilon + \varphi_{\rho\lambda} g_{\sigma\mu} - \varphi_{\rho\mu} g_{\sigma\lambda} - \varphi_{\sigma\lambda} g_{\rho\mu} + \varphi_{\sigma\mu} g_{\rho\lambda}\,,$$

$$(1/2)\varphi^{\alpha\beta} R_{\alpha\beta\lambda\mu} = R_{\lambda\varepsilon}\varphi_\mu{}^\varepsilon + (\mathrm{n}-2)\varphi_{\lambda\mu}\,,$$

$$R_{\mu\varepsilon}\varphi_\lambda{}^\varepsilon = -R_{\lambda\varepsilon}\varphi_\mu{}^\varepsilon, \qquad R_\mu{}^\varepsilon \varphi_\varepsilon{}^\lambda = R_\varepsilon{}^\lambda \varphi_\mu{}^\varepsilon\,.$$

LEMMA 3.1. *For any skew-symmetric tensors $u^{\alpha\beta}$ and $w^{\lambda\mu}$ we have*

$$\varphi_\lambda^{\ \sigma} R_{\sigma\alpha\beta\mu} u^{\alpha\beta} w^{\lambda\mu} = R_{\beta\lambda\mu\sigma} \varphi_\alpha^{\ \sigma} u^{\alpha\beta} w^{\lambda\mu} .$$

Now we define two differential forms φ and η by

$$\varphi = (1/2)\varphi_{\lambda\mu} dx^\lambda \wedge dx^\mu , \qquad \eta = \eta_\lambda dx^\lambda ,$$

then we have

$$d\eta = 2\varphi .$$

About harmonic tensors in a compact Sasakian space the following theorems are known [8].

THEOREM A. *In an $n(=2m+1)$ dimensional compact Sasakian space, a harmonic p-form u is orthogonal to η, i.e., $i(\eta)u=0$, if $p \leqq m$.*

THEOREM B. *In an n dimensional compact Sasakian space, if u is a harmonic p-form ($p \leqq m$), then so is Φu.*

THEOREM C[6]. *The $(2p+1)$-th Betti number of an n dimensional compact Sasakian space is even, if $0 < 2p+1 \leqq m$.*

From Theorem A we have

LEMMA 3.2. *Any harmonic p-form ($p \leqq m$) in a compact M^n is effective.*

4. *C*-harmonic form in a compact Sasakian space. Let M^n be an n $(=2m+1)$ dimensional compact Sasakian space. We shall call a p-form u in M^n to be C-harmonic, if it satisfies

(i) $$i(\eta)\, u = 0 ,$$

(ii) $$du = 0 ,$$

(iii) $$\delta u = \eta \wedge i(2\varphi)\, u .$$ [7]

By definition, a C-harmonic form of degree 0 or 1 is nothing but harmonic. It is easily seen that the form φ itself is a C-harmonic 2-form.

By virtue of Theorem A and Lemma 3.2, we have

6) S. Tachibana and Y. Ogawa, [9]. S. Tanno [10].
7) Y. Ogawa [4] proved that if $p \leqq m$ then (i) is a consequence of (ii) and (iii).

THEOREM 4.1. *In a $2m+1$ dimensional compact Sasakian space, a p-form $(0 \leqq p \leqq m)$ is harmonic if and only if it is effective C-harmonic.*

Next we have

LEMMA 4.1. *If u is a C-harmonic p-form, then $v = i(2\varphi)u$ is a C-harmonic $(p-2)$-form, $(p \geqq 2)$.*

PROOF. $i(\eta)v=0$ is trivial. Putting

$$w = i(2\varphi)v \quad : \quad w_{\lambda_5\cdots\lambda_p} = \varphi^{\alpha\beta} v_{\alpha\beta\lambda_5\cdots\lambda_p},$$

we can get

$$\delta v = \eta \wedge w = \eta \wedge i(2\varphi)v$$

by a straightforward computation.

Next we shall prove that $dv = 0$. At first we have

$$\varphi^{\lambda_1\lambda_2}(\Delta u)_{\lambda_1\cdots\lambda_p} = A_1 + A_2 + A_3,$$

where

$$\begin{aligned} A_1 &= -\varphi^{\lambda_1\lambda_2} \nabla^\alpha \nabla_\alpha u_{\lambda_1\cdots\lambda_p} \\ &= -\nabla^\alpha \nabla_\alpha v_{\lambda_3\cdots\lambda_p} + 2v_{\lambda_3\cdots\lambda_p}, \\ A_2 &= \varphi^{\lambda_1\lambda_2} \sum R_{\lambda_i}{}^\sigma u_{\lambda_1\cdots\sigma\cdots\lambda_p} \\ &= 2\varphi^{\lambda_1\lambda_2} R_{\lambda_1}{}^\sigma u_{\sigma\lambda_2\cdots\lambda_p} + \sum R_{\lambda_i}{}^\sigma v_{\lambda_3\cdots\sigma\cdots\lambda_p}, \\ A_3 &= \varphi^{\lambda_1\lambda_2} \sum_{j<i} R_{\lambda_j\lambda_i}{}^{\rho\sigma} u_{\lambda_1\cdots\rho\cdots\sigma\cdots\lambda_p} \\ &= \varphi^{\lambda_1\lambda_2} R_{\lambda_1\lambda_2}{}^{\rho\sigma} u_{\rho\sigma\lambda_3\cdots\lambda_p} + \varphi^{\lambda_1\lambda_2} \sum R_{\lambda_1\lambda_i}{}^{\rho\sigma} u_{\rho\lambda_2\cdots\sigma\cdots\lambda_p} \\ &\quad + \varphi^{\lambda_1\lambda_2} \sum R_{\lambda_2\lambda_i}{}^{\rho\sigma} u_{\lambda_1\rho\cdots\sigma\cdots\lambda_p} + \varphi^{\lambda_1\lambda_2} \sum_{j<i} R_{\lambda_j\lambda_i}{}^{\rho\sigma} u_{\lambda_1\cdots\rho\cdots\sigma\cdots\lambda_p} \\ &= \{-2\varphi^{\lambda_1\lambda_2} R_{\lambda_1}{}^\sigma u_{\sigma\lambda_2\cdots\lambda_p} + 2(n-2)v_{\lambda_3\cdots\lambda_p}\} - 2(p-2)v_{\lambda_3\cdots\lambda_p} \\ &\quad - 2(p-2)v_{\lambda_3\cdots\lambda_p} + \sum_{2<j<i} R_{\lambda_j\lambda_i}{}^{\rho\sigma} v_{\lambda_3\cdots\rho\cdots\sigma\cdots\lambda_p}. \end{aligned}$$

Thus we can get

$$i(2\varphi)\,\Delta u = \Delta v + 2(n-2p+3)\,v\,. \tag{4.1}$$

On the other hand, operating d to $\delta u = \eta\wedge v$ we have

$$\triangle u = 2\varphi \wedge v - \eta \wedge dv\,,$$

from which it follows that

$$\begin{aligned} i(2\varphi)\,\Delta u &= 2i(2\varphi)\,Lv - i(2\varphi)(\eta\wedge dv) \\ &= 2(n-2p+3)\,v + 2\varphi \wedge i(2\varphi)v - \eta \wedge i(2\varphi)\,dv\,. \end{aligned} \tag{4.2}$$

Comparing (4.1) and (4.2) we have

$$\Delta v = 2\varphi \wedge i(2\varphi)\,v - \eta \wedge i(2\varphi)\,dv\,.$$

Consequently we obtain

$$<\Delta v, v> = <2\varphi \wedge i(2\varphi)\,v, v>\,.$$

Integrating the last equation we have

$$(\Delta v, v) = (2\varphi \wedge i(2\varphi)\,v, v) = \|i(2\varphi)\,v\|^2\,. \tag{4.3}$$

On the other hand we have

$$\|\delta v\|^2 = \|i(2\varphi)\,v\|^2 \tag{4.4}$$

by taking account of Lemma 2.5. Thus by (4.3), (4.4) and (1.1), we have $\|dv\|^2 = 0$. Q.E.D.

LEMMA 4.2. *If u is a C-harmonic p-form, then so is Φu.*

PROOF. Put $\overset{*}{u} = \Phi u$. $i(\eta)\overset{*}{u} = 0$ is evident. We put $v = i(2\varphi)\,u$ and calculate $\delta\overset{*}{u}$, then we have

$$\begin{aligned} (\delta\overset{*}{u})_{\lambda_2\cdots\lambda_p} &= -\nabla^{\lambda_1}\Big(\sum \varphi_{\lambda_i}{}^{\alpha} u_{\lambda_1\cdots\alpha\cdots\lambda_p}\Big) \\ &= B_1 + B_2 + B_3 + B_4\,, \end{aligned}$$

where

$$B_1 = -\nabla^{\lambda_1}\varphi_{\lambda_1}{}^{\alpha}u_{\alpha\lambda_2\cdots\lambda_p} = 0, \quad (\because\ i(\eta)u = 0),$$

$$B_2 = -\varphi_{\lambda_1}{}^{\alpha}\nabla^{\lambda_1}u_{\alpha\lambda_2\cdots\lambda_p} = 0, \quad (\because\ dv = 0),$$

$$B_3 = -\sum_{i=2}^{p}\nabla^{\lambda_1}\varphi_{\lambda_i}{}^{\alpha}u_{\lambda_1\cdots\alpha\cdots\lambda_p} = 0,$$

$$B_4 = -\sum_{i=2}^{p}\varphi_{\lambda_i}{}^{\alpha}\nabla^{\lambda_1}u_{\lambda_1\cdots\alpha\cdots\lambda_p} = (\eta\wedge\overset{*}{v})_{\lambda_2\cdots\lambda_p}.$$

Hence we get

$$\delta\overset{*}{u} = \eta\wedge\overset{*}{v} = \eta\wedge\Phi v = \eta\wedge i(2\varphi)\overset{*}{u}.$$

To prove that $\overset{*}{u}$ is closed, we calculate $<\Delta\overset{*}{u}, \overset{*}{u}>$. At first we have

$$\nabla^{\alpha}\nabla_{\alpha}\overset{*}{u}_{\lambda_1\cdots\lambda_p} = \sum\{\varphi_{\lambda_i}{}^{\alpha}u_{\lambda_1\cdots\alpha\cdots\lambda_p} + \eta_{\lambda_i}\nabla^{\alpha}u_{\lambda_1\cdots\alpha\cdots\lambda_p}$$
$$+\nabla^{\alpha}\varphi_{\lambda_i}{}^{\sigma}\nabla_{\alpha}u_{\lambda_1\cdots\sigma\cdots\lambda_k} + \varphi_{\lambda_i}{}^{\sigma}\nabla^{\alpha}\nabla_{\alpha}u_{\lambda_1\cdots\sigma\cdots\lambda_p}\},$$

from which we can get

$$-\overset{*}{u}{}^{\lambda_1\cdots\lambda_p}\nabla^{\alpha}\nabla_{\alpha}\overset{*}{u}_{\lambda_1\cdots\lambda_p} = -\overset{*}{u}{}^{\lambda_1\cdots\lambda_p}\sum\varphi_{\lambda_i}{}^{\sigma}\nabla^{\alpha}\nabla_{\alpha}u_{\lambda_1\cdots\sigma\cdots\lambda_p}.$$

As u is C-harmonic, we have

$$\Delta u = d(\eta\wedge v) = 2\varphi\wedge v - \eta\wedge dv$$

and hence

$$-\nabla^{\alpha}\nabla_{\alpha}u_{\lambda_1\cdots\sigma\cdots\lambda_p} = -\sum_{j\neq i}R_{\lambda_j}{}^{\rho}u_{\lambda_1\cdots\rho\cdots\sigma\cdots\lambda_p} - R_{\sigma}{}^{\rho}u_{\lambda_1\cdots\rho\cdots\lambda_p}$$
$$-\sum_{k<j}R_{\lambda_k\lambda_j}{}^{\alpha\beta}u_{\lambda_1\cdots\alpha\cdots\sigma\cdots\beta\cdots\lambda_p}$$
$$-\sum_{j>i}R_{\sigma\lambda_j}{}^{\alpha\beta}u_{\lambda_1\cdots\alpha\cdots\beta\cdots\lambda_p} - \sum_{k<i}R_{\lambda_k\sigma}{}^{\alpha\beta}u_{\lambda_1\cdots\alpha\cdots\beta\cdots\lambda_p}$$
$$+(2\varphi\wedge v - \eta\wedge dv)_{\lambda_1\cdots\sigma\cdots\lambda_p}.$$

Thus complicated computations show that we can have

$$<\Delta\overset{*}{u}, \overset{*}{u}> = |\overset{*}{v}|^2.$$

On the other hand, we have $|\delta \overset{*}{u}|^2 = |\overset{*}{v}|^2$, because of $\delta \overset{*}{u} = \eta \wedge \overset{*}{v}$. Hence it follows that $\|d\overset{*}{u}\|^2 = 0$ by (1. 1), from which $\overset{*}{u}$ is closed. Q.E.D.

LEMMA 4.3. *If v is a C-harmonic $(p-2)$-form, then $u = Lv$ is a C-harmonic p-form.*

PROOF. It is evident that $i(\eta)u = 0$ and $du = d(\varphi \wedge v) = 0$ hold good. As we have

$$u_{\alpha\beta\lambda_1\cdots\lambda_{p-2}} = \varphi_{\alpha\beta} v_{\lambda_1\cdots\lambda_{p-2}} - \sum \varphi_{\alpha\lambda_i} v_{\lambda_1\cdots\beta\cdots\lambda_p}$$

$$- \sum \varphi_{\lambda_j\beta} v_{\lambda_1\cdots\alpha\cdots\lambda_{p-2}} + \sum_{j<i} \varphi_{\lambda_j\lambda_i} v_{\lambda_1\cdots\alpha\cdots\beta\cdots\lambda_p},$$

$\nabla^\alpha u_{\alpha\beta\lambda_1\cdots\lambda_{p-2}}$ is the sum of the following eight terms $C_1, \cdots, C_8$:

$$C_1 = \nabla^\alpha \varphi_{\alpha\beta} v_{\lambda_1\cdots\lambda_{p-2}} = -(n-1)\,\eta_\beta v_{\lambda_1\cdots\lambda_{p-2}},$$

$$C_2 = \varphi_{\alpha\beta} \nabla^\alpha v_{\lambda_1\cdots\lambda_{p-2}} = -\sum \nabla_{\lambda_i}(\varphi_\beta{}^\alpha v_{\lambda_1\cdots\alpha\cdots\lambda_{p-2}}) + (p-2)\,\eta_\beta v_{\lambda_1\cdots\lambda_{p-2}},$$

$$C_3 = -\sum \nabla^\alpha \varphi_{\alpha\lambda_i} v_{\lambda_1\cdots\beta\cdots\lambda_{p-2}} = (n-1)\sum \eta_{\lambda_i} v_{\lambda_1\cdots\beta\cdots\lambda_{p-2}},$$

$$C_4 = -\sum \varphi_{\alpha\lambda_i} \nabla^\alpha v_{\lambda_1\cdots\beta\cdots\lambda_{p-2}} = \sum \varphi_{\lambda_i}{}^\alpha \nabla_\alpha v_{\lambda_1\cdots\beta\cdots\lambda_{p-2}}$$

$$= -\sum \{\nabla_{\lambda_j} \overset{*}{v}_{\lambda_1\cdots\beta\cdots\lambda_{p-2}} - \nabla_j(\varphi_\beta{}^\alpha v_{\lambda_1\cdots\alpha\cdots\lambda_{p-2}})\}$$

$$+ \nabla_\beta \overset{*}{v}_{\lambda_1\cdots\lambda_{p-2}} - (p-2)\sum \eta_{\lambda_j} v_{\lambda_1\cdots\beta\cdots\lambda_{p-2}},$$

$$C_5 = -\sum \nabla^\alpha \varphi_{\lambda_j\beta} v_{\lambda_1\cdots\alpha\cdots\lambda_{p-2}}$$

$$= (\eta \wedge v)_{\beta\lambda_1\cdots\lambda_{p-2}} + (p-3)\,\eta_\beta v_{\lambda_1\cdots\lambda_{p-2}},$$

$$C_6 = -\sum \varphi_{\lambda_j\beta} \nabla^\alpha v_{\lambda_1\cdots\alpha\cdots\lambda_{p-2}}$$

$$= \sum (-1)^j \varphi_{\beta\lambda_j} (\delta v)_{\lambda_1\cdots\hat{\lambda}_i\cdots\lambda_{p-2}},$$

$$C_7 = \sum_{j<i} \nabla^\alpha \varphi_{\lambda_j\lambda_i} v_{\lambda_1\cdots\alpha\cdots\beta\cdots\lambda_{p-2}}$$

$$= (p-3)\{(\eta \wedge v)_{\beta\lambda_1\cdots\lambda_{p-2}} - \eta_\beta v_{\lambda_1\cdots\lambda_{p-2}}\},$$

$$C_8 = \sum_{j<i} \varphi_{\lambda_j\lambda_i} \nabla^\alpha v_{\lambda_1\cdots\alpha\cdots\beta\cdots\lambda_{p-2}} = \sum_{j<i} (-1)^j \varphi_{\lambda_j\lambda_i} (\delta v)_{\lambda_1\cdots\hat{\lambda}_j\cdots\beta\cdots\lambda_{p-2}}.$$

Thus we can get

$$\begin{aligned}\delta u &= (n-2p+3)\,\eta\wedge v+\varphi\wedge\delta v\\ &= \eta\wedge\{(n-2p+3)\,v+\varphi\wedge i(2\varphi)v\}\\ &= \eta\wedge i(2\varphi)\,u\,. \end{aligned}$$

Q.E.D.

5. Main theorems.

THEOREM 5.1. *In an n $(=2m+1)$ dimensional compact Sasakian space, any C-harmonic p-form u_p, $0\leqq p\leqq m+1$, can be written uniquely in the following form:*

$$u_p=\sum_{k=0}^{r}L^k\phi_{p-2k}\,,\qquad r=\left[\frac{p}{2}\right],$$

where ϕ_{p-2k} are harmonic $(p-2k)$-forms.

PROOF. We use the notations in the proof of Theorem 2.2. Assuming its validity for p, $2\leqq p\leqq m'<m$, we shall prove it for $p+2$. Let u_{p+2} be C-harmonic, then

$$i(2\varphi)\,u_{p+2}=u_p$$

is C-harmonic ($\because$ Lemma 4.1). By the assumption of the induction, u_p is written uniquely in the form:

$$u_p=\sum L^k\psi_{p-2k}\,,$$

where ψ_{p-2k} are harmonic. The equation

$$i(2\varphi)\,Lv_p=u_p\,,\qquad i(\eta)\,v_p=0$$

admits unique solution

$$v_p=\sum L^k\phi_{p-2k}\,,$$

where

$$\phi_{p-2k}=\frac{1}{2(k+1)(m-p+k)}\,\psi_{p-2k}$$

are harmonic, so v_p is C-harmonic by virtue of Lemma 4.3. By putting $\phi_{p+2}=u_{p+2}-Lv_p$, the proof is completed. Q.E.D.

$A^p(M)$ is the vector space of p-forms such that $i(\eta)u = 0$. Let $C^p(M)$ and $H^p(M)$ be the vector space of C-harmonic p-forms and harmonic p-forms respectively. Then we have

$$A^p(M) \supset C^p(M) \supset H^p(M), \qquad p \leqq m.$$

The p-th Betti number b_p is $\dim H^p(M)$. Now we introduce c_p by

$$c_p = \dim C^p(M), \qquad p \leqq m.$$

Then we can obtain the following theorem by the analogous way as that of Kählerain spaces.

THEOREM 5.2. *In an n $(=2m+1)$ dimensional compact Sasakian space, we have*

$$b_0 = c_0 = 1, \qquad b_1 = c_1,$$

$$c_{2k} \geqq 1, \qquad k = 1, \cdots, \left[\frac{m}{2}\right],$$

$$b_p = c_p - c_{p-2} \geqq 0, \qquad 2 \leqq p \leqq m,$$

$$c_p = b_p + b_{p-2} + \cdots + b_{p-2r}, \qquad 2 \leqq p \leqq m, \qquad r = \left[\frac{p}{2}\right].$$

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