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ON A CLASS OF CONVOLUTION TRANSFORM II

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1. Introduction. In the preceding paper [7], we studied the inversion theory for some restricted class of convolution transform

$$(1) \qquad f(x) = \int_{-\infty}^{\infty} G(x-t)\, e^{ct}\, d\alpha(t) \qquad (c : \text{real}),$$

for which the kernel $G(t)$ is of the form

$$(2) \qquad G(t) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} [F(s)]^{-1} e^{st}\, ds .$$

Here $F(s)$ is the meromorphic function with real zeros and poles only, and is of the form $F(s) = E_1(s)/E_2(s)$,

$$(3) \qquad E_1(s) = e^{bs} \prod_1^{\infty} (1-s/a_k)\, e^{s/a_k}, \quad E_2(s) = \prod_1^{\infty} (1-s/c_k)\, e^{s/c_k},$$

where b, $\{a_k\}_1^{\infty}$, $\{c_k\}_1^{\infty}$ are constants such that $\sum_1^{\infty} a_k^{-2} < \infty$, $\sum_1^{\infty} c_k^{-2} < \infty$.

When $E_1(s)$ and $E_2(s)$ are reciprocals of the generating functions of kernels of class I and class II (or I, II) [3], [4], respectively, and $F(s)$ satisfies some conditions, we knew that the properties of the transform (1) are similar to those of the convolution transform with the class I kernel.

In this paper we shall study the case in which both $E_1(s)$ and $E_2(s)$ are reciprocals of the generating functions of class II (or III) kernels and satisfies the conditions:

$$(4\text{a}) \qquad a_k c_k > 0, \quad |a_k| \leqq |c_k| \quad \text{for all } k \text{ and } \left|\sum_1^{\infty} (a_k^{-1} - c_k^{-1})\right| < \infty ;$$

(4 b) for some positive α and any positive number R,

$$[F(s)]^{-1} = O(|\tau|^{-(2+\alpha)}) \qquad |\tau| \to \infty,\ s = \sigma + i\tau,$$

uniformly in the strip $|\sigma| \leqq R$.

In this case we shall know that our kernel has similar properties to those of the class III kernel and vanishes from a certain point on. If both $E_1(s)$ and $E_2(s)$ are corresponding to class III kernels, then the last condition of (4 a) is satisfied necessarily and the inversion theorem obtained without the condition for $\alpha(t)$.

However, if $E_1(s)$ and $E_2(s)$ are corresponding to class II kernels and satisfy (4a), then for the inversion theorem it is necessary to assume some order condition for $\alpha(t)$. Further, in the last section we shall show that even if we remove the last condition of (4 a), our method cannot be applied, in general, to the convolution transform without some order condition for $\alpha(t)$.

2. Properties of the kernel. For brevity, we assume hereafter that the constants a_k, c_k, are positive for all k.

Let us define

$$g_k(t) = \begin{cases} a_k\, e^{a_k t - 1} & (-\infty < t < 1/a_k) \\ 0 & (1/a_k < t < \infty), \end{cases}$$

$$h_k(t) = \int_{-\infty}^{t} (1 - a_k/c_k)\, g_k(u + 1/c_k)\, du + \frac{a_k}{c_k} j[t - (1/a_k - 1/c_k)],$$

where $j(t)$ is the standard jump function, that is, $j(t) = 1$ for $t > 0$, $1/2$ for $t = 0$, and 0 for $t < 0$. It is easily verified that $h_k(t)$ is a normalized distribution function with mean 0 and variance $a_k^{-2} - c_k^{-2}$, and

$$\int_{-\infty}^{\infty} e^{-st}\, dh_k(t) = \frac{(1 - s/c_k)\, e^{s/c_k}}{(1 - s/a_k)\, e^{s/a_k}},$$

the bilateral Laplace transform converging absolutely for $\Re s < a_k$.

THEOREM 2.1. *If*

1. $F(s)$ *satisfies conditions* (4 a), (4 b) *of* § 1,
2. μ_2 = *multiplicities of* α_2 *as a zero of* $F(s)$, *where* $\alpha_2 = \min a_k$,
3. $G(t) = \dfrac{1}{2\pi i} \displaystyle\int_{-i\infty}^{i\infty} [F(s)]^{-1} e^{st}\, ds \qquad (-\infty < t < \infty)$,

then:

A. $G(t)$ *is a frequency function with mean* b *and variance* $\sum_1^\infty a_k^{-2} - \sum_1^\infty c_k^{-2}$;

B. $\int_{-\infty}^{\infty} e^{-st}G(t)\,dt = [F(s)]^{-1}$, *the bilateral Laplace transform converging absolutely in the half plane* $\Re s < \alpha_2$;

C. $G(t) \in C^1(-\infty, \infty)$;

D. $$[G(t)]^{(n)} = \begin{cases} 0 & t \geqq b + \sum_1^{\infty} (a_k^{-1} - c_k^{-1}), \\ [q(t)\, e^{\alpha_2 t}]^{(n)} + O(e^{(\alpha_2+\varepsilon)t}) & t \to -\infty, \end{cases} \qquad (n = 0, 1)$$

for some $\varepsilon > 0$, *where* $q(t)$ *is a real polynomial of degree* $\mu_2 - 1$.

PROOF. If we set

$$H_n(t) = h_1 \# h_2 \# \cdots \# h_n(t-b),$$

where operation $\#$ denotes the Stieltjes convolution for distribution functions, then $H_n(t)$ is a distribution function with mean b and variance $\sum_1^n a_k^{-2} - \sum_1^n c_k^{-2}$. From the condition (4 a), (4 b) of §1, it is easily seen [7] that distribution function $H(t) = \lim_{n\to\infty} H_n(t)$ is twice differentiable and $G(t) = \frac{d}{dt} H(t)$. Hence, $G(t)$ is a frequency function and by Humburger's theorem [8] we see that

$$\int_{-\infty}^{\infty} G(t)\, e^{-st}\, dt = 1/F(s),$$

the integral converging absolutely for $\Re s < \alpha_2$. From these facts the conclusion A, B, C and the second part of conclusion D are obvious. However, by the direct calculation, it is seen that

$$H_n(t) = \prod_1^{n-1} \left(1 - \frac{1}{2} \frac{a_k}{c_k}\right) \qquad \text{for} \quad t > b + \sum_1^{n} (a_k^{-1} - c_k^{-1})$$

and

$$H(t) = \prod_1^{\infty} \left(1 - \frac{1}{2} \frac{a_k}{c_k}\right) \qquad \text{for} \quad t > b + \sum_1^{\infty} (a_k^{-1} - c_k^{-1}).$$

This infinite product is either converges or diverges to 0 and

$$G(t) = H'(t) = 0 \qquad \text{for} \quad t > b + \sum_1^{\infty} (a_k^{-1} - c_k^{-1}).$$

From the conclusion C we have $[G(t)]^{(n)}=0$ $(n=0,1)$ for $t \geqq b + \sum_1^\infty (a_k^{-1} - c_k^{-1})$. This completes the proof.

The 'degeneracy' phenomenon of conclusion D of this theorem is the characteristic property of class III kernels. In our case, however, $G(t)$ is the kernel which is generated by the ratio of two generating functions of class II (or III) kernels.

THEOREM 2.2. [4, p. 108] *If*

1. $E_1(s)$ *is defined as in* §1,
2. $G_1(t) = \dfrac{1}{2\pi i}\displaystyle\int_{-i\infty}^{i\infty} [E_1(s)]^{-1} e^{st}\, ds \qquad -\infty < t < \infty,$

then:

A. $G_1(t)$ *is a frequency function with mean* 0 *and variance* $\sum_1^\infty a_k^{-2}$;

B. $\displaystyle\int_{-\infty}^{\infty} G_1(t)\, e^{-st}\, dt = [E_1(s)]^{-1}$, *the bilateral Laplace transform converging absolutely in the half plane* $\Re s < \alpha_2$;

C. $G_1(t) \in C^\infty(-\infty, \infty)$;

D. $G_1^{(n)}(t) = O(e^{kt}) \quad t \to \infty,\ n = 0, 1, 2, \cdots,$ *where* k *is an arbitrary* (*negative*) *real number*,

$$G_1^{(n)}(t) = [e^{\alpha_2 t} q_1(t)]^{(n)} + O(e^{(\alpha_2+\varepsilon)t}) \quad t \to -\infty,\ n = 0, 1, 2, \cdots,$$

for some $\varepsilon > 0$, *where* $q_1(t)$ *is a real polynomial of degree* $\mu_2 - 1$.

If $G_1(t)$ is the class III kernel, then it vanishes for $t \geqq b + \sum_1^\infty a_k^{-1}$.

The properties of kernel $G_2(t)$ which is generated by $[E_2(s)]^{-1}$ are similar to those of $G_1(t)$.

3. Convergence. In our case, $G(t) = 0$ for $t \geqq b + \sum_1^\infty (a_k^{-1} - c_k^{-1})$, hence it follows that we need not suppose $\alpha(t)$ defined for all t and it is enough to assume $\alpha(t)$ defined for $T < t < \infty$ and of bounded variation in every subinterval provided that we consider only those x for which $x > T + b + \sum_1^\infty (a_k^{-1} - c_k^{-1})$.

THEOREM 3.1. *If*

1. *$\alpha(t)$ is defined for $T < t < \infty$ and is of bounded variation in every interval $T < t_1 \leqq t \leqq t_2 < \infty$,*

2. $\int_{-\infty}^{\infty} G(x_0-t)\,d\alpha(t)$ *converges for* $x_0 > T + b + \sum_1^{\infty} (a_k^{-1} - c_k^{-1})$,

then

$$\int_{-\infty}^{\infty} G(x-t)\,d\alpha(t)$$

converges uniformly for x in any finite interval lying to the right of $T + b + \sum_1^{\infty} (a_k^{-1} - c_k^{-1})$; i.e. for $T + b + \sum_1^{\infty} (a_k^{-1} - c_k^{-1}) < x_1 \leqq x \leqq x_2 < \infty$.

PROOF. Since $G(t)$ vanishes for $t \geqq b + \sum_1^{\infty} (a_k^{-1} - c_k^{-1})$, it is enough to show that

$$\lim_{A, B\to\infty} \int_A^B G(x-t)\,d\alpha(t) = 0, \tag{1}$$

uniformly for x, $x_1 \leqq x \leqq x_2$. From the conclusion D of Theorem 2.1, $G(t) \neq 0$ for negative t with sufficiently large absolute value and from the same theorem

$$G(x-t)/G(x_0-t) = O(1) \qquad t \to \infty,$$

$$\frac{d}{dt}\,[G(x-t)/G(x_0-t)] = O(1/t^2) \qquad t \to \infty,$$

uniformly for x, $x_1 \leqq x \leqq x_2$. Using these equations we see by the usual method that equation (1) holds uniformly for x, $x_1 \leqq x \leqq x_2$.

The following two theorems are well known [4, p. 124].

THEOREM 3.2. *If*

1. *$G_1(t) \in$ class II,*
2. *$\alpha(t)$ is of bounded variation in every finite interval,*
3. $\int_{-\infty}^{\infty} G_1(x_0-t)\,d\alpha(t)$ *converges (conditionally),*

then

$$\int_{-\infty}^{\infty} G_1(x-t)\,d\alpha(t)$$

converges uniformly for x in any finite interval bounded on the left by x_0; i.e. for $x_0 \leqq x \leqq x_1 < \infty$.

THEOREM 3.3. *If*

1. $G_1(t) \in$ *class III*,
2. $\alpha(t)$ *is defined and of bounded variation in every finite interval* $T < t_1 \leqq t \leqq t_2 < \infty$,
3. $\int_{-\infty}^{\infty} G_1(x_0 - t)\, d\alpha(t)$ *converges for* $x_0 > T + \sum_k a_k^{-1}$,

then

$$\int_{-\infty}^{\infty} G_1(x-t)\, d\alpha(t)$$

converges uniformly for x in any finite interval lying to the right of $T + b + \sum_k a_k^{-1}$; i.e. for $T + b + \sum_k a_k^{-1} < x_1 \leqq x \leqq x_2 < \infty$.

THEOREM 3.4. *If*

1. $G(t)$, $G_1(t)$ *are defined as in* § 2,
2. $\alpha(t)$ *is defined for $t > T$ and is of bounded variation in every finite interval $T < t_1 \leqq t \leqq t_2 < \infty$, then the transform $\int_{-\infty}^{\infty} G(x_0 - t)\, d\alpha(t)$ converges for $x_0 > T + b + \sum_k (a_k^{-1} - c_k^{-1})$ if and only if the transform $\int_{x}^{\infty} G_1(x_0 - t) d\alpha(t)$ converges for all x, $x > T$.*

PROOF. We suppose that the first transform converges. By Theorem 2.1 and Theorem 2.2, we have

$$G_1(x_0 - t)/G(x_0 - t) = O(1) \qquad t \to \infty\,,$$

$$\frac{d}{dt}\,[G_1(x_0 - t)/G(x_0 - t)] = O(1/t^2) \qquad t \to \infty\,.$$

From these facts it is easily seen that

$$\lim_{A, B \to \infty} \int_{A}^{B} G_1(x_0 - t)\, d\alpha(t) = 0$$

and that the second transform converges for all x, $x > T$. Noticing that

$G(x_0-t)=0$ for $t<x_0-b-\sum_1^\infty(a_k^{-1}-c_k^{-1})$, we may similarly establish only if part.

Under the same condition of this theorem, if $G_1(t)\in$ class III, then the transform $\int_{-\infty}^{\infty}G_1(x_0-t)\,d\alpha(t)$ converges, though $\alpha(t)$ is defined for all t, $-\infty<t<\infty$ (i.e. $T=-\infty$). However, if $G_1(t)\in$ class II and $T=-\infty$, this convergence is not guaranteed without some order condition for $\alpha(t)$, because $G_1(t)$ does not vanish on $(-\infty,\infty)$ (see §5). Concerning to this fact we have the following theorem.

THEOREM 3.5. *If $\alpha(t)$ is of bounded variation in every finite interval and for some x_0*

$$\alpha(t)=O[G_1(x_0-t)\,e^{ct}]^{-1}\qquad t\to-\infty\ (c:\ real), \tag{1}$$

then

$$\int_{-\infty}^{0}G_1(x-t)\,e^{ct}d\alpha(t) \tag{2}$$

converges uniformly for $(x_1\leqq x<\infty)$ for any $x_1>x_0$.

PROOF. Let $\chi_1(t)=-\log G_1(t)$, then by the properties of $\chi_1(t)$ it is known [3] that

$$\lim_{t\to-\infty}G_1(x-t)/G_1(x_0-t)=0\,, \tag{3}$$

uniformly for $(x_1\leqq x<\infty)$ for any $x_1>x$, and that

$$\int_{-\infty}^{0}|G_1'(x-t)/G_1(x_0-t)|\;dt \tag{4}$$

converges uniformly for $(x_1\leqq x<\infty)$. Concerning to the equation (1) it is also easily verified that

$$\int_{-\infty}^{0}G_1(x-t)/G_1(x_0-t)\,dt \tag{5}$$

converges uniformly for $(x_1\leqq x<\infty)$. Integrating by parts we have

$$\int_A^B G_1(x-t)\,e^{ct}\;d\alpha(t)=[G_1(x-t)\,e^{ct}\;\alpha(t)]_A^B+\int_A^B\{G_1'(x-t)-cG_1(x-t)\}\,e^{ct}\,\alpha(t)\,dt.$$

Using (1), (3), (4) and (5) it is obvious that

$$\lim_{A, B \to -\infty} \int_A^B G_1(x-t)\, e^{ct}\, d\alpha(t) = 0,$$

uniformly for $(x_1 \leqq x < \infty)$. This completes the proof.

In this theorem, when the transform (2) converges for some $x = x_0$, it is also familiar that

$$\alpha(t) = o[G_1(x_0-t)\, e^{ct}]^{-1} \qquad t \to -\infty,$$

hence, for the convergence of (2), the condition (1) for $\alpha(t)$ is the best one.

4. Inversion theorem. We suppose that we are given a sequence $\{b_n\}_0^\infty$ of real numbers such that $b_0 = 0$, $\lim_{n\to\infty} b_n = 0$.

We define, as usual,

$$E_{1,n}(D) = e^{(b-b_n)D} \prod_1^\infty (1-D/a_k)\, e^{D/a_k},$$

where D stands for differentiation and we interpret e^{lD} the operation of translation through a distance l [3], [4].

On the other hand, we define

$$[E_2(D)]^{-1} f(x) = \int_{-\infty}^{\infty} f(x-t)\, G_2(t)\, dt,$$

whenever this integral converges [9, p. 121].

Our main result in this section is the following.

THEOREM 4.1. *If*

1. $G_1(t)$, $G_2(t) \in$ *class II,*
2. $\alpha(t)$ *is of bounded variation in every finite interval and for some* x_0

$$\alpha(t) = O[G_1(x_0-t)\, e^{ct}]^{-1} \qquad t \to \infty,$$

3. $f(x) = \int_{-\infty}^{\infty} G(x-t)\, e^{ct}\, d\alpha(t)$ *converges,*
4. $\alpha(t)$ *is continuous at* x_1, x_2 $(x_1, x_2 > x_0)$,

then:

A. $c < \alpha_2$ *implies that*

$$\lim_{n\to\infty}\int_{x_1}^{x_2} e^{-cx} E_{1,n}(D)([E_2(D)]^{-1} f(x))\, dx = \alpha(x_2) - \alpha(x_1);$$

B. $c \geqq \alpha_2$ *implies that* $\alpha(+\infty)$ *exists and that*

$$\lim_{n\to\infty}\int_{x_1}^{\infty} e^{-cx} E_{1,n}(D)([E_2(D)]^{-1} f(x))\, dx = \alpha(+\infty) - \alpha(x_1).$$

PROOF. From Theorem 2.1 and Theorem 2.2 for $G_2(t)$ it follows that the bilateral Laplace transform of $G(t)$ and $G_2(t)$ have a common region of absolute convergence $\Re s < \alpha_2$ and hence by the product theorem [8] that

$$(1)\quad \begin{aligned} &\frac{1}{F(s)}\frac{1}{E_2(s)} = \frac{1}{E_1(s)} = \int_{-\infty}^{\infty} e^{-st} G_1(t)\, dt, \quad \sigma < \alpha_2, \ s = \sigma + i\tau, \\ &G_1(x) = \int_{-\infty}^{\infty} G(x-t) G_2(t)\, dt = \int_{x-\lambda}^{\infty} G(x-t) G_2(t)\, dt \qquad -\infty < x < \infty, \end{aligned}$$

both integrals converging absolutely, where $\lambda = b + \sum_1^{\infty} (a_k^{-1} - c_k^{-1})$. From Theorem 3.1 for arbitrary finite number A, B ($A<0, B>0$) we have

$$(2)\quad \int_A^B \left(\int_{-\infty}^{\infty} G(x-t-u)\, e^{cu}\, d\alpha(u)\right) G_2(t)\, dt = \int_{-\infty}^{\infty} \left(\int_A^B G(x-t-u)\, G_2(t)\, dt\right) e^{cu}\, d\alpha(u).$$

Using Theorem 2.1 and Theorem 2.2 for $G_2(t)$ again, it is easily seen that

$$(3)\quad \int_A^B G(x-t) G_2(t)\, dt = P(A, B) q^*(x)\, e^{\alpha_2 x} + Q(A, B) O(e^{(\alpha_2+\varepsilon)x}) \quad x \to -\infty,$$

where $0 < \varepsilon < \alpha_2' - \alpha_2$ and $q^*(x)$ is a real polynomial of degree $\mu_2 - 1$ and $P(A, B)$, $Q(A, B)$ are bounded uniformly for A, B ($\alpha_2' = \min c_k$).

Further, it follows that if $\varepsilon > 0$ then

$$(4)\quad \frac{d}{dx}\int_A^B G(x-t) G_2(t)\, dt = O(G_1'(x)) + O(G(x-\varepsilon)) \qquad x \to \infty,$$

independently of A, B with sufficiently large absolute value. In fact, $G(x-t)=0$

for $x \geqq B+\lambda$, therefore let $A+\lambda < x < B+\lambda$ and let $L_1(t)$ and $L_2(t)$ be defined by the relations

$$t = b + \sum_1^\infty \frac{L_1}{a_k(a_k+L_1)}, \quad t = \sum_1^\infty \frac{L_2}{c_k(c_k+L_2)},$$

then it is well known ([3], [4]) that $L_1(t), L_2(t) \in \uparrow$, $L_1(\infty) = \infty$, $L_2(\infty) = \infty$ and

$$\begin{aligned} &-[\log G_1(t)]' = L_1(t+o(1)) \qquad t \to \infty, \\ &-[\log G_2(t)]' = L_2(t+o(1)) \qquad t \to \infty. \end{aligned} \tag{5}$$

The relation defining L_2 may be written in the form

$$L_1^{-1}[L_2(t)] = t + b + \sum_1^\infty \frac{L_2}{a_k(a_k+L_2)} - \sum_1^\infty \frac{L_2}{c_k(c_k+L_2)},$$

because

$$t + b + \sum_1^\infty \frac{L_2}{a_k(a_k+L_2)} - \sum_1^\infty \frac{L_2}{c_k(c_k+L_2)} = b + \sum_1^\infty \frac{L_2}{a_k(a_k+L_2)},$$

and thus

$$L_2(t) = L_1\left(t + b + \sum_1^\infty \frac{L_2}{a_k(a_k+L_2)} - \sum_1^\infty \frac{L_2}{c_k(c_k+L_2)}\right).$$

Since

$$t + b + \sum_1^\infty \frac{L_2}{a_k(a_k+L_2)} - \sum_1^\infty \frac{L_2}{c_k(c_k+L_2)} \to t + \lambda \qquad L_2 \to \infty,$$

$$L_2(t) = L_1(t+\lambda+o(1)). \tag{6}$$

Consequently, by (5) and (6)

$$\frac{d}{dx} \log G_2(x-\lambda)/G_1(x-\varepsilon) = -L_1(x+o(1)) + L_1(x-\varepsilon+o(1)),$$

hence, if x is sufficiently large,

$$\frac{d}{dx} \log G_2(x-\lambda)/G_1(x-\varepsilon) \leqq 0,$$

or equivalently

$$G_2(x-\lambda)/G_1(x-\varepsilon) \in \downarrow .$$

Thus we have

$$(7) \qquad G_1(x-\lambda)/G_1(x-\varepsilon) = O(1) \qquad x \to \infty .$$

Now, integrating by parts we obtain

$$(8) \qquad \frac{d}{dx}\int_A^B G(x-t)\,G_2(t)\,dt = -G(x-B)G_2(B) + \int_{x-\lambda}^B G(x-t)\,G_2'(t)\,dt .$$

Since $G(x)$ is bounded and $G_2(t) \in \downarrow$ for sufficiently large t, using (7) we have

$$G(x-B)\,G_2(B) = O(G_1(x-\varepsilon)) \qquad x \uparrow B+\lambda .$$

Consequently,

$$(9) \qquad G(x-B)\,G_2(B) = O(G_1(x-\varepsilon)) \qquad x \to \infty ,$$

independently of sufficiently large B. On the other hand, $G_2'(t) < 0$ for sufficiently large t and $G(x) \geqq 0$ for all x, therefore,

$$(10) \qquad \left|\int_{x-\lambda}^B G(x-t)\,G_2'(t)\,dt\right| \leqq \left|\int_{x-\lambda}^\infty G(x-t)\,G_2'(t)\,dt\right| = |G_1'(x)| .$$

Combining (8), (9) and (10) we have the estimation (4). From Theorem 3.4 and the properties of the convolution transform with class II kernel we have

$$\alpha(t) = o[e^{(\alpha_2-c)t}\,t^{-(\mu_2-1)}] \qquad t\to\infty, \quad c<\alpha_2$$

$$\alpha(+\infty) - \alpha(t) = o[e^{(\alpha_2-c)t}\,t^{-(\mu_2-1)}] \qquad t\to\infty, \quad c \geqq \alpha_2$$

and

$$\int_0^\infty e^{-\alpha_2 t+ct}\,t^i\,\alpha(t)\,dt \qquad (i=0,1,2,\cdots,\mu_2-1),\ c<\alpha_2$$

$$\int_0^\infty e^{-\alpha_2 t+ct}\,t^i(\alpha(\infty)-\alpha(t))\,dt \quad (i=0,1,2,\cdots,\mu_2-1),\ c\geqq\alpha_2$$

converges. From these facts and (3), integrating by parts, it is easily seen that

$$\lim_{D\to\infty}\int_D^\infty\left(\int_A^B G(x-t-u)\,G_2(t)\right)e^{cu}\,d\alpha(u)=0,$$

independently of A, B. Now, if we take positive number ε such that $\varepsilon < x-x_0$ and use the estimation (4) and equations in the proof of Theorem 3.5, then it is also easily seen that

$$\lim_{C\to-\infty}\int_{-\infty}^C\left(\int_A^B G(x-t-u)\,G_2(t)\,dt\right)e^{cu}\,d\alpha(u)=0\,,$$

indepedently of A, B with sufficiently large absolute value. Thus, for all x the integral of right hand side of (2) converges uniformly for A, B with sufficiently large absolute value and we obtain

$$\begin{aligned}[E_2(D)]^{-1}f(x) &= \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} G(x-t-u)\,e^{cu}\,d\alpha(u)\right)G_2(t)\,dt\\ &= \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} G(x-t-u)\,G_2(t)\,dt\right)e^{cu}\,d\alpha(u)\\ &= \int_{-\infty}^{\infty} G_1(x-u)\,e^{cu}\,d\alpha(u)\,.\end{aligned}$$

Thus obtained transform is the one with class II kernel $G_1(t)$ and appealing the familiar theorem [4, p. 136], we obtain our desired result.

COROLLARY 4.1. *If*

1. $G_1(t)$, $G_2(t)\in$ *class II,*
2. $\varphi(t)$ *is integrable on every finite interval and for some* x_0

$$\int^t\varphi(u)\,du = O[G_1(x_0-t)]^{-1}\quad as\quad t\to-\infty\,,$$

3. $f(x)=\int_{-\infty}^{\infty}G(x-t)\,\varphi(t)\,dt$ *converges,*

4. $\varphi(t)$ *is continuous at* $t=x$,

then

$$\lim_{n\to\infty}E_{1,n}(D)([E_2(D)]^{-1}f(x))=\varphi(x)\,.$$

We now turn to the case $G_1(t)$, $G_2(t)\in$ class III. The demonstration of

,the following results follows in the pattern of the preceding theorem, however, it is more easy, because both $G_1(x)$ and $\int_A^B G(x-t)G_2(t)\,dt$ vanish for $t \geqq b + \sum_1^\infty a_k^{-1}$ and for all A, B.

THEOREM 4.2. *If*

1. $G_1(t)$, $G_2(t) \in$ *class III,*
2. $\alpha(t)$ *is defined for* $T < t < \infty$ *and is of bounded variation in every finite interval* $T < t_1 \leqq t \leqq t_2 < \infty$,
3. $f(x) = \int_{-\infty}^{\infty} G(x-t)\,e^{ct}\,d\alpha(t)$ *converges for* $x > T + b + \sum^\infty a_k^{-1}$,
4. $\alpha(t)$ *is continuous at* x_1, x_2 $(x\ , x_2 > T)$,

then:

A. $c < \alpha_2$ *implies that*

$$\lim_{n\to\infty}\int_{x_1}^{x_2} e^{-cx}E_{1,n}(D)([E_2(D)]^{-1}f(x))\,dx = \alpha(x_2) - \alpha(x_1)\,;$$

B. $c \geqq \alpha_2$ *implies that* $\alpha(\infty)$ *exists and that*

$$\lim_{n\to\infty}\int_{x_1}^{\infty} e^{-cx}\,E_{1,n}(D)([E_2(D)]^{-1}f(x))\,dx = \alpha(\infty) - \alpha(x_1)\,.$$

COROLLARY 4.2. *If*

1. $G_1(t)$, $G_2(t) \in$ *class III,*
2. $\varphi(t)$ *is defined for* $T < t < \infty$ *and is integrable on every finite interval* $T < t_1 \leqq t \leqq t_2 < \infty$,
3. $f(x) = \int_{-\infty}^{\infty} G(x-t)\,\varphi(t)\,dt$ *converges for* $x > T + b + \sum_1^\infty a_k^{-1}$,
4. $\varphi(t)$ *is continuous at* $t = x$ $(x > T)$,

then

$$\lim_{n\to\infty} E_{1,n}(D)([E_2(D)]^{-1}f(x)) = \varphi(x)\,.$$

5. Applications. (a) The Riemann-Liouville fractional integral equation of order $\nu+1$ $(\nu > -1)$

$$H(X) = \frac{1}{\Gamma(\nu+1)}\int_0^X (X-T)^\nu\,\Phi(T)\,dT \quad (X > 0) \tag{1}$$

becomes after an exponential change of variables X and T, putting $f(x) = H(e^x)\, e^{-(\nu+1)x}$ and $\varphi(t) = \Phi(e^t)$,

(2) $$f(x) = \int_{-\infty}^{\infty} G(x-t)\,\varphi(t)\,dt$$

(3) $$G(t) = \begin{cases} \dfrac{1}{\Gamma(\nu+1)}[1-e^{-t}]^{\nu} e^{-t} & t \geqq 0 \\ 0 & t < 0\,. \end{cases}$$

The inversion function $F(s)$ of integral equation (2) as a convolution transform is given by

$$[F(s)]^{-1} = \int_{-\infty}^{\infty} G(t)\, e^{-st}\, dt = \frac{1}{\Gamma(\nu+1)} \int_0^{\infty} [1-e^{-t}]^{\nu} e^{-st}\, dt = \Gamma(s+1)/\Gamma(s+\nu+2)$$

for $\sigma > -1$, $s = \sigma + i\tau$. By Stirling's formula

$$\Gamma(\sigma+i\tau) \sim \sqrt{2\pi}\, e^{-\pi|\tau|/2} |\tau|^{\sigma-1/2} \quad |\tau| \to \infty\,,$$

$$|F(\sigma+i\tau)|^{-1} = O(|\tau|^{-\nu-1}) \quad |\tau| \to \infty$$

for all σ. Hence, if $\nu > 1$, $F(s)$ satisfies the condition (4b) of §1. $F(s)$ has zeros at $-k$ $(k=1, 2, \cdots)$ and poles $-(\nu+k)$ $(k=1, 2, \cdots)$, and satisfies the condition (4a) of §1. Therefore, our general theorem is applicable to this transform (2). However, when the transform (1) converges for any ν $(-1 < \nu \leqq 1)$, $H_\beta(X)$, the β-th integral of $H(x)$, always exists and

$$H_\beta(X) = \frac{1}{\Gamma(\nu+\beta+1)} \int_0^X (X-T)^{\nu+\beta}\, \Phi(T)\, dT\,.$$

Hence, if we choose β such that $\beta > 2 - \nu$, then our general theorem is applicable to this integral equation as a convolution transform [6].

We may similarly discuss the Weyl's fractional integral of order $\nu+1$ $(\nu > -1)$

$$H(X) = \frac{1}{\Gamma(\nu+1)} \int_X^{\infty} (T-X)^{\nu}\, \Phi(T)\, dT\,.$$

In this case,

(4) $$G(t) = \begin{cases} \dfrac{1}{\Gamma(\nu+1)}[1-e^{t}]^{\nu} e^{t} & t < 0 \\ 0 & t \geqq 0 \end{cases}$$

and $F(s)=\Gamma(\nu+2-s)/\Gamma(1-s)$, hence we may regard as $E_1(s)=[\Gamma(1-s)]^{-1}$, $E_2(s)=[\Gamma(\nu+2-s)]^{-1}$, $G_1(t)=\exp(-e^t)e^t$. Let us take $\varphi(t)=\exp(e^{e^{-t}})$, then the transform (1) with the kernel function (4) converges but $\int_{-\infty}^{\infty}G_1(x-t)\varphi(t)\,dt$ does not exist for any x $(-\infty<x<\infty)$. From this fact we see that some order condition for $\varphi(t)$ is necessary for our inversion theorem.

(b) More generally, the integral equation

$$(5)\qquad H(X)=\frac{1}{\Gamma(\nu+1)}\int_0^X (X^h-T^h)^\nu\,\Phi(T)\,dT \quad (h>0, X>0, \nu>-1)$$

becomes, putting $f(x)=H(e^x)\,e^{-(h\nu+1)x}$, $\varphi(t)=\Phi(e^t)$,

$$(6)\qquad f(x)=\int_{-\infty}^{\infty}G(x-t)\,\varphi(t)\,dt,$$

$$G(t)=\begin{cases}\dfrac{1}{\Gamma(\nu+1)}(1-e^{-ht})^\nu e^{-t} & t\geqq 0\\ 0 & t<0.\end{cases}$$

The inversion function $F(s)$ of transform (6) is given by

$$[F(s)]^{-1}=\frac{1}{\Gamma(\nu+1)}\int_0^\infty[1-e^{-ht}]^\nu e^{-t}e^{-st}\,dt=\Gamma\left(\frac{s+1}{h}\right)\Big/h\Gamma\left(\nu+1+\frac{s+1}{h}\right)$$

$$\sigma>-1.$$

$F(s)$ has zeros at $-(kh+1)$ and poles $-(kh+h\nu+1)$ $(k=1,2,\cdots)$ and $|F(\sigma+i\tau)|^{-1}=O(|\tau|^{-\nu-1})$ as $|\tau|\to\infty$ for all σ. Therefore, if $\nu>1$, $F(s)$ satisfies the condition (4a), (4b) of §1 and our theorem applicable to (6). However, we may similarly discuss the integral equation (5) for $-1<\nu\leqq 1$ as in example (a).

(c) T.P.Higgins [2] introduced a hypergeometric function transform

$$H(X)=\frac{X^{-b}}{\Gamma(c)}\int_X^1 (T-X)^{c-1}F(a,b,c\,;\,1-T/X)\,Y(T)\,dT \qquad (X>0)$$

and obtained the inversion formula. More generally, we introduce the integral transform

$$H(X)=\frac{X^{-b}}{\Gamma(c)}\int_X^\infty (T-X)^{c-1}F(a,b;c\,;\,1-T/X)\,Y(T)\,dT \qquad (X>0)\,.$$

This becomes after an exponential change of variables, putting $f(x)=H(e^x)e^{bx}$, $\varphi(t)=Y(e^t)e^{ct}$,

$$(7) \qquad f(x)=\int_{-\infty}^\infty G(x-t)\,\varphi(t)\,dt,$$

$$G(t)=\begin{cases}\dfrac{1}{\Gamma(c)}(1-e^t)^{c-1}F(a,b;c;1-e^{-t}) & t<0\\ 0 & t\geqq 0\end{cases}.$$

The inversion function of (7) is given by

$$\begin{aligned}[F(s)]^{-1}&=\frac{1}{\Gamma(c)}\int_{-\infty}^0(1-e^t)^{c-1}F(a,b;c;1-e^{-t})\,e^{-st}\,dt\\ &=\frac{\Gamma(a-s)\,\Gamma(b-s)}{\Gamma(c-s)\,\Gamma(a+b-s)},\end{aligned}$$

provided that $c>0$, $\Re s<a$ and $\Re s<b$.

$F(s)$ has zeros at $a, a+1, a+2, \cdots; b, b+1, b+2, \cdots$ and poles at $c, c+1, c+2, \cdots; a+b, a+b+1, a+b+2, \cdots$; and $|F(\sigma+i\tau)|^{-1}=O(|\tau|^{-c})$ as $|\tau|\to\infty$ for all σ. Hence, for instance, if $c>2$, $c>b>0$, $c>a+b$, then our theorem is applicable to this transform. We may also discuss for $0<c\leqq 2$ in usual way. Further, we may similarly discuss the integral transform

$$H(X)=\frac{1}{\Gamma(c)}\int_0^X (X-T)^{c-1}F(a,b;c;1-T/X)\,Y(T)\,dT \qquad (X>0)\,.$$

(d) K. N. Srivastava [5] introduced the integral equation

$$H(X)=\int_X^1 (T^2-X^2)^{k+\alpha}\,I(n,k+\alpha,-\beta,T/X)\,\Phi(T)\,dT,$$

where $I(n,k+\alpha,-\beta,T/X)$ is Jacobi polynomial and $=F(-n,n+k+\alpha-\beta+1; k+\alpha+1; 1-T^2/X^2)$, n is positive integer, k is zero or positive integer, $k>\alpha>-1$ and $-1<\beta<1$. Under some conditions for $H(X)$ he obtained the inversion formula, using the relations of fractional integrations. More generally, we introduce the integral equation

$$H(X) = \int_X^\infty (T^2-X^2)^{k+\alpha} F(-n, n+k+\alpha-\beta+1; k+\alpha+1; 1-T^2/X^2)\, \Phi(T)\, dT.$$

This becomes after an exponential change of variables, putting $f(x)=e^{2(n+1)x}F(e^x)$, $\varphi(t)=\Phi(e^t)$, the convolution transform with the kernel

$$G(t) = \begin{cases} (1-e^{2t})^{k+\alpha} F(-n, n+k+\alpha-\beta+1; k+\alpha+1; 1-e^{-2t})\, e^{2(n+1)t} & t<0 \\ 0 & t \geqq 0 \end{cases}$$

and the inversion function

$$F(s) = \frac{2\Gamma\left(n+k+\alpha+2-\frac{s}{2}\right)\Gamma\left(n+k+\alpha-\beta+2-\frac{s}{2}\right)}{\Gamma(k+\alpha+1)\Gamma\left(1-\frac{s}{2}\right)\Gamma\left(2n+k+\alpha-\beta+2-\frac{s}{2}\right)}.$$

$F(s)$ has zeros at $2(m+1)$, $2(2n+k+\alpha-\beta+2)+2m$ $(m=0,1,2,\cdots)$; and poles at $2(n+k+\alpha+2)+2m$; $2(n+k+\alpha-\beta)+2m$ $(m=0,1,2,\cdots)$.

Since $|F(\sigma+i\tau)|^{-1} = O(|\tau|^{-k-\alpha-2})$ as $|\tau| \to \infty$ for all σ, our theorem can be applied to this transform, if necessary, after some modification.

6. In this section, concerning to the last condition of (4a) of §1, we show that even if we remove this condition, our method cannot be applied, in general, without some order condition for $\varphi(t)$. To see this, we introduce the integral transform

$$H(X) = \int_0^\infty \exp\left[-\frac{1}{4}X^2T^2\right] D_{-\nu}(XT)\, \Phi(T)\, dT \qquad X>0, \nu>0,$$

where $D_{-\nu}(z)$ is the parabolic cylinder function [1, p. 116]. This becomes after an exponential change of variables X and T

$$(1) \qquad H(e^x)\, e^x = \int_{-\infty}^\infty G(x-t)\, \Phi(e^{-t})\, dt,$$

$$G(t) = \exp\left(-\frac{1}{4}e^{2t}\right) D_{-\nu}(e^t)\, e^t.$$

Since for $\Re s < 1$

$$\int_{-\infty}^{\infty} e^{-st} \exp\left(-\frac{1}{4}e^{2t}\right) D_{-\nu}(e^t)\, e^t\, dt = \int_0^{\infty} e^{-\frac{1}{4}v^2} D_{-\nu}(v)\, v^{-s}\, dv$$

$$= \frac{\pi^{1/2}\Gamma(1-s)}{2^{\frac{1+\nu-s}{2}}\Gamma\left(\frac{1+\nu-s}{2}\right)},$$

the inversion function $F(s)$ is the reciprocal of the last function and we may suppose $E_1(s)$ be $[\Gamma(1-s)]^{-1}$, which is the inversion function of the Laplace transform as a convolution transform [4, p. 66]. It is obvious that $F(s)$ has zeros at k, poles at $\nu+2k$ $(k=1, 2, \cdots)$ and satisfies the condition (4b) of §1 but not (4a). Now, if we take $e^{(1/4)T^2}$ for $\Phi(T)$, that is, $\varphi(t) = \exp(1/4\, e^{-2t})$, then $\int_{-\infty}^{\infty} G_1(x-t)\, \varphi(t)\, dt$ diverges, because $\int_0^{\infty} e^{-XT} e^{\frac{1}{4}T^2}\, dT = \infty$ for all X.

On the other hand, since [1, p. 122]

$$D_\nu(z) \sim e^{-\frac{z^2}{4}} z^\nu \qquad |z| \to \infty,$$

it is clear that

$$\int_{-\infty}^{\infty} \exp\left[-\frac{1}{4}X^2T^2\right] D_{-\nu}(XT)\, e^{\frac{1}{4}T^2}\, dT$$

converges for $X > 2^{-1/2}$ and accordingly the integral (1) converges for $x > \log 2^{-1/2}$. This difficulties may be caused by the fact that $G(t)$ decreases more rapidly than $G_1(t)$ as $t \to \infty$.

It is shown by direct computations that if $\alpha(t)$ is of bounded variation in every finite interval and (1) converges (conditionally) at $x = x_0$, then (1) converges uniformly for x in any finite interval bounded on the left by x_0. Therefore, what is interesting is that (1) has an abscissa of covergence. Of course, if the transform (1) and $\int_{-\infty}^{\infty} G_1(x-t)\, \varphi(t)\, dt$ converges absolutely, then Widder's method [9] and ours can be applied.

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