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ON THE LITTLEWOOD-PALEY FUNCTION g^* OF MULTIPLE FOURIER INTEGRALS AND HANKEL MULTIPLIER TRANSFORMATIONS.

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1. Introduction. The function g^* introduced by Littlewood and Paley is important in their work. A generalized Littlewood-Paley function g^*_α is essentially the same as the function

$$\left\{\sum_{n=1}^{\infty} |\sigma_n^\alpha(x)-\sigma_n^{\alpha-1}(x)|^2/n\right\}^{1/2}$$

where $\sigma_n^\alpha(x)$ denotes the n-th (C,α)-mean of Fourier series of $f(x)$. Hence we denote this by $g^*_\alpha(x)=g^*_\alpha(x,f)$. One of the most important results of them is that, if $f(x)\in L^p(1<p\leqq 2)$ and $\alpha>1/p$ then

$$\int_{-\pi}^{\pi} |g^*_\alpha(x)|^p dx \leqq A_{p,\alpha} \int_{-\pi}^{\pi} |f(x)|^p dx.$$

The known proofs of this inequality depend upon complex method and at least depend upon M.Riesz's theorem. In the present note, the author gives a real proof which is independent from M.Riesz's theorem. In section 3, we extend this to multiple Fourier integrals. Specifically, when the function is radial, we can give a heuristic proof of the Hankel multiplier theorem. This is done in section 4. D.L.Guy [2] has proved already this theorem by transplantation technique and B.Muckenhoupt and E.M.Stein [5] have proved by the method of generalized conjugate function. In the last section, we shall give the theorem in multiple Fourier series.

2. One variable case. Let $f(x)$ be an integrable function with period 2π and its Fourier series be

$$S(f)=a_0/2+\sum_{\nu=1}^{\infty}(a_\nu \cos \nu x + b_\nu \sin \nu x)$$

$$= \sum_{\nu=0}^{\infty} A_\nu(x).$$

Set $A_n^\alpha = \binom{n+\alpha}{n}$, and

$$\tau_n^\alpha(x) = \tau_n^\alpha(x, f) = \frac{1}{A_n^\alpha} \sum_{\nu=1}^{n} A_{n-\nu}^{\alpha-1} \nu A_\nu(x),$$

then

$$\tau_n^\alpha(x) = \alpha\{\sigma_n^\alpha(x) - \sigma_n^{\alpha-1}(x)\} \qquad (\alpha > 0)$$

where $\sigma_n^\alpha(x)$ is the n-th (C,α)-mean of $S(f)$.

We consider now the operation T_α such that

$$T_\alpha f = \left\{\sum_{n=1}^{\infty} \left| \frac{\tau_n^\alpha(x,f)}{\sqrt{n}} \right|^2 \right\}^{1/2}.$$

Applying Bessel's inequality, it is easy to see that T_α is strong type (2, 2), provided $\alpha > 1/2$. Next we consider for $\delta > 0$, $\tau_n^{1+\delta}(x)$. Denote by $\widetilde{K}_n^\delta(t)$ the conjugate (C, δ)-kernel, that is,

$$\widetilde{K}_n^\delta(t) = \frac{1}{A_n^\delta} \sum_{\nu=1}^{n} A_{n-\nu}^\delta \sin \nu t,$$

then

$$\tau_n^{1+\delta}(x) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x+t) \frac{A_n^\delta}{A_n^{1+\delta}} \{\widetilde{K}_n^\delta(t)\}' \, dt$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x+t) H_n(t) dt$$

say. Since

$$H_n(t) = \frac{C}{n} \{\widetilde{K}_n^\delta(t)\}'$$

we have an estimation of the kernel $H_n(t)$ such that

$$(1) \qquad |H_n(t)| \leqq \frac{C}{n^\delta t^{1+\delta}}$$

which is proved by the method of Zygmund's book [4, p.94]. And we have also

$$|H_n'(t)| \leqq \frac{Cn^{1-\delta}}{t^{1+\delta}}, \tag{2}$$

since, when $nt \leqq 1$

$$|H_n'(t)| \leqq Cn^2$$

and $nt \geqq 1$

$$|H_n'(t)| \leqq C\left(\frac{n^{1-\delta}}{t^{1+\delta}} + \frac{1}{t^2}\right) \leqq \frac{Cn^{1-\delta}}{t^{1+\delta}}.$$

Following the method of J.T.Schwarz [1, pp.1164-1184], denote by $\mathfrak{H}(x)$ the l_2-valued kernel such that

$$\mathfrak{H}(x) = \{H_n(x)/\sqrt{n}\} \quad (n=1, 2\cdots).$$

Then, by (1) for $|x| > 2|y|$,

$$\left|\frac{H_n(x+y)}{\sqrt{n}} - \frac{H_n(x)}{\sqrt{n}}\right| \leqq \frac{|H_n(x+y)|}{\sqrt{n}} + \frac{|H_n(x)|}{\sqrt{n}} \tag{3}$$

$$\leqq \frac{C}{n^{1/2+\delta}|x|^{1+\delta}}.$$

On the other hand, from the mean value theorem and (2), we have also

$$\left|\frac{H_n(x+y)}{\sqrt{n}} - \frac{H_n(x)}{\sqrt{n}}\right| \leqq \frac{|y|}{\sqrt{n}} |H_n'(x+\theta y)| \qquad (0 \leqq \theta \leqq 1) \tag{4}$$

$$\leqq \frac{Cn^{1-\delta}|y|}{\sqrt{n}|x+\theta y|^{1+\delta}} \leqq \frac{Cn^{1/2-\delta}|y|}{|x|^{1+\delta}}$$

provided $|x| > 2|y|$. Therefore

$$\int_{\pi \geqq |x| > 2|y| > 0} \left\{\sum_{n=1}^{\infty} \left|\frac{H_n(x+y)}{\sqrt{n}} - \frac{H_n(x)}{\sqrt{n}}\right|^2\right\}^{1/2} dx$$

$$\leqq C\int_{\pi\geqq x>2y>0}\left\{\sum_{n=1}^{[y^{-1}]}\left(\frac{n^{1/2-\delta}y}{x^{1+\delta}}\right)^2+\sum_{n=[y^{-1}]+1}^{\infty}\left(\frac{1}{n^{1/2+\delta}x^{1+\delta}}\right)^2\right\}^{1/2}dx$$

$$\leqq C\int_{\pi\geqq x>2y>0}\left(\frac{y^{\delta}}{x^{1+\delta}}+\frac{y^{\delta}}{x^{1+\delta}}\right)dx\leqq C.$$

By the Hörmander test [1, p.1169], $T_{1+\delta}$ is of weak type (1, 1). Hence applying Marcinkiewicz's interpolation theorem, we have that $T_{1+\delta}$ is of strong type (p, p) for $1<p<2$. However in the L^2-case as mentioned above, we have rather stronger result, that is to say, $T_\alpha(\alpha>1/2)$ is of strong type (2, 2). We take now any function $g_n(x)$ such that

$$\sum_{n=1}^{\infty}|g_n(x)|^2/n\leqq 1$$

for all x and consider the linear operation

$$\boldsymbol{T}_\alpha f=\sum_{n=1}^{\infty}\frac{\tau_n^\alpha(x)g_n(x)}{n}.$$

Moreover we extend the index α to complex $\sigma+i\tau$, then the norm increases with $e^{2\tau^2}$. If we interpolate between $p=1+\varepsilon$ $(\varepsilon>0)$ and $p=2$ changing index $\mathrm{Re}\,\alpha$ between $1+\delta$ $(\delta>0)$ and $1/2+\eta$ $(\eta>0)$, we get finally the following theorem.

THEOREM 1. *If* $\alpha>1/p$ $(1<p\leqq 2)$, *then*

$$\int_{-\pi}^{\pi}\left\{\sum_{n=1}^{\infty}\frac{|\tau_n^\alpha(x)|^2}{n}\right\}^{p/2}dx\leqq A_p\int_{-\pi}^{\pi}|f(x)|^p dx.$$

This theorem has been proved in the author's note [6] by the complex method.

3. Spherical means of multiple Fourier integrals. Our real proof has an advantage to be able to extend the result to the spherical mean of multiple Fourier integrals and Fourier series.

Let x and y be vectors in k-dimensional euclidean space E_k, and set

$$x=(x_1,x_2,\cdots,x_k),\quad (x,y)=x_1y_1+x_2y_2+\cdots+x_ky_k,$$
$$|x|^2=x_1^2+x_2^2+\cdots+x_k^2,\quad dx=dx_1dx_2\cdots dx_k.$$

When $f(x)=f(x_1,x_2,\cdots,x_k)$ belongs to the class L_p $(1\leqq p\leqq 2)$, we consider its Fourier transform defined by

$$F(y)=\int_{E_k} f(x)e^{i(x,y)}\,dx$$

and the spherical Riesz means of order $\alpha=(k-1)/2+\beta$ of $F(y)$, that is to say

$$S_R^\alpha(x,f)=(2\pi)^{-k}\int_{|y|<R}\left(1-\frac{|y|^2}{R^2}\right)^\alpha F(y)e^{-i(y,x)}\,dy$$

$$=(2\pi)^{-k}\int_{E_k} f(x+u)K_R^\alpha(u)du$$

where

$$K_R^\alpha(x)=\int_{|y|<R}\left(1-\frac{|y|^2}{R^2}\right)^\alpha e^{-i(y,x)}dy$$

$$=C_\beta R^{1/2-\beta}\frac{J_{k-1/2+\beta}(R|y|)}{|y|^{k-1/2+\beta}},\ C_\beta=2^{k-1/2-\beta}\Gamma\left\{\frac{k+2}{2}+\beta\right\}\pi^{k/2}.$$

Following the notation of the preceding section, we set

$$T_\alpha f=\left\{\int^\infty \frac{|S_R^\alpha(x,f)-S_R^{\alpha-1}(x,f)|^2}{R}\,dR\right\}^{1/2},$$

$$=\left\{\int_0^\infty\left|\frac{\tau_R^\alpha(x,f)}{\sqrt{R}}\right|^2 dR\right\}^{1/2},$$

where

$$\tau_R^\alpha(x,f)=S_R^\alpha(x,f)-S_R^{\alpha-1}(x,f)$$

$$=(2\pi)^{-k}\int_{|y|<R}\left(1-\frac{|y|^2}{R^2}\right)^\alpha F(y)e^{-i(y,x)}dx$$

$$-(2\pi)^{-k}\int_{|y|<R}\left(1-\frac{|y|^2}{R^2}\right)^{\alpha-1} F(y)e^{-i(y,x)}dy$$

$$=\frac{(2\pi)^{-k}}{R^2}\int_{|y|<R}\left(1-\frac{|y|^2}{R^2}\right)^{\alpha-1}|y|^2F(y)e^{-i(y,x)}dy$$

$$= \frac{(2\pi)^{-k}}{R^2} \Delta \int_{E_k} f(x+u) K_R^{\alpha-1}(u) du$$

$$= \frac{(2\pi)^{-k}}{R^2} \int_{E_k} f(x+u) \Delta K_R^{\alpha-1}(u) du ,$$

where Δ is the Laplacian operator. By the Parseval relation, T_α is of the strong type $(2,2)$ if $\alpha > 1/2$. We consider now $\alpha = \frac{k-1}{2} + 1 + \delta$, and set

$$\tau_R^\alpha(x, f) = (f * H_R)(x)$$

where

$$H_R(x)dx = C_\delta \frac{1}{R^2} \Delta K_R^{\frac{k-1}{2}+\delta}(x) dx$$

$$= C_\delta \frac{R^{1/2-\delta}}{R^2} \left\{ \Delta_r \frac{J_{k-1/2+\delta}(Rr)}{r^{k-1/2+\delta}} \right\} r^{k-1} dr d\omega$$

$$= C_\delta R^{k-2} \{ \Delta_r V_{k-1/2+\delta}(Rr) \} r^{k-1} dr d\omega .$$

Here we set

$$V_\mu(x) = J_\mu(x)/x^\mu \qquad (\mu > -1/2) .$$

Then it is well known that

$$\frac{d}{dx} \{V_\mu(x)\} = -x V_{\mu+1}(x)$$

$$V_\mu(x) = \begin{cases} O(1) & \text{as } x \to 0 \\ O\{x^{-(1/2+\mu)}\} & \text{as } x \to \infty \end{cases}$$

and $V_\mu(x)$ is finite in any compact range. When the function is radial, the Laplacian is transformed by polar coordinates to

$$\Delta_r = \frac{d^2}{dr^2} + \frac{(k-1)}{r} \frac{d}{dr} .$$

Differentiating

$$\frac{d}{dr}V_{k-1/2+\delta}(Rr) = -R^2rV_{k-1/2+\delta+1},$$

$$\frac{d^2}{dr^2}V_{k-1/2+\delta}(Rr) = -R^2V_{k-1/2+\delta+1} + R^4r^2V_{k-1/2+\delta+2}\,.$$

Hence

$$H_R(r)r^{k-1}dr$$

$$\leqq C_\delta R^{k-1}\left\{\frac{d^2}{dr^2}V_{k-1/2+\delta}(Rr) + \frac{k-1}{r}\frac{d}{dr}V_{k-1/2+\delta}(Rr)\right\}r^{k-1}dr$$

$$\leqq C_\delta R^{k-2}r^{k-1}\{R^2V_{k-1/2+\delta+1}(Rr) + R^4r^2V_{k-1/2+\delta+2}(Rr)\}\,dr$$

If $Rr \geqq 1$

$$H_R(r)r^{k-1}dr = O(R^{-(\delta+1)}r^{-(\delta+2)} + R^{-\delta}r^{-(\delta+1)})dr.$$
$$= O(R^{-\delta}r^{-(\delta+1)})dr$$

and if $Rr \leqq 1$

$$H_R(r)r^{k-1}dr = O(R^kr^{k-1} + R^{k+2}r^{k+1})dr$$
$$= O(R^{-\delta}r^{-(\delta+1)})dr,$$

because $k \geqq 2$. Thus we get

$$H_R(r)r^{k-1}dr = O(R^{-\delta}r^{-(1+\delta)})dr\,. \tag{1}$$

Once more differentiating $H_R(r)$, we have

$$\left|\frac{dH_R(r)}{dr}\right|r^{k-1}dr = O\,[r^{k-1}R^{k-2}\{R^2\cdot R^2rV_{k-1/2+\delta+2}(Rr)$$
$$+ R^4r^2 + R^2rV_{k-1/2+\delta+3}(Rr)\}\,]dr.$$

When $Rr \leqq 1$,

$$= O(R^{k+2}r^k + R^{k+4}r^{k+2})dr$$

$$= O\left\{\frac{R^{1-\delta}}{r^{1+\delta}}(R^{k-\delta}r^{k-\delta} + R^{k+2-\delta}r^{k+2-\delta})\right\}dr$$

$$=O\left(\frac{R^{1-\delta}}{r^{1+\delta}}\right) dr$$

and when $Rr \geqq 1$

$$=O(R^{-\delta}r^{-\delta-2} + R^{-\delta+1}r^{-\delta-1})dr$$

$$=O\left\{\frac{R^{1-\delta}}{r^{1+\delta}}\left(\frac{1}{Rr} + 1\right)\right\} dr$$

$$=O\left(\frac{R^{1-\delta}}{r^{1+\delta}}\right) dr.$$

Thus we have

$$\left|\frac{dH_R(r)}{dr}\right| r^{k-1}dr = O\left(\frac{R^{1-\delta}}{r^{1+\delta}}\right) dr. \tag{2}$$

It is remarkable that $H_R(r)r^{k-1}dr$ has the same estimation to the one-dimensional case. Consider now the L^2-valued kernel

$$\mathfrak{H}(r) = \left\{\frac{H_R(r)}{\sqrt{R}}\right\} r^{k-1}dr,$$

then by (1) for $|r| > 2|s|$

$$\left|\frac{H_R(r+s)}{\sqrt{R}} - \frac{H_R(r)}{\sqrt{R}}\right| r^{k-1}dr \tag{3}$$

$$\leqq \frac{C}{R^{1/2+\delta}r^{1+\delta}} dr \qquad (\delta > 0)$$

and by (2)

$$\left|\frac{H_R(r+s)}{\sqrt{R}} - \frac{H_R(r)}{\sqrt{R}}\right| r^{k-1}\, dr \tag{4}$$

$$= \left|\frac{dH_R(r+\theta s)}{dr}\right| |s| r^{k-1}\, dr$$

$$\leqq \frac{C|s|R^{1/2-\delta}}{r^{1+\delta}} dr.$$

Hence, by (3) and (4)

$$\int_{0<2s<r}\left\{\int_0^\infty\left|\frac{H_R(r+s)}{\sqrt{R}}-\frac{H_R(r)}{\sqrt{R}}\right|^2 dR\right\}^{1/2} r^{k-1}dr$$

$$\leqq C\int_{0<2s<r}\left\{\int_0\left(\frac{sR^{1/2-\delta}}{r^{1+\delta}}\right)^2 dR+\int_{s^{-1}}^\infty\left(\frac{1}{R^{1/2+\delta}r^{1+\delta}}\right)^2 dR\right\}^{1/2} dr$$

$$\leqq C\int_{0<2s<r}\left\{\frac{s^2s^{-2(1-\delta)}}{r^{2(1+\delta)}}+\frac{s^{-2\delta}}{r^{2(1+\delta)}}\right\}^{1/2} dr$$

$$\leqq C\int_{0<2s<r}\left(\frac{s^\delta}{r^{1+\delta}}\right)dr\leqq C.$$

Thus by the Hörmander test we can prove that $T_{1+\frac{k-1}{2}+\delta}$ $(\delta>0)$ is of weak type (1, 1). Now the operation $T_{1+\frac{k-1}{2}+\delta}$ is of weak type (1, 1) and of strong type (2, 2) as mentioned above. Applying Marcinkiewicz's interpolation theorem, we have that $T_{1+\frac{k-1}{2}+\delta}$ is of strong type (p, p) for $1<p\leqq 2$.

Next we take any function $g(r, R)$ such that

$$\int_0^\infty \frac{|g(r,R)|^2}{R}\,dR\leqq 1$$

for all r and consider the linear operation

$$\boldsymbol{T}_\alpha f=\int_0^\infty \frac{\tau_R^\alpha(r,f)g(r,R)}{R}\,dR.$$

However, in the L^2-case, $\boldsymbol{T}_\alpha$ is of strong type (2, 2) if $\alpha>1/2$. We extend the index α to complex $\sigma+i\tau$, then the norm increases with $e^{\pi|\tau|}$. If we interpolate between $p=1+\varepsilon$ $(\varepsilon>0)$ and $p=2$ changing the index Re α between $1+\dfrac{k-1}{2}+\delta$ $(\delta>0)$ and $\dfrac{1}{2}+\eta$ $(\eta>0)$, we get finally the following theorem.

THEOREM 2. *If* $\alpha>\dfrac{k}{p}+\dfrac{1}{2}(1-k)$, $(1<p\leqq 2)$ *then*

$$\int_{E_k}\left\{\int_0^\infty \frac{|S_R^\alpha(x,f)-S_R^{\alpha-1}(x,f)|^2}{R}\,dR\right\}^{p/2} dx\leqq A_p\int_{E_k}|f(x)|^p dx.$$

COROLLARY. *If* $\frac{2k}{k+1} < p \leqq 2$, *then*

$$\int_{E_k} \left\{ \int_0^\infty \frac{|S_R^1(x,f)-S_R^0(x,f)|^2}{R} dR \right\}^{p/2} dx \leqq A_p \int_{E_k} |f(x)|^p dx .$$

This is a k-dimensional extension of the original g^* function of Littlewood-Paley and the range $\frac{2k}{k+1} < p \leqq 2$ is the conjectured range of the validity of mean convergence of spherical means.

4. Radial functions. If $f(x)$ is radial, that is

$$f(x)=\varphi(|x|)=\varphi(\xi)$$

then $F(y)$ is also radial and

$$\begin{aligned} F(y)&=\Phi(|y|)=\Phi(\eta) \\ &=(2\pi)^{k/2}\int_0^\infty \varphi(\xi)\xi^{k-1}\frac{J_{(k-2)/2}(\eta\xi)}{(\eta\xi)^{(k-2)/2}}d\xi \\ &=(2\pi)^{k/2}\int_0^\infty \varphi(\xi)\xi^{k-1}V_{(k-2)/2}(\eta\xi)d\xi \end{aligned}$$

Now let us set the weight function

$$dm_\nu(\xi)=\xi^{2\nu}d\xi$$

where

$$\nu=(k-1)/2 \geqq 0$$

is the critical index, and consider

$$\varphi(\xi)\in L(dm_\nu).$$

Then the above Fourier transforms reduce to the Hankel transforms

$$\Phi(\eta)=(2\pi)^{k/2}\int_0^\infty \varphi(\xi)V_{\nu-1/2}(\xi\eta)dm_\nu(\xi).$$

For the partial integrals defined by

$$S_a(\xi,\varphi)=(2\pi)^{k/2}\int_0^a \Phi(\eta)V_{\nu-1/2}(\xi\eta)dm_\nu(\eta),$$

C.S.Herz [3] proved the norm inequality such as

$$(1)\qquad \int_0^\infty |S_a(\xi,\varphi)|^p dm_\nu(\xi)\leqq A_p\int_0^\infty |\varphi(\xi)|^p dm_\nu(\xi)$$

provided that

$$\frac{2\nu+1}{\nu+1}=\frac{2k}{k+1}<p\leqq 2 \qquad (\nu\geqq 0).$$

Hence g^* gets a full power and it is a routine argument to prove the following theorem.

THEOREM 3. *Let T be the multiplier transformation defined by*

$$(T\varphi)(\xi)=\int_0^\infty \Phi(\eta)\mu(\eta)V_{\nu-1/2}(\xi\eta)dm_\nu(\eta)$$

where

$$\Phi(\eta)=(2\pi)^{k/2}\int_0^\infty \varphi(\xi)V_{\nu-1/2}(\xi\eta)dm_\nu(\xi)$$

and the multiplier $\mu(\eta)$ satisfies

$$|\mu(\eta)|\leqq M,\ \int_0^\eta t|d\mu(t)|\leqq M\eta, \qquad 0<\eta<\infty.$$

Then the transformation $\varphi\rightarrow T\varphi$ has a bounded extension from $L^p(dm_\nu)$ to $L^p(dm_\nu)$ provided that $(2\nu+1)/(\nu+1)<p<(2\nu+1)/\nu$.

PROOF. At first, we shall reduce the corollary of Theorem 1 to the radial function $\varphi(\xi)$. Thus, if $\frac{2\nu+1}{\nu+1}<p\leqq 2$, then

$$(2)\qquad \int_0^\infty \left\{ \int_0^\infty \left| \int_0^R \eta^2 \Phi(\eta) V_{\nu-1/2}(\xi\eta)\, dm_\nu(\eta) \right|^2 R^{-5}\, dR \right\}^{p/2} dm_\nu(\xi)$$

$$\leqq A_{p,\nu} \int_0^\infty |\varphi(\xi)|^p dm_\nu(\xi).$$

Next we have to generalize the norm inequality (1) to vector-valued functions. This is done by the Herz method [3], and we get

$$(3)\quad \int_0^\infty \left| \int_0^\infty |S_{a(t)}\{\xi, \varphi(\cdot, t)\}|^2 dt \right|^{p/2} dm_\nu(\xi) \leqq A_p \int_0^\infty \left| \int_0^\infty |\varphi(\xi, t)|^2 dt \right|^{p/2} dm_\nu(\xi).$$

From (2) and (3), it is easy to get the following proposition. (See Zygmund [9]).

PROPOSITION. *If* $\dfrac{2\nu+1}{\nu+1} < p < \dfrac{2\nu+1}{\nu}$, *then*

$$\int_0^\infty \left\{ \sum_{k=-\infty}^{\infty} \left| |S_{2^k, 2^{k+1}}(\xi, \varphi)| \right|^2 \right\}^{p/2} dm_\nu(\xi) \leqq A_p \int_0^\infty |\varphi(\xi)|^p dm_\nu(\xi),$$

$$(k=0, \pm 1, \pm 2, \cdots)$$

where

$$S_{a,b}(\xi, \varphi) = \int_a^b \Phi(\eta) V_{\nu-1/2}(\xi\eta)\, dm_\nu(\eta).$$

From this and Herz's theorem we can prove easily the multiplier theorem of Marcinkiewicz type.

REMARK 1. We can extend the original g^*-theorem to the weighted norm. Hence the weight function $m_\nu(x)$ is extensible to wider range.

REMARK 2. In the above theorem, $\nu=(k-1)/2$ and k is any positive integer. By an interpolation argument, we can extend ν to any positive real number in that range.

5. Spherical means of multiple Fourier series. For the sake of simplicity we consider only two variables case. Let $f(x)=f(x_1, x_2)$ be integrable on the unit cube Q and its Fourier series be

$$S(f)=\sum c_{n_1,n_2}e^{2\pi i(n_1x_1+n_2x_2)}$$

and set

$$K_R^\alpha(x_1,x_2)=\sum_{n_1^2+n_2^2<R^2}\left(1-\frac{n_1^2+n_2^2}{R^2}\right)^\alpha e^{-2\pi i(n_1x_1+n_2x_2)},$$

then the spherical (R,n^2,α)-mean of the Fourier series is representable by convolution such as

$$S_R^\alpha(x)=S_R^\alpha(x,f)=(2\pi)^{-2}\int_Q f(x_1+u_1,x_2+u_2)\ K_R^\alpha(u_1,u_2)du_1du_2.$$

The kernel is transformed into (See [2]),

$$(1)\qquad \sum_{n_1^2+n_2^2<R^2}\left(1-\frac{n_1^2+n_2^2}{R^2}\right)^\alpha e^{-2\pi i(n_1x_1+n_2x_2)}$$

$$=CR^{1-\alpha}\sum_{m_1,m_2=-\infty}^{\infty}\frac{J_{1+\alpha}[2\pi R\{(m_1-x_1)^2+(m_2-x_2)^2\}^{1/2}]}{\{(m_1-x_1)^2+(m_2-x_2)^2\}^{(1+\alpha)/2}}$$

where $C=\Gamma(1+\alpha)/\pi^\alpha$. We consider now

$$T_\alpha f=\left\{\int_1^\infty\frac{|S_R^\alpha(x,f)-S_R^{\alpha-1}(x,f)|^2}{R}dR\right\}^{1/2}$$

$$=\left\{\int_1^\infty\left|\frac{\tau_R^\alpha(x,f)}{\sqrt{R}}\right|^2dR\right\}^{1/2}.$$

This corresponds to the T_α in the section 3. It is easy to see that T_α is of strong type (2, 2) provided that $\alpha>1/2$. Next we take $\alpha=1+1/2+\delta(\delta>0)$ and set

$$\tau^{1+1/2+\delta}(x,f)=\int_Q f(x+t)H_R(t)dt$$

where the corresponding kernel is

$$H_R(x)=\frac{1}{R^2}\ \Delta K_R^{1/2+\delta}(x)$$

where Δ is the Laplacian, that is to say

$$\Delta=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}.$$

At first, we consider for $m_1^2+m_2^2>0$

$$(2)\qquad R^{-(1+1/2+\delta)}\sum_{m_1^2+m_2^2>0}\Delta\frac{J_{1+1/2+\delta}[2\pi R\{(m_1-x_1)^2+(m_2-x_2)^2\}]}{\{(m_1-x_1)^2+(m_2-x_2)^2\}^{(1+1/2+\delta)}}.$$

Change to the polar coordinate, then for a radial u, Δu is transformed to

$$\frac{d^2u}{dr^2}+\frac{1}{r}\frac{du}{dr}.$$

Hence the term in the summation has the form such as

$$\Delta V_\alpha(2\pi Rr)$$

where $\alpha=1+1/2+\delta$. Differentiating this

$$\frac{d}{dr}V_\alpha(2\pi Rr)=-R^2rV_{1+\alpha}(2\pi Rr)$$

$$\frac{1}{r}\frac{d}{dr}V_\alpha(2\pi Rr)=-R^2V_{1+\alpha}(2\pi Rr)$$

$$\frac{d^2}{dr^2}V_\alpha(2\pi Rr)=-R^2V_{1+\alpha}(2\pi Rr)+R^4r^2V_{2+\alpha}(2\pi Rr)$$

and

$$\Delta V_\alpha(2\pi Rr)=O(R^{-\delta}r^{-(\delta+2)}).$$

Hence the formula (2) is less than

$$\sum_{m_1^2+m_2^2>0}\Delta V_{1+1/2+\delta}$$

$$=O\left\{R^{-\delta}\sum_{m_1^2+m_2^2>0}\frac{1}{(m_1^2+m_2^2)^{(\delta+2)/2}}\right\}=O(R^{-\delta})$$

Thus we have the estimation

$$H_R(x_1, x_2) = C\Delta(V_{1+1/2+\delta})(2\pi Rr) + O(R^{-\delta})$$

and since

$$\left\{\int_1^\infty \frac{dR}{R^{2\delta+1}}\right\}^{1/2} = C$$

we can neglect the ramainder terms in the following argument. Hence we can proceed to the same as Fourier integral and get the following theorem.

THEOREM 4. *If* $\alpha > \frac{2}{p} - \frac{1}{2}$ $(1 < p \leqq 2)$, *then*

$$\int_Q \left\{\int_1^\infty \frac{|S_R^\alpha(x) - S_R^{\alpha-1}(x)|^2}{R} dR\right\}^{p/2} dx \leqq A_p \int_Q |f(x)|^p dx.$$

From Theorem 4, we can prove easily some of the almost everywhere summability and strong summability theorems such as given in Stein [7].

COROLLARY. *If* $4/3 < p \leqq 2$, *then*

$$\int_Q \left\{\int_1^\infty \frac{|S_R^0(x) - S_R^1(x)|^2}{R} dR\right\}^{p/2} dx \leqq A_p \int_Q |f(x)|^p dx.$$

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