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DUALITY OF CYCLIC MODULES

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Quasi-Frobenius rings which possess so many interesting properties, have been studied by a number of authors. J. Dieudonné pointed out that the duality of R-modules is most closely related to a quasi-Frobenius ring R, stating that, for a both right and left Noetherian ring R, R is quasi-Frobenius if and only if all left and all right finitely generated modules over R are reflexive [3] (see also [5]). H. Bass introduced the terminologies "reflexive" and "torsionless" [4]. These seem most important in duality theory. In this paper we shall study the duality of cyclic modules over rings without any finiteness assumptions generalizing the above theorem (Theorem 12 and Theorem 15) and the Ikeda-Nakayama's theorem [1] (Theorem 13, Corollary 14, and Theorem 15). It seems to me that the duality of cyclic modules is essential in duality theory.

1. Introduction. Throughout this paper, we shall assume that R is a ring with identity element and that every module over R is unitary. If A is a left (right) R-module, the dual $A^* = \mathrm{Hom}_R(A, R)$ becomes a right (left) R-module ([6], p. 65). Thus the dual operation $*$ is a contravariant left exact functor on the category of R-modules to that of R-modules. Considering the element of A as homomorphisms from A^* to R, we get the natural R-homomorphism

$$\delta_A\colon A \longrightarrow A^{**}.$$

We shall say that A is torsionless if δ_A is a monomorphism and reflexive if δ_A is an isomorphism. A is torsionless if and only if $A \subset \Pi R$ (direct product of copies of R) ([6], p. 68). It is well known that every finitely generated projective module P is reflexive and that P^* is also finitely generated projective ([6], p. 68). If we are given the diagram $A \to B$, then we have the following commutative diagram

$$\begin{array}{ccc} A^{**} & \longrightarrow & B^{**} \\ \delta_A\big\uparrow & & \big\uparrow\delta_B \\ A & \longrightarrow & B\,. \end{array}$$

For each subset X of R we shall denote by $l(X)$ (resp. $r(X)$) the set of all left (resp. right) annihilators in R of the set X. R satisfies the left (resp. right) annihilator condition if $l(r(L))=L$ (resp. $r(l(I))=I$) for all left (resp. right) ideals L (resp. I). If R satisfies both left and right annihilator conditions, then we say that R satisfies the annihilator conditions. L, L_1, L_2, (resp. I, I_1, I_2,) denote left (resp. right) ideals of R.

2. Duality of cyclic modules.

THEOREM 1. *The following conditions are equivalent:*
1. *R/L is torsionless.*
2. *$l(r(L))=L$.*

PROOF. Consider the commutative diagram

$$\begin{array}{ccc} (R/L)^* & \longrightarrow & ({}_R R)^* \\ \wr & & \wr \\ r(L) & \longrightarrow & R_R. \end{array}$$

From this we get the exact sequence

$$0 \longrightarrow (R/L)^* \longrightarrow ({}_R R)^* \longrightarrow R/r(L) \longrightarrow 0.$$

Then we can form the commutative diagram in which the upper row is exact

$$\begin{array}{ccccccc} 0 & \longrightarrow & (R/r(L))^* & \longrightarrow & ({}_R R)^{**} & \longrightarrow & (R/L)^{**} \\ & & \wr & & \wr & & \uparrow \delta_{R/L} \\ & & l(r(L)) & \longrightarrow & {}_R R & \longrightarrow & R/L. \end{array}$$

By examining the diagram we see that $\delta_{R/L}$ is a monomorphism if and only if $l(r(L))=L$. This completes the proof.

COROLLARY 2. *Let R be a left Noetherian ring and let all left and all right cyclic modules over R be torsionless. Then any finitely generated (right or left) R-module is reflexive.*

PROOF. Cyclic left (resp. right) R-modules are of the form R/L (resp. R/I). Since all cyclic modules over R are torsionless, R satisfies the annihilator conditions by Theorem 1 ("left" and "right" interchanged if necessary). Then R is quasi-Frobenius [2]. Hence each finitely generated R-module is reflexive [3], [5].

COROLLARY 3. *Cyclic torsionless left R-modules are of the form $R/l(I)$.*

PROOF. Since $l(r(l(I)))=l(I)$, the statement is clear by Theorem 1.

In the following we shall study the properties of δ_I and $\delta_{R/L}$. We recall in mind that a module A is a W module if $\mathrm{Ext}^1_R(A,R)=0$.

PROPOSITION 4. *Let R/I be a* W *module. Then $I^*\approx R/l(I)$.*

PROOF. Take the exact sequence

$$0 \longrightarrow I \longrightarrow R_R \longrightarrow R/I \longrightarrow 0\,.$$

Now dualize this sequence. Then we have the commutative diagram

$$\begin{array}{ccccccc} 0 \longrightarrow & (R/I)^* & \longrightarrow & (R_R)^* & \longrightarrow & I^* & \longrightarrow 0 \\ & \wr & & \wr & & & \\ & l(I) & \longrightarrow & {}_RR & & & \end{array}$$

where the upper row is exact since R/I is a W module. Hence $I^*\approx R/l(I)$.

REMARK. The element of I^* is considered as a left multiplication of an element of R modulo $l(I)$ in the condition that R/I is a W module.

PROPOSITION 5. *Let R/I be a* W *module. Then $I^{**}\approx r(l(I))$ and δ_I is nothing but the inclusion*

$$\delta_I : I \longrightarrow r(l(I)).$$

PROOF. $I^*\approx R/l(I)$ by Proposition 4. Then $I^{**}\approx(R/l(I))^*\approx r(l(I))$. It is easy to check that δ_I is the inclusion $I\to r(l(I))$.

PROPOSITION 6. *Let $R/r(L)$ be a* W *module. Then $(R/L)^{**}\approx R/l(r(L))$ and $\delta_{R/L}$ is merely the natural homomorphism*

$$\delta_{R/L} : R/L \longrightarrow R/l(r(L)).$$

PROOF. Since $(R/L)^*\approx r(L)$, $(R/L)^{**}\approx r(L)^*\approx R/l(r(L))$ by Proposition 4. It is then easy to see that $\delta_{R/L}$ is the natural homomorphism $R/L\to R/l(r(L))$.

The next proposition tells us when $\delta_{R/L}$ is an epimorphism.

PROPOSITION 7. *The following conditions are equivalent*:
1. *$\delta_{R/L}$ is an epimorphism.*
2. $\mathrm{Ext}^1_R(R/r(L),R)=0$.

PROOF. Consider the exact sequence

$$0 \longrightarrow (R/L)^* \longrightarrow ({}_RR)^* \longrightarrow R/r(L) \longrightarrow 0\,.$$

Dualizing this we get the commutative diagram with exact rows

$$\begin{array}{ccccccc} ({}_RR)^{**} & \longrightarrow & (R/L)^{**} & \longrightarrow & \mathrm{Ext}_R^1(R/r(L),R) & \longrightarrow & 0 \\ \wr & & \uparrow \delta_{R/L} & & & & \\ {}_RR & \longrightarrow & R/L & \longrightarrow & 0. & & \end{array}$$

Then $\delta_{R/L}$ is an epimorphism if and only if $\mathrm{Ext}_R^1(R/r(L),R)=0$ proving the proposition.

REMARK. By the above proposition, we see that the converse of Proposition 6 also holds.

The following theorem gives a criterion for reflexivity of cyclic modules, which is the key theorem in this paper.

THEOREM 8. *The following conditions are equivalent*:
1. *R/L is reflexive.*
2. *$l(r(L))=L$ and* $\mathrm{Ext}_R^1(R/r(L),R)=0$.

PROOF. The result is clear by Theorem 1, Proposition 6, and Proposition 7.

REMARK. If R/L is reflexive, $\delta_{R/L}$ must be of the form in Proposition 6. But if I is reflexive, δ_I need not be of the form in Proposition 5. For if we take $R=Z$, the ring of rational integers, and $I=2Z$, we have $r(l(2Z))=Z$.

3. Self-injective rings.

The following proposition was studied by J.P.Jans [5], [6] under the condition that R is Noetherian.

PROPOSITION 9. *The following conditions are equivalent*:
1. *All finitely generated torsionless left modules are reflexive.*
2. *All finitely generated torsionless right modules are* W *modules.*

PROOF. Assume 2 and let A be a finitely generated torsionless left R-module. Choose a finitely generated projective left R-module P such that

$$P \longrightarrow A \longrightarrow 0$$

is exact. Then we have the exact sequence

$$0 \longrightarrow A^* \longrightarrow P^* \longrightarrow B \longrightarrow 0$$

where B is a finitely generated torsionless right R-module. From this we get the commutative diagram with an exact row

$$\begin{array}{ccccc} P^{**} & \longrightarrow & A^{**} & \longrightarrow & 0 \\ \wr\wr & & \uparrow \delta_A & & \\ P & \longrightarrow & A & & \end{array}$$

since B is a W module by the assumption. Hence A is reflexive. Conversely assume 1 and let B be a finitely generated torsionless right R-module. Look at the exact sequence

$$0 \longrightarrow B^* \longrightarrow P^* \longrightarrow A \longrightarrow 0$$

where P is finitely generated projective and A finitely generated torsionless. From this we get the exact sequence

$$0 \longrightarrow A^* \longrightarrow P^{**} \longrightarrow B \longrightarrow 0$$

since B is torsionless. Now dualize this sequence to get the commutative diagram with exact rows

$$\begin{array}{ccccccc} P^{***} & \longrightarrow & A^{**} & \longrightarrow & \mathrm{Ext}^1_R(B,R) & \longrightarrow & 0 \\ \wr\wr & & \wr\wr & & & & \\ P^* & \longrightarrow & A & \longrightarrow & 0 & & \end{array}$$

where A is reflexive by the assumption. Hence we have $\mathrm{Ext}^1_R(B,R)=0$ which completes the proof.

The rings with the annihilator conditions are similar to the self-injective rings. In fact they coincide if the rings considered are Noetherian [7].

THEOREM 10. *Let R satisfy the annihilator conditions. Then both $R/r(L)$ and $R/l(I)$ are reflexive for all finitely generated L and I.*

PROOF. We see easily that $r(L_1 \cap L_2)=r(L_1)+r(L_2)$, $l(I_1 \cap I_2)=l(I_1)+l(I_2)$ since R satisfies the annihilator conditions. Then $\mathrm{Ext}^1_R(R/L,R)=0=\mathrm{Ext}^1_R(R/I,R)$ for all finitely generated L and I by Ikeda-Nakayama [1]. Now the result follows by Theorem 8.

REMARK. Under the assumption of Theorem 10, both L and I are reflexive for all finitely generated L, I by Proposition 5.

Now we give characterizations of self-injective rings in terms of duality. Consider the following condition (a):

(a) Every diagram

$$\begin{tikzcd} 0 \arrow[r] & I \arrow[r] \arrow[d] & r(l(I)) \\ & R_R & \end{tikzcd}$$

in which $I \longrightarrow r(l(I))$ is inclusion, can be imbedded in a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & I \arrow[r] \arrow[d] & r(l(I)) \arrow[dl] \\ & R_R. & \end{tikzcd}$$

Clearly the rings with the right annihilator condition satisfy the condition (a).

THEOREM 11. *Let R satisfy the condition* (a). *Then the following statements are equivalent*:

1. *$R/l(I)$ is reflexive for all I.*
2. *R_R is injective.*

PROOF. Assume 1. Then $\mathrm{Ext}^1_R(R/r(l(I)),R)=0$ in view of Theorem 8. Now let the following diagram, in which $I \longrightarrow R_R$ is inclusion, be given

$$\begin{tikzcd} 0 \arrow[r] & I \arrow[r] \arrow[d] & R_R \\ & R_R. & \end{tikzcd}$$

Then we can imbed this in a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & I \arrow[r] \arrow[d] & r(l(I)) \arrow[r] \arrow[dl] & R_R \arrow[dll] \\ & R_R & & \end{tikzcd}$$

since $\mathrm{Ext}^1_R(R/r(l(I)),R)=0$ and since by the condition (a). Hence R_R is injective. The converse follows immediately by Theorem 8 and by $l(r(l(I)))=l(I)$.

THEOREM 12. *The following statements are equivalent*:

1. *R/L and R/I are reflexive for all L and I.*
2. *${}_RR$ and R_R are injective and R satisfies the annihilator conditions.*

PROOF. Assume 1. Since both R/L and R/I are torsionless for all L and I, R satisfies the annihilator conditions by Theorem 1. Hence ${}_RR$ and R_R are injective by Theorem 11 ("left" and "right" interchanged). The converse is clear by Theorem 8 ("left" and "right" interchanged if necessary).

REMARK. Under the condition of Theorem 12, L and I are always reflexive by Proposition 5.

THEOREM 13. *Let R_R be injective and L finitely generated. Then L and R/L are reflexive.*

PROOF. Since L is finitely generated and torsionless, L is reflexive by Proposition 9. Next, consider the exact sequence of right R-modules

$$0 \longrightarrow (R/L)^* \longrightarrow ({}_R R)^* \longrightarrow L^*.$$

From this we get the commutative diagram with exact rows

$$\begin{tikzcd}
 & L^{**} \arrow[r] & ({}_I R)^{**} \arrow[r] & (R/L)^{**} \arrow[r] & 0 \\
0 \arrow[r] & L \arrow[r] \arrow[u, phantom, "\wr"] & {}_R R \arrow[r] \arrow[u, phantom, "\wr"] & R/L \arrow[r] \arrow[u, "\delta_{R/L}"'] & 0
\end{tikzcd}$$

since R is right self-injective. By the diagram chasing, we see easily that $\delta_{R/L}$ is an isomorphism and this completes the proof.

REMARK. If R_R is injective and L finitely generated, then R/L is torsionless by Theorem 13. Hence $l(r(L))=L$ by Theorem 1. This is the Ikeda-Nakayama's main theorem [1].

COROLLARY 14. *Let R be right self-injective and L reflexive. Then R/L is reflexive and hence $l(r(L))=L$.*

PROOF. The result follows by the same argument as in Theorem 13.

A module A is called finitely related if $A \approx P/T$ where P is projective and T is finitely generated.

THEOREM 15. *Let R_R be injective. Then finitely generated, finitely related left R-modules are reflexive.*

PROOF. Let A be a finitely generated, finitely related left R-module. Then we have an exact sequence

$$0 \longrightarrow T \longrightarrow P \longrightarrow A \longrightarrow 0$$

where P is projective and T is finitely generated. Since T and A are finitely generated, P is also finitely generated and then P is reflexive. Since R_R is injective, T is also reflexive by Proposition 9. By the same argument as in Theorem 13, we get the commutative diagram with exact rows

$$\begin{array}{ccccccccc} & & T^{**} & \longrightarrow & P^{**} & \longrightarrow & A^{**} & \longrightarrow & 0 \\ & & \wr\wr & & \wr\wr & & \uparrow \delta_A & & \\ 0 & \longrightarrow & T & \longrightarrow & P & \longrightarrow & A & \longrightarrow & 0 \end{array}$$

Thus A is reflexive which proves the Theorem.

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