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WEAKENED BERTRAND CURVES

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1. Introduction. In the discussion of Bertrand curves (in fact nearly all curves) in elementary classical differential geometry, it is always assumed (explicitly or implicitly) that the curvature is nowhere zero. In this paper we drop this requirement on the Bertrand curves and investigate into the properties of two types of similar curves (the Frenet-Bertrand curves and the weakened Bertrand curves) under weakened conditions. The properties of the Frenet-Bertrand curves turn out to be strikingly similar to those of the Bertrand curves.

We first put down our convention of defining regular curves. We also use the convention that if f is a real function on a set A, by $f=0$ we mean that f is everywhere zero on A, and by $f\neq 0$ we mean that f is nowhere zero on A.

DEFINITION 1.1. A *parametrized curve* (simply called a *curve*) in E^3 is a point set Γ in E^3 together with an equivalence class of continuous and locally injective surjections

$$(\phi, \psi, \chi):\ L \to \Gamma$$

defined by

$$(\phi, \psi, \chi)(t) = (\phi(t), \psi(t), \chi(t)),$$

where two such mappings (ϕ, ψ, χ), $(\widetilde{\phi}, \widetilde{\psi}, \widetilde{\chi})$ which are defined on two intervals L, $\widetilde{L}$ respectively are said to be equivalent if there exists a continuous, strictly monotonic increasing function

$$\sigma:\ L \to \widetilde{L}$$

with $\widetilde{L}$ as image set (σ must then be a bijection from L onto $\widetilde{L}$) such that

$$(\phi, \psi, \chi) = (\widetilde{\phi}, \widetilde{\psi}, \widetilde{\chi}) \circ \sigma .$$

Each of the mappings (ϕ, ψ, χ) is called a *parametrization* of Γ.

DEFINITION 1.2. A C^r *curve* ($r = 1, 2, \cdots$, or ∞ or ω) is a curve which admits a parametrization (ϕ, ψ, χ) in which ϕ, ψ, χ are of class C^r.

DEFINITION 1.3. A C^r *regular curve* is a curve which admits a parametrization (ϕ, ψ, χ) in which ϕ, ψ, χ are of class C^r and $\dot{\phi}^2+\dot{\psi}^2+\dot{\chi}^2 \neq 0$ throughout the interval L.

The following proposition is easily established:

PROPOSITION 1.1. *Each of the following conditions is necessary and sufficient for the curve* Γ *to be a* C^r *regular curve*:

(i) *Each point of* Γ *has a neighbourhood in which the curve can be represented (in a suitable co-ordinate system) by equations of the form*

$$y = f(x), \qquad z = g(x),$$

where f, g *are functions of class* C^r.
(ii) Γ *admits a parametrization* (ϕ, ψ, χ) *in terms of its arc length* s, *where* ϕ, ψ, χ *are of class* C^r. *(It follows that* $\phi'^2+\psi'^2+\chi'^2=1$, *where the dash denotes differentiation with respect to* s*).*

REMARK 1. A C^1 regular curve which is also a C^r curve need not be a C^r regular curve if $r > 1$. For example, consider the analytic curve Γ:

$$x = t^3, \quad y = t^4, \quad z = 0,$$

which is a C^1 regular curve because it admits the representation

$$y = x^{4/3}, \quad z = 0,$$

and the function $x^{4/3}$ is of class C^1. However, Γ cannot be a C^r regular curve for any $r > 1$.

REMARK 2. A C^r regular curve can have a parametrization (ϕ, ψ, χ) which is of class C^r and such that $\dot{\phi}^2+\dot{\psi}^2+\dot{\chi}^2$ is zero at some points. For example, consider the straight line

$$x = t^3, \quad y = 0, \quad z = 0.$$

In the following, we shall only consider C^∞ regular curves, although the results obtained also hold for any C^r regular curve with sufficiently large r.

DEFINITION 1.4. A *Bertrand curve* Γ: $\boldsymbol{x}(s)$, $s \in L$ is a C^∞ regular curve with non-zero curvature for which there exists another (different) C^∞ regular curve $\tilde{\Gamma}$: $\tilde{\boldsymbol{x}}(s)$, where $\tilde{\boldsymbol{x}}(s)$ is of class C^∞ and $\tilde{\boldsymbol{x}}'(s) \neq 0$ (s being the arc length of Γ only), also with non-zero curvature, in bijection with it in such a manner that the principal normals to Γ, $\tilde{\Gamma}$ at each pair of corresponding points coincide with the line joining the corresponding points. The curve $\tilde{\Gamma}$ is called a *Bertrand conjugate* of Γ.

It easily follows that the relation between a Bertrand curve and its conjugate is symmetric.

In this paper we weaken the conditions. We shall adopt the definition of a C^∞ *Frenet curve* ([2]) as a C^∞ regular curve Γ: $\boldsymbol{x}(s)$, $s \in L$, for which there exists a C^∞ family of Frenet frames, that is, right-handed orthonormal frames $\boldsymbol{t}(s)$, $\boldsymbol{n}(s)$, $\boldsymbol{b}(s)$, where $\boldsymbol{t}(s) = \boldsymbol{x}'(s)$, satisfying the Frenet equations

$$\begin{cases} \boldsymbol{t}' = & k_1\boldsymbol{n}\,, \\ \boldsymbol{n}' = -k_1\boldsymbol{t} & + k_2\boldsymbol{b}\,, \\ \boldsymbol{b}' = & -k_2\boldsymbol{n}\,, \end{cases}$$

for some C^∞ scalar functions $k_1(s)$, $k_2(s)$ which are called the curvature and pseudo-torsion of Γ.

DEFINITION 1.5. A *Frenet-Bertrand curve* Γ: $\boldsymbol{x}(s)$ (briefly called a *FB* curve) is a C^∞ Frenet curve for which there exists another C^∞ Frenet curve $\tilde{\Gamma}$: $\tilde{\boldsymbol{x}}(s)$, where $\tilde{\boldsymbol{x}}(s)$ is of class C^∞ and $\tilde{\boldsymbol{x}}'(s) \neq 0$, in bijection with it so that, by suitable choice of the Frenet frames (see [4]), the principal normal vectors $\boldsymbol{n}$, $\tilde{\boldsymbol{n}}$ at corresponding points on Γ, $\tilde{\Gamma}$ both lie on the line joining the corresponding points. The curve $\tilde{\Gamma}$ is called a *FB conjugate* of Γ.

Again the relation between a *FB* curve and its conjugate is symmetric.

DEFINITION 1.6. A *weakened Bertrand curve* Γ : $\boldsymbol{x}(s)$, $s \in L$ (briefly called a *WB* curve) is a C^∞ regular curve for which there exists another C^∞ regular curve $\tilde{\Gamma}$: $\tilde{\boldsymbol{x}}(\tilde{s})$, $\tilde{s} \in \tilde{L}$, where $\tilde{s}$ is the arc length of $\tilde{\Gamma}$, and a homeomorphism $\sigma : L \to \tilde{L}$ such that

(i) there exist two (disjoint) closed subsets Z, N of L with void interiors such that $\sigma \in C^\infty$ on $L \backslash N$, $(d\tilde{s}/ds) = 0$ on Z, $\sigma^{-1} \in C^\infty$ on $\sigma(L \backslash Z)$, and $(ds/d\tilde{s}) = 0$ on $\sigma(N)$,

(ii) the line joining corresponding points s, $\tilde{s}$ of Γ and $\tilde{\Gamma}$ is orthogonal to Γ

and $\tilde{\Gamma}$ at the points $s, \tilde{s}$ respectively, and is along the principal normal to Γ or $\tilde{\Gamma}$ at the points $s, \tilde{s}$ whenever it is well defined. The curve $\tilde{\Gamma}$ is called a *WB conjugate* of Γ.

Thus for a WB curve we not only drop the requirement of Γ being a Frenet curve, but also allow $(d\tilde{s}/ds)$ to be zero on a subset with void interior ($(d\tilde{s}/ds)=0$ on an interval would destroy the injectivity of the mapping σ). Since $(d\tilde{s}/ds)=0$ implies that $(ds/d\tilde{s})$ does not exist, the apparently artificial requirements in (i) are in fact quite natural. The relation between a WB curve and its conjugate is again symmetric.

It is clear that a Bertrand curve is necessarily a FB curve, and a FB curve is necessarily a WB curve. It will be proved in Theorem 3.1 that under certain conditions a WB curve is also a FB curve, while it will be clear from Lemma 2.2 that a FB curve need not be a Bertrand curve.

2. Frenet-Bertrand curves. In this section we study the structure and characterization of FB curves. We begin with a lemma, the method used in which is classical (see [3]).

LEMMA 2.1. *Let Γ: $\boldsymbol{x}(s)$, $s \in L$ be a FB curve and $\tilde{\Gamma}$: $\tilde{\boldsymbol{x}}(s)$ a FB conjugate of Γ. Let all quantities belonging to $\tilde{\Gamma}$ be marked with a tilde. Let*

$$\tilde{\boldsymbol{x}}(s) = \boldsymbol{x}(s) + \lambda(s)\,\boldsymbol{n}(s)\,. \tag{2.1}$$

Then the distance $|\lambda|$ between corresponding points of Γ, $\tilde{\Gamma}$ is constant, and there is a constant angle α such that $\boldsymbol{t}\cdot\tilde{\boldsymbol{t}} = \cos\alpha$ and

$$\text{(i)}\qquad (1-\lambda k_1)\sin\alpha = \lambda k_2\cos\alpha\,,$$

$$\text{(ii)}\qquad (1+\varepsilon\lambda\tilde{k}_1)\sin\alpha = \lambda\tilde{k}_2\cos\alpha\,,\ \textit{with}\ \varepsilon=\pm 1,$$

$$\text{(iii)}\qquad (1-\lambda k_1)(1+\varepsilon\lambda\tilde{k}_1) = \cos^2\alpha\,,$$

$$\text{(iv)}\qquad k_2\tilde{k}_2 = \frac{1}{\lambda^2}\sin^2\alpha\,.$$

PROOF. From (2.1) it follows that $\lambda(s) = (\tilde{\boldsymbol{x}}(s) - \boldsymbol{x}(s))\cdot\boldsymbol{n}(s)$ is of class C^∞. Differentiation of (2.1) with respect to s gives

$$\tilde{s}'\tilde{\boldsymbol{t}} = (1-\lambda k_1)\boldsymbol{t} + \lambda'\boldsymbol{n} - \lambda k_2\boldsymbol{b}\,. \tag{2.2}$$

Since by hypothesis we have $\tilde{\boldsymbol{n}} = \varepsilon\boldsymbol{n}$ with $\varepsilon=\pm 1$, scalar multiplication of (2.2) by $\boldsymbol{n}$ gives

$$\lambda' = 0, \qquad \lambda = \text{constant}.$$

Therefore

$$\tilde{s}'\tilde{\boldsymbol{t}} = (1-\lambda k_1)\boldsymbol{t} + \lambda k_2 \boldsymbol{b}\,. \tag{2.3}$$

But by definition of FB curve we have $\tilde{s}' \neq 0$, so that $\tilde{\boldsymbol{t}}$ is C^∞ function of s. Hence

$$\begin{aligned}\frac{d}{ds}(\boldsymbol{t}\cdot\tilde{\boldsymbol{t}}) &= k_1 \boldsymbol{n}\cdot\tilde{\boldsymbol{t}} + \boldsymbol{t}(\tilde{k}_1\tilde{s}'\tilde{\boldsymbol{n}}) \\ &= 0\,.\end{aligned}$$

Consequently $\boldsymbol{t}\cdot\tilde{\boldsymbol{t}}$ is constant, and there exists a constant angle α such that

$$\tilde{\boldsymbol{t}} = \cos\alpha \boldsymbol{t} + \sin\alpha \boldsymbol{b}\,. \tag{2.4}$$

Taking the vector product of (2. 3) and (2. 4), we obtain

$$(1-\lambda k_1)\sin\alpha = \lambda k_2 \cos\alpha\,,$$

which is (i).

Now write

$$\boldsymbol{x} = \tilde{\boldsymbol{x}} - \varepsilon\lambda\tilde{\boldsymbol{n}}\,.$$

Therefore

$$\boldsymbol{t} = \tilde{s}'[(1+\varepsilon\lambda\tilde{k}_1)\tilde{\boldsymbol{t}} - \varepsilon\lambda\tilde{k}_2\tilde{\boldsymbol{b}}]\,. \tag{2.3'}$$

On the other hand, equation (2. 4) gives

$$\begin{aligned}\tilde{\boldsymbol{b}} &= \tilde{\boldsymbol{t}} \times \tilde{\boldsymbol{n}} \\ &= -\varepsilon\sin\alpha\boldsymbol{t} + \varepsilon\cos\alpha\boldsymbol{b}\,.\end{aligned}$$

Using (2. 4) again, we get

$$\boldsymbol{t} = \cos\alpha\,\tilde{\boldsymbol{t}} - \varepsilon\sin\alpha\,\tilde{\boldsymbol{b}}\,. \tag{2.4'}$$

Taking the vector product of (2. 3′) and (2. 4′), we obtain

$$(1+\varepsilon\lambda\tilde{k}_1)\sin\alpha = \lambda\tilde{k}_2\cos\alpha\,,$$

which is (ii).

On the other hand, comparison of (2.3) and (2.4) gives

$$1-\lambda k_1=\tilde{s}'\cos\alpha, \tag{2.5}$$

$$\lambda k_2=\tilde{s}'\sin\alpha. \tag{2.6}$$

Similarly (2.3′), (2.4′) give

$$\tilde{s}'(1+\varepsilon\lambda\tilde{k}_1)=\cos\alpha, \tag{2.5′}$$

$$\tilde{s}'\lambda\tilde{k}_2=\sin\alpha. \tag{2.6′}$$

The properties (iii) and (iv) then easily follow from (2.5) and (2.5′), (2.6) and (2.6′).

LEMMA 2.2. *A necessary and sufficient condition for a C^∞ regular curve Γ to be a FB curve with a FB conjugate which is a line-segment is that Γ should be either a line-segment or a non-planar circular helix.*

PROOF. *Necessity*: Let Γ: $\boldsymbol{x}(s)$ have a *FB* conjugate $\tilde{\Gamma}$: $\tilde{\boldsymbol{x}}(s)$ which is a line-segment. Then $\tilde{k}_1=0$. Using Lemma 2.1 (iii) and (i), (ii), we have

$$1-\lambda k_1=\cos^2\alpha, \tag{2.7}$$

and then

$$\cos^2\alpha\sin\alpha=\lambda k_2\cos\alpha, \tag{2.8}$$

$$\sin\alpha=\lambda\tilde{k}_2\cos\alpha. \tag{2.9}$$

From (2.9) it follows that $\cos\alpha\neq 0$. Hence (2.8) is equivalent to

$$\lambda k_2=\sin\alpha\cos\alpha. \tag{2.8′}$$

Case 1. $\sin\alpha=0$. Then $\cos\alpha=\pm 1$, so that (2.7) implies that $k_1=0$, and Γ is a line-segment. We note also that (2.8′) implies that $k_2=0$.

Case 2. $\sin\alpha\neq 0$. Then $\cos\alpha\neq\pm 1$, and (2.7), (2.8′) imply that k_1, k_2 are non-zero constants, and Γ is a non-planar circular helix.

Sufficiency: If Γ is a non-planar circular helix

$$\boldsymbol{x}=(a\cos t,\ a\sin t,\ bt),\quad \text{where}\quad t=s/\sqrt{a^2+b^2},$$

we may take

$$\boldsymbol{n} = (-\cos t, \sin t, 0)\,. \quad \text{Now put } \lambda = a\,.$$

Then the curve $\widetilde{\Gamma}$ with

$$\widetilde{\boldsymbol{x}} = \boldsymbol{x} + \lambda \boldsymbol{n}$$

will be a line-segment along the z-axis, and can be made into a *FB* conjugate of Γ if $\widetilde{\boldsymbol{n}}$ is defined to be equal to $\boldsymbol{n}$.

The case where Γ is a line-segment is trivial.

DEFINITION 2.1. By a *plane arc* we mean a plane curve which contains no line-segments.

LEMMA 2.3. *If a FB curve Γ has a FB conjugate which is a plane arc, then Γ is a plane arc on the same plane.*

PROOF. Let a *FB* curve Γ: $\boldsymbol{x}(s)$, $s \in L$, have a *FB* conjugate $\widetilde{\Gamma}$: $\widetilde{\boldsymbol{x}}(s)$ which is a plane arc. Since $\widetilde{\Gamma}$ does not contain any line-segments, the (closed) subset M of L on which $\widetilde{k}_1 = 0$ has a void interior, so that $L\backslash M$ is dense in L. But since $\widetilde{\Gamma}$ is a plane curve, we must have

$$\widetilde{s}'^6\, \widetilde{k}_1^2 \widetilde{k}_2 = |\widetilde{\boldsymbol{x}}', \widetilde{\boldsymbol{x}}'', \widetilde{\boldsymbol{x}}'''| = 0$$

(see [4]). Therefore $\widetilde{k}_2 = 0$ on $L\backslash M$ and hence also on L, by continuity. Therefore Lemma 2.1 (iv), (i) give first

$$\sin \alpha = 0\,,$$

and then

$$k_2 = 0\,.$$

Thus Γ is a plane curve. Moreover, Γ cannot contain any line-segments, otherwise by Lemma 2.2 $\widetilde{\Gamma}$ must also contain a line-segment. Thus Γ is a plane arc, which obviously lies on the same plane as $\widetilde{\Gamma}$.

DEFINITION 2.2. A *Bertrand arc* is a *FB* curve which contains no line-segments or plane arcs and whose *FB* conjugate also has this property.

Notations: We use the symbols $\mathfrak{L}$, $\mathfrak{P}$, $\mathfrak{H}$, $\mathfrak{B}$ to denote line-segment, plane arc, non-planar circular helix and Bertrand arc respectively.

DEFINITION 2.3. A *FB* curve is said to be of $\mathfrak{H}\mathfrak{L}$ type if it is a non-planar circular helix and its *FB* conjugate is a line-segment. Similarly we define *FB* curves of $\mathfrak{L}\mathfrak{H}$, $\mathfrak{L}\mathfrak{L}$, $\mathfrak{P}\mathfrak{P}$ and $\mathfrak{B}\mathfrak{B}$ types.

Lemma 2.2 can then be restated: a *FB* curve which is a line-segment is of either $\mathfrak{L}\mathfrak{L}$ or $\mathfrak{L}\mathfrak{H}$ type. On the other hand, a *FB* curve which is a non-planar circular helix need not be of $\mathfrak{H}\mathfrak{L}$ type, for it is well-known that a non-planar circular helix has c Bertrand conjugates (c being the cardinality of the continuum) which are also non-planar circular helices.

Theorems 2.1 and 2.2 below show that, apart from the cases treated in Lemma 2.2, a *FB* curve either is a plane curve consisting of arcs of $\mathfrak{L}\mathfrak{L}$ or $\mathfrak{P}\mathfrak{P}$ type or is a non-planar curve consisting of arcs of $\mathfrak{H}\mathfrak{L}$, $\mathfrak{L}\mathfrak{H}$ or $\mathfrak{B}\mathfrak{B}$ type.

We shall have to use the following ([4])

LEMMA A. *Let $f_1, \cdots, f_n$ be continuous (real-valued) functions of which f_1 is defined on a proper interval L of the real line, and f_i $(2 \leqq i \leqq n)$ is defined on the set $G_{i-1} \equiv \{s \in L: f_1(s) \neq 0, \cdots, f_{i-1}(s) \neq 0\}$. Then there exist $n+1$ open sets $B_1, B_2, \cdots, B_n, G_n$ of L with the following properties:*

$$f_1 = 0 \quad on \quad B_1,$$

$$f_1 \neq 0 \quad and \quad f_2 = 0 \quad on \quad B_2,$$

$$\cdots\cdots\cdots\cdots\cdots$$

$$f_1 \neq 0, \quad f_2 \neq 0, \cdots, \quad f_{n-1} \neq 0 \quad and \quad f_n = 0 \quad on \quad B_n,$$

$$f_1 \neq 0, \quad f_2 \neq 0, \cdots, \quad and \quad f_n \neq 0 \quad on \quad G_n;$$

and

$$\bar{L} = \bar{B}_1 \cup \bar{B}_2 \cup \cdots \cup \bar{B}_n \cup \bar{G}_n,$$

where the closure operation is taken in R. Thus, the component intervals of $B_1, B_2, \cdots, B_n, G_n$, taken together, form a countable family of disjoint proper intervals each of which is open in L, and the union of these component intervals is a dense subset of L.

THEOREM 2.1. *If a FB curve Γ contains a plane arc, then* (i) *it is a plane curve with zero pseudo-torsion and has a dense subset which is the union of a countable number of FB curves of $\mathfrak{L}\mathfrak{L}$ or $\mathfrak{P}\mathfrak{P}$ type, and the FB conjugate $\tilde{\Gamma}$ lies on the same plane as Γ;* (ii) *the curvatures of Γ and of $\tilde{\Gamma}$ are bounded below or bounded above.*

PROOF. (i). Let Γ: $\boldsymbol{x}(s)$, $s \in L$ be a *FB* curve which contains a plane arc corresponding to an interval $I \subset L$. Let $\widetilde{\Gamma}$: $\widetilde{\boldsymbol{x}}(s)$ be the FB conjugate of Γ under discussion.

As proved in Lemma 2.3, we have $\sin\alpha=0$ (see notations in Lemma 2.1). Consequently by Lemma 2.1 (i) and (ii), $k_2 = \widetilde{k_2} = 0$ on L, and Γ, $\widetilde{\Gamma}$ are plane curves on the same plane.

By Lemma A, L has a dense subset which is the union of two countable families $\mathfrak{B}_1$, $\mathfrak{B}_2$ of disjoint intervals open in L such that $k_1=0$ on each member of $\mathfrak{B}_1$ and $k_1\neq 0$ on each member of $\mathfrak{B}_2$. By Lemma 2.2, the part of $\widetilde{\Gamma}$ corresponding to any component interval of $\mathfrak{B}_1$ is a line-segment, and by Lemma 2.3, the part of $\widetilde{\Gamma}$ corresponding to any component interval of $\mathfrak{B}_2$ is a plane arc.

(ii) Since $\sin\alpha = 0$, $\cos\alpha = \pm 1$. Hence by Lemma 2.1 (iii) we have $1-\lambda k_1 \neq 0$. It follows that k_1 is either bounded below or bounded above by $\dfrac{1}{\lambda}$. Similar arguments apply to $\widetilde{\Gamma}$.

THEOREM 2.2. *If a FB curve Γ contains no plane arc and is not a line segment, then* (i) *it is non-planar and has a dense subset which is the union of a countable number of FB curves of $\mathfrak{L}\mathfrak{H}$, $\mathfrak{H}\mathfrak{L}$ or $\mathfrak{B}\mathfrak{B}$ type,* (ii) *the pseudo-torsion k_2 is nowhere zero.*

PROOF. Let Γ: $\boldsymbol{x}(s)$, $s \in L$ be a FB curve which contains no plane arcs and is not a line-segment. Then there exists $s_0 \in L$ such that $k_1(s_0) \neq 0$. In a neighbourhood of s_0 on which $k_1\neq 0$, we have at least one point where k_2 is non-zero, otherwise this part of Γ would be a plane arc. It follows from Lemma 2.1 (i) that $\sin\alpha\neq 0$, and then from Lemma 2.1 (iv) and (iii) that k_2 is nowhere zero on L and that k_1, $\widetilde{k}_1$ cannot be both zero at any point. The latter fact shows that Γ contains no arc of $\mathfrak{L}\mathfrak{L}$ type. It follows from Lemma 2.2 that any interval of L on which $k_1=0$ corresponds to an arc of $\mathfrak{L}\mathfrak{H}$ type, and any interval on which $\widetilde{k}_1=0$ corresponds to an arc of $\mathfrak{H}\mathfrak{L}$ type.

Now consider an interval $I\subset L$ on which $k_1\neq 0$, $\widetilde{k_1}\neq 0$. The corresponding parts Γ_1, $\widetilde{\Gamma}_1$ of Γ, $\widetilde{\Gamma}$ then contain no line-segments. Since they also cannot contain any plane arcs by hypothesis and Lemma 2.3, Γ_1 is a Bertrand arc of $\mathfrak{B}\mathfrak{B}$ type. Moreover, since Γ is not a line-segment, it must contain at least one arc of $\mathfrak{H}\mathfrak{L}$ or $\mathfrak{B}\mathfrak{B}$ type, and is therefore non-planar.

The proof is then completed by applying Lemma A with $n = 2$, $f_1 = k_1$, $f_2 = \widetilde{k}_1$.

REMARK. It should be noted that although a non-planar FB curve may contain arcs of $\mathfrak{L}\mathfrak{H}$ and $\mathfrak{H}\mathfrak{L}$ types, no two of these arcs can be directly joined together at a point. This follows immediately from consideration of the continuity of k_1 (or $\widetilde{k}_1$). It also follows from Lemma 2.1 (i) that for a Bertrand curve containing an arc of $\mathfrak{L}\mathfrak{H}$ type, $\cos\alpha \neq 0$.

Following the methods of [4], it is not difficult to construct a *FB* curve with a countably infinite number of arcs of $\mathfrak{H}\mathfrak{L}$, $\mathfrak{L}\mathfrak{H}$ and $\mathfrak{B}\mathfrak{B}$ types. We need only to notice that on an arc of $\mathfrak{L}\mathfrak{H}$ (respectively $\mathfrak{H}\mathfrak{L}$) type, we have $(k_1, k_2, \widetilde{k}_1, \widetilde{k}_2) = \left(0, \frac{1}{\lambda}\tan\alpha, \frac{1}{\lambda}\sin^2\alpha, \frac{1}{\lambda}\sin\alpha\cos\alpha\right)$ (respectively $= \left(\frac{1}{\lambda}\sin^2\alpha, \frac{1}{\lambda}\sin\alpha\cos\alpha, 0, \frac{1}{\lambda}\tan\alpha\right)$), and that for fixed α with $0<\alpha<\frac{\pi}{2}$, the functions k_2, $\widetilde{k}_1$, $\widetilde{k}_2$ are uniquely determined by the function k_1 using equations (i) to (iv) of Lemma 2.1.

We now study the converse problem: to find sufficient conditions for a Frenet curve to be a *FB* curve. In view of Theorems 2.1, 2.2, we need only consider plane Frenet curves with zero pseudo-torsion whose curvatures are bounded below or bounded above, and non-planar Frenet curves whose pseudo-torsions are nowhere zero. (The case of a line-segment has already been treated in Lemma 2.2.)

THEOREM 2.3. *Let Γ: $\boldsymbol{x}(s)$, $s \in L$ be a plane C^∞ Frenet curve with zero pseudo-torsion and whose curvature is either bounded below or bounded above (which is always the case if L is compact). Then Γ is a FB curve, and has c FB conjugates which are plane curves. (c being the cardinality of the continuum).*

PROOF. Let Γ be a curve satisfying the conditions of the hypothesis. Then there are c non-zero numbers λ such that $k_1(s)<1/\lambda$ on L or $k_1(s)>1/\lambda$ on L. For any such λ, consider the plane curve $\widetilde{\Gamma}$ (since $\boldsymbol{n}$ always lies on a plane containing Γ) with position vector

$$\widetilde{\boldsymbol{x}} = \boldsymbol{x} + \lambda\boldsymbol{n}.$$

Then

$$\widetilde{\boldsymbol{x}}' = (1-\lambda k_1)\boldsymbol{t}.$$

Since $1-\lambda k_1 \neq 0$, $\widetilde{\Gamma}$ is a C^∞ regular curve, and $\widetilde{\boldsymbol{t}} = \boldsymbol{t}$. It is then a straightforward matter to verify that $\widetilde{\Gamma}$ is a *FB* conjugate of Γ.

THEOREM 2.4. *Let Γ: $\boldsymbol{x}(s)$, $s \in L$ be a C^∞ Frenet curve with k_2 nowhere zero and satisfying the equation*

$$(*) \qquad (1-\lambda k_1)\sin\alpha = \lambda k_2\cos\alpha$$

for some constants λ, α with $\lambda \neq 0$. Then Γ is a non-planar FB curve.

PROOF. Define the curve $\widetilde{\Gamma}$ with position vector

$$\widetilde{\boldsymbol{x}} = \boldsymbol{x} + \lambda\,\boldsymbol{n}\,.$$

Then, denoting differentiation with respect to s by a dash, we have

$$\widetilde{\boldsymbol{x}}' = (1-\lambda k_1)\boldsymbol{t} + \lambda k_2 \boldsymbol{b}\,.$$

Since $k_2 \neq 0$, it follows that $\widetilde{\Gamma}$ is a C^∞ regular curve. Let quantities belonging to $\widetilde{\Gamma}$ be marked with a tilde. Then

$$\widetilde{s}'\widetilde{\boldsymbol{t}} = (1-\lambda k_1)\boldsymbol{t} + \lambda k_2 \boldsymbol{b}\,.$$

Hence

$$\widetilde{s}' = [(1-\lambda k_1)^2 + (\lambda k_2)^2]^{1/2} > 0\,,$$

and, using (*),

$$\varepsilon\widetilde{\boldsymbol{t}} = \cos\alpha\,\boldsymbol{t} + \sin\alpha\,\boldsymbol{b}\,,$$

where $\varepsilon = +1$ or -1 according as λk_2 and $\sin\alpha$ have the same or opposite signs. (Notice that from (*) we have $\sin\alpha \neq 0$). Therefore

$$\varepsilon\widetilde{s}'(d\widetilde{\boldsymbol{t}}/d\widetilde{s}) = (k_1\cos\alpha - k_2\sin\alpha)\,\boldsymbol{n}\,.$$

Now define $\widetilde{\boldsymbol{n}} = \boldsymbol{n}$, $\widetilde{k}_1 = (\varepsilon/\widetilde{s}')(k_1\cos\alpha - k_2\sin\alpha)$. These are C^∞ functions of s (and hence of $\widetilde{s}$), and

$$(d\widetilde{\boldsymbol{t}}/d\widetilde{s}) = \widetilde{k}_1\widetilde{\boldsymbol{n}}\,.$$

Further define $\widetilde{\boldsymbol{b}} = \widetilde{\boldsymbol{t}}\times\boldsymbol{n}$ and $\widetilde{k}_2 = (-d\widetilde{\boldsymbol{b}}/d\widetilde{s})\cdot\widetilde{\boldsymbol{n}}$. These are also C^∞ functions on L. It is then easy to verify that with the frame $\{\widetilde{\boldsymbol{t}}, \widetilde{\boldsymbol{n}}, \widetilde{\boldsymbol{b}}\}$ and the functions $\widetilde{k}_1, \widetilde{k}_2$, the curve $\widetilde{\Gamma}$ becomes a C^∞ Frenet curve. But $\boldsymbol{n}$ and $\widetilde{\boldsymbol{n}}$ lie on the line joining corresponding points of Γ and $\widetilde{\Gamma}$. Thus Γ is a *FB* curve and $\widetilde{\Gamma}$ a *FB* conjugate of Γ.

3. Weakened Bertrand curves.

DEFINITION 3.1. Let D be a subset of a topological space X. A function on X into a set Y is said to be *D-piecewise constant* if it is constant on each component of D.

We shall use the following lemma ([1]).

LEMMA B. *Let X be a proper interval on the real line and D an open subset of X. Then a necessary and sufficient condition for every continuous, D-piecewise constant real function on X to be constant is that $X\backslash D$ should have empty dense-in-itself kernel.*

We notice, however, that if D is dense in X, any C^1 and D-piecewise constant real function on X must be constant, even if D has non-empty dense-in-itself kernel.

THEOREM 3.1. *A WB curve for which N and Z have empty dense-in-itself kernels* (*notations as in Definition* 1.6) *is a FB curve.*

PROOF. Let $\Gamma: \boldsymbol{x}(s)$, $s \in L$ be a WB curve and $\widetilde{\Gamma}: \widetilde{\boldsymbol{x}}(\widetilde{s})$, $\widetilde{s} \in \widetilde{L}$ a WB conjugate of Γ. It follows from the definition that Γ and $\widetilde{\Gamma}$ each has a C^∞ family of tangent vectors $\boldsymbol{t}(s)$, $\widetilde{\boldsymbol{t}}(\widetilde{s})$. Let

$$\widetilde{\boldsymbol{x}}(s) \equiv \widetilde{\boldsymbol{x}}(\sigma(s)) = \boldsymbol{x}(s) + \lambda(s)\,\boldsymbol{n}(s)\,, \tag{3.1}$$

where $\boldsymbol{n}(s)$ is some unit vector function and $\lambda(s) \geqslant 0$ is some scalar function.

Let $D = L\backslash N$, $\widetilde{D} = \widetilde{L}\backslash\sigma(Z)$. Then $\widetilde{s}(s) \in C^\infty$ on D, and $s(\widetilde{s}) \in C^\infty$ on $\widetilde{D}$.

Step 1. To prove λ = constant.

Since $\lambda = |\widetilde{\boldsymbol{x}} - \boldsymbol{x}|$, it is continuous on L and is of class C^∞ on every interval of D on which it is nowhere zero. Let $P = \{s \in L : \lambda(s) \neq 0\}$ and X any component of P. Then P, and hence also X, is open in L. Let I be any component interval of $X \cap D$. Then on I, $\lambda(s)$ and $\boldsymbol{n}(s)$ are of class C^∞, and from (3.1) we have

$$\widetilde{\boldsymbol{x}}'(s) = \boldsymbol{x}'(s) + \lambda'(s)\,\boldsymbol{n}(s) + \lambda(s)\,\boldsymbol{n}'(s)\,.$$

Now by definition of a WB curve we have $\boldsymbol{x}'(s)\cdot\boldsymbol{n}(s) = 0 = \widetilde{\boldsymbol{x}}'(s)\cdot\boldsymbol{n}(s)$. Hence, using the identity $\boldsymbol{n}'(s)\cdot\boldsymbol{n}(s) = 0$, we have

$$0 = \lambda'(s)\cdot\boldsymbol{n}(s)\cdot\boldsymbol{n}(s)\,.$$

Therefore λ=constant on I.

Hence λ is constant on each component (interval) of the set $X \cap D$. But by hypothesis $X\backslash D$ has empty dense-in-itself kernel. It follows from Lemma B that λ is constant (and non-zero) on X. Since λ is continuous on L, X

must be closed in L. But X is also open in L. Therefore by connectedness we must have $X=L$, that is, λ is constant on L.

Step 2. To prove the existence of two frames $\{\boldsymbol{t}(s), \boldsymbol{n}(s), \boldsymbol{b}(s)\}$, $\{(\tilde{\boldsymbol{t}}(\tilde{s}), \tilde{\boldsymbol{n}}(\tilde{s}), \tilde{\boldsymbol{b}}(\tilde{s})\}$ which are Frenet frames for Γ, $\tilde{\Gamma}$ on D, $\tilde{D}$ respectively.

Since λ is a non-zero constant, it follows from (3.1) that $\boldsymbol{n}(s)$ is continuous on L and C^∞ on D, and is always orthogonal to $\boldsymbol{t}(s)$ (by definition of WB curve). Now define $\boldsymbol{b}(s)=\boldsymbol{t}(s)\times\boldsymbol{n}(s)$. Then $\{\boldsymbol{t}(s), \boldsymbol{n}(s), \boldsymbol{b}(s)\}$ forms a right-handed orthonormal frame for Γ which is continuous on L and C^∞ on D.

Now from the definition of WB curve we see that there exists a scalar function $k_1(s)$ such that

$$\boldsymbol{t}'(s)=k_1(s)\,\boldsymbol{n}(s) \quad \text{on} \quad L\,.$$

Hence $k_1(s)=\boldsymbol{t}'(s)\cdot\boldsymbol{n}(s)$ is continuous on L and C^∞ on D. Thus the first Frenet formula holds on D. It is then straightforward to show that there exists a C^∞ function $k_2(s)$ on D such that the Frenet formulas hold. Thus $\{\boldsymbol{t}(s), \boldsymbol{n}(s), \boldsymbol{b}(s)\}$ is a Frenet frame for Γ on D.

Similarly there exists a right-handed orthonormal frame $\{\tilde{\boldsymbol{t}}(\tilde{s}), \tilde{\boldsymbol{n}}(\tilde{s}), \tilde{\boldsymbol{b}}(\tilde{s})\}$ for $\tilde{\Gamma}$ which is continuous on $\tilde{L}$ and is a Frenet frame for $\tilde{\Gamma}$ on $\tilde{D}$. Moreover, we can choose

$$\tilde{\boldsymbol{n}}(\sigma(s))=\boldsymbol{n}(s)\,.$$

Step 3. To prove that $N=\emptyset$, $Z=$.

We first notice that on D we have

$$(\boldsymbol{t}\cdot\tilde{\boldsymbol{t}})'=\boldsymbol{t}\cdot\tilde{s}'\,\tilde{k}_1\tilde{\boldsymbol{n}}+k_1\boldsymbol{n}\cdot\tilde{\boldsymbol{t}}=0\,,$$

so that $\boldsymbol{t}\cdot\tilde{\boldsymbol{t}}$ is constant on each component of D and hence on L by Lemma B. Consequently there exists a constant angle α such that

$$\tilde{\boldsymbol{t}}(s)=\cos\alpha\,\boldsymbol{t}(s)+\sin\alpha\,\boldsymbol{b}(s) \quad \text{on} \quad L\,.$$

Further,

$$\tilde{\boldsymbol{n}}(s)=\boldsymbol{n}(s)$$

and so

$$\tilde{\boldsymbol{b}}(s)=-\sin\alpha\,\boldsymbol{t}(s)+\cos\alpha\,\boldsymbol{b}(s)\,.$$

Thus $\tilde{\boldsymbol{t}}(s)$, $\boldsymbol{n}(s)$, $\tilde{\boldsymbol{b}}(s)$ are also of class C^∞ on D. On the other hand, $\tilde{\boldsymbol{t}}$, $\tilde{\boldsymbol{n}}$, $\tilde{\boldsymbol{b}}$ are of class C^∞ with respect to $\tilde{s}$ on $\tilde{D}$. Writing (3.1) in the form

$$\boldsymbol{x} = \widetilde{\boldsymbol{x}} - \lambda \widetilde{\boldsymbol{n}}$$

and differentiating with respect to s on $D \cap \sigma^{-1}(\widetilde{D})$, we have

$$\boldsymbol{t} = \widetilde{s}'[(1+\lambda \widetilde{k}_1)\widetilde{\boldsymbol{t}} - \lambda \widetilde{k}_2 \widetilde{\boldsymbol{b}}].$$

But

$$\boldsymbol{t} = \cos \alpha \, \widetilde{\boldsymbol{t}} - \sin \alpha \, \widetilde{\boldsymbol{b}}.$$

Hence

$$\widetilde{s}'(1+\lambda \widetilde{k}_1) = \cos \alpha. \tag{3.2}$$

Since $\widetilde{k}_1(\widetilde{s}) = (d\widetilde{\boldsymbol{t}}/d\widetilde{s}) \cdot \widetilde{\boldsymbol{n}}$ is defined and continuous on $\widetilde{L}$ and $\sigma^{-1}(\widetilde{D})$ is dense, it follows by continuity that (3.2) holds throughout D.

Case 1. $\cos \alpha \neq 0$. Then (3.2) implies that $\widetilde{s}' \neq 0$ on D. Hence $Z = \phi$. Similarly $N = \emptyset$.

Case 2. $\cos \alpha = 0$. Then

$$\widetilde{\boldsymbol{t}} = \pm \boldsymbol{b}. \tag{3.3}$$

Differentiation of (3.1) with respect to s in D gives

$$\widetilde{s}' \widetilde{\boldsymbol{t}} = (1-\lambda k_1)\boldsymbol{t} + \lambda k_2 \boldsymbol{b}.$$

Hence using (3.3) we have

$$1 - \lambda k_1 = 0.$$

Therefore

$$k_1 = \frac{1}{\lambda} \quad \text{on} \quad D,$$

and so also on L, by Lemma B. It follows that k_1 is nowhere zero on L.

Let $\varepsilon = \pm 1$ be the sign of k_1. Then

$$\boldsymbol{n}(s) = \frac{1}{k_1(s)} \boldsymbol{t}'(s) = \varepsilon \frac{\boldsymbol{t}'(s)}{|\boldsymbol{t}'(s)|},$$

and is of class C^∞ on L. Consequently

$$\widetilde{\boldsymbol{x}}(s) = \boldsymbol{x}(s) + \lambda \boldsymbol{n}(s) \quad \text{is of class } C^\infty \text{ on } L.$$

Hence $N = \emptyset$. Similarly $Z = \emptyset$.

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