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INDEX AND COBORDISM.

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Introduction. In this paper we shall generalize the results obtained in the paper [5]. In [5] we stated many conditions under which a differentiable manifold becomes bord. In this paper we shall study the conditions under which two differentiable manifolds become cobordant.

1. Let X_{4n} be an orientable compact differentiable $4n$-manifold. Suppose that X_{4n} be differentiably imbedded in the $(4n+4)-$dimensional euclidean space E_{4n+4}. Then we have $\bar{p}_i=0$ $(i\geqq 2)$ where $\bar{p}_i$ $(\in H^{4i}(X_{4n},Z))$ denotes the dual-Pontrjagin class. Next we have

$$(1.\ 1)\qquad (1+\bar{p}_1)^{-1}=\sum_{i\geqq 0}(-1)^i p_i$$

where p_i $(\in H^i(X_{4n},Z))$ denotes the Pontrjagin class. We have from (1.1)

$$(1.\ 2)\qquad p_k=\bar{p}_1^{\,k}\qquad k\geqq 1.$$

Let $\tau(X_{4n})$ be the index of X_{4n}. Then $\tau(X_{4n})$ takes the form:

$$(1.\ 3)\qquad \tau(X_{4n})=\alpha_n\bar{p}_1^{\,n}[X_{4n}]$$

where α_n denotes some rational number depending only on n. When $n\leqq 5$, α_n is not zero, because it is known that

$$(1.\ 4)\quad \begin{cases} \tau(X_4)=\dfrac{1}{3}\,p_1[X_4], \\[2mm] \tau(X_8)=\dfrac{1}{45}\,(7p_2-p_1^2)[X_8], \\[2mm] \tau(X_{12})=\dfrac{1}{3^3\cdot 5\cdot 7}\,(62p_3-13p_2p_1+2p_1^3)[X_{12}], \\[2mm] \tau(X_{16})=\dfrac{1}{3^4\cdot 5^2\cdot 7}\,(381p_4-71p_3p_1-19p_2^2+22p_2p_1^2-3p_1^4)[X_{16}], \end{cases}$$

$$\tau(X_{20})=\frac{1}{3^5\cdot 5^2\cdot 7\cdot 11}(5110p_5-919p_4p_1-336p_3p_2+237p_3p_1^2+127p_2^2p_1-83p_2p_1^3+10p_1^5)[X_{20}]$$

([8] p.13).

We have from (1.2) and (1.4)

$$\text{(1. 5)}\qquad \begin{cases} \tau(X_4)=\dfrac{1}{3}p_1[X_4], \\ \tau(X_8)=\dfrac{2}{15}p_1^2[X_8], \\ \tau(X_{12})=\dfrac{17}{3^2\cdot 5\cdot 7}p_1^3[X_{12}], \\ \tau(X_{16})=\dfrac{62}{3^4\cdot 5\cdot 7}p_1^4[X_{16}], \\ \tau(X_{20})=\dfrac{1382}{3^4\cdot 5^2\cdot 7\cdot 11}p_1^5[X_{20}]. \end{cases}$$

Thus if $\alpha_n\neq 0$, then all Pontrjagin numbers become $p_1^n[X_{4n}]$ and they are completely determined by the index. Therefore we have from the cobordism theory ([6]) the

THEOREM 1. *Let both X_{4n} and Y_{4n} be compact orientable differentiable 4n-manifolds which are differentiably imbedded in the $(4n+4)$-dimensional euclidean space. If X_{4n} and Y_{4n} have a same index and $\alpha_n\neq 0$, then they are cobordant mod torsion, i.e. $2(X_{4n}-Y_{4n})$ is bord.*

REMARKS. (i) When $n\leqq 5$, the assumption $\alpha_n\neq 0$ is not necessary.
(ii) When $n\leqq 3$, "mod torsion" is not necessary.
(iii) If we replace "differentiably imbedded in the $(4n+4)$-dimensional euclidean space" by "of constant 4-sectional curvature"([4]) or by "admit a continuous field of $(4n-3)$-frame"([2]), then the index is expressed in the form $\lambda_n p_1^n[X_{4n}]$, where λ_n denotes some rational number depending only on n. Hence we have similar results in these cases.

Next we consider the case where X_{4n} is connected and almost parallelizable ([1]). In this case we have

$$\text{(1. 6)}\qquad p_i=0 \qquad 1\leqq i\leqq n-1.$$

Hence the only non-zero Pontrjagin number is $p_n[X_{4n}]$. Therefore we have

$$(1.\ 7) \qquad \tau(X_{4n})=\beta_n p_n[X_{4n}]\,,$$

where β_n denotes some rational number depending only on n. We see from (1. 4) that $\beta_n \neq 0$ for $n \leqq 5$.
Hence we have the

THEOREM 2. *Let both X_{4n} and Y_{4n} be compact orientable differentiable $4n$-manifolds. Suppose that they are connected and almost parallelizable and have a same index and $\beta_n \neq 0$. Then they are cobordant mod torsion, i.e. $2(X_{4n}-Y_{4n})$ is bord.*

REMARKS. (i) When $n \leqq 3$, "mod torsion" is not necessary. (ii) When $n \leqq 5$, the assumption $\beta_n \neq 0$ is not necessary. (iii) We can replace "connected and almost parallelizable" by "$(4n-4)$-parallelizable" ([3]).

Next we consider the case where X_{12} is differentialy imbedded in the E_{18}. In this case we have $\bar{p}_3=0$. Hence we have

$$(1.\ 8) \qquad 1-p_1+p_2-p_3=\frac{1}{1+\bar{p}_1+\bar{p}_2}=1-\bar{p}_1+(\bar{p}_1{}^2-\bar{p}_2)+(2\bar{p}_1\bar{p}_2-\bar{p}_1{}^3),\ \text{i.e.}$$

$$(1.\ 9) \qquad p_1=\bar{p}_1,\ p_2=\bar{p}_1{}^2-\bar{p}_2,\ p_3=\bar{p}_1{}^3-2\bar{p}_1\bar{p}_2$$

which leads to

$$(1.10) \qquad \tau(X_{12})=\frac{1}{3^2\cdot 5\cdot 7}(17\bar{p}_1{}^3-37\bar{p}_1\bar{p}_2)[X_{12}].$$

Moreover the A-genus becomes

$$(1.11) \qquad A(X_{12})=\frac{-4}{3^2\cdot 5\cdot 7}(16p_3-44p_2p_1+31p_1{}^3)[X_{12}]$$

$$=\frac{-4}{3\cdot 5\cdot 7}(\bar{p}_1{}^3+4\overline{p_1\bar{p}_2})[X_{12}] \qquad ([8]\ p.14).$$

Therefore all Pontrjagin numbers of X_{12} are determined by τ and A. Hence we have the

THEOREM 3. *Let both X_{12} and Y_{12} be compact orientable differentiable 12-manifolds which are differentiably imbedded in the 18-dimensional euclidean space. If they have same index and A-genus, then they are cobordant.*

2. We would like to state the results obtained in the paper [9] from another point of view. It was proved in [9] that

$$\bar{p}_n[X_{4n}]\equiv 0 \qquad \text{mod } 3 \tag{2.1}$$

and if X_{4n} is a product of two compact orientable differentiable manifolds, then

$$\bar{p}_n[X_{4n}]\equiv 0 \qquad \text{mod } 9. \tag{2.2}$$

Meanwhile it is known that ([10])

$$\frac{(-1)^{q/2}}{(2\pi)^q q!}\,\delta^2\begin{pmatrix} i_1\cdots i_q \\ j_1\cdots j_q\end{pmatrix}\Omega_{i_1 j_1}\cdots\Omega_{i_q j_q}=\bar{p}_{q/2} \qquad (q \textit{ is even}) \tag{2.3}$$

where Ω_{ij} denotes the curvature form of a compact orientable Riemannian manifold and $\delta\begin{pmatrix} i_1\cdots i_q \\ j_1\cdots j_q\end{pmatrix}$ denotes the generalized Kronecker symbol. Hence we have the

THEOREM 4. *Let X_{4n} be a compact orientable Riemannian $4n$-manifold. Then we have*

$$\frac{1}{(2\pi)^{2n}(2n)!}\int_{X_{4n}}\delta^2\begin{pmatrix} i_1\cdots i_{2n} \\ j_1\cdots j_{2n}\end{pmatrix}\Omega_{i_1 j_1}\cdots\Omega_{i_{2n} j_{2n}}\equiv 0 \qquad \text{mod } 3 \tag{2.4}$$

and if moreover X_{4n} is a product of two compact orientable differentiable manifold, then the integral (2.4) *is divisible by* 9.

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