Tôhoku Math. Journ.
Vol. 19, No. 4, 1967

SATURATION OF LOCAL APPROXIMATION BY LINEAR POSITIVE OPERATORS OF BERNSTEIN TYPE

YOSHIYA SUZUKI

(Received June 19, 1967)

1. Introduction. The Bernstein polynomial

$$B_n(f;x)=\sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k}x^k(1-x)^{n-k} \tag{1}$$

is constructed in correspondence with a function $f(x)\in C[0,1]$ and the saturation problem with the polynomial was studied by K. de Leeuw [3] and G.G. Lorentz [4], independently. In the same way that the Bernstein polynomial originates in the identity

$$\sum_{k=0}^{n}\binom{n}{k}x^k(1-x)^{n-k}=1,$$

starting from the identity

$$e^{-nx}\sum_{k=0}^{\infty}\frac{1}{k!}(nx)^k=1,$$

we are led to the operator introduced and investigated by O.Szász [7];

$$S_n(f;x)=e^{-nx}\sum_{k=0}^{\infty} f\left(\frac{k}{n}\right)\frac{1}{k!}(nx)^k. \tag{2}$$

In 1957, V.A.Baskakov [1] gave an example of a sequence of linear positive operators of which the Bernstein polynomial (1) and Szász operator (2) are particular cases. In this paper, we shall determine the order of saturation and its class in the local approximation by a special form of the Baskakov operator which is defined in the following.

In the sequence of real functions

$$\varphi_n(y) \quad (n=1,2,\cdots),$$

each function has the following properties:

1) $\varphi_n(y)$ can be expanded in Taylor's series in $[0, \infty)$,
2) $\varphi_n(0) = 1$,
3) $(-1)^k\varphi_n^{(k)}(x) \geqq 0$ $(k = 0, 1, 2, \cdots)$, for $x \in [0, \infty)$,
4) $-\varphi_n^{(k)}(x) = n\varphi_{n+c}^{(k-1)}(x)$ $(k = 1, 2, \cdots)$, $x \in [0, \infty)$,

where c is an integer.
We expand the function $\varphi_n(y)$ in a Taylor's series with $x \in [0, \infty)$, that is

$$\varphi_n(y) = \sum_{k=0}^{\infty} \frac{\varphi_n^{(k)}(x)}{k!}(y-x)^k.$$

By 1) and 2),

$$\sum_{k=0}^{\infty}(-1)^k \frac{\varphi_n^{(k)}(x)}{k!} x^k = 1. \tag{3}$$

Now we define for $x \in [0, \infty)$ the linear operator $M_n(f; x)$ by

$$M_n(f; x) = \sum_{k=0}^{\infty}(-1)^k \frac{\varphi_n^{(k)}(x)}{k!} x^k f\left(\frac{k}{n}\right), (n = 1, 2, \cdots). \tag{4}$$

It has a meaning for each function $f(x)$ which is continuous on $[0,R]$ and zero on (R, ∞), and it is positive on account of 3), where $R(\geqq 1)$ is an arbitrarily fixed positive number.

REMARK 1. We say a linear operator L_n is positive if the positivity of $f(x)$ implies the positivity of $L_n(f : x)$. In this case we note that

$$L_n(f; x) \leqq L_n(g; x), \quad x \in [a, b]$$

if $f(x) \leqq g(x)$ on the finite interval $[a, b]$.

2. Auxiliary theorems. In this section, we consider the local approximation of continuous functions by linear positive operators $L_n(f, x)$, which have the follwoing properties;

P_1: If $f(x)$ is a linear function on a finite interval $[a, b]$ $(0 \leqq a < b < \infty)$, then $L_n(f; x) = f(x)$, $x \in [a, b]$.

P_2: If $f(x) = Ax^2$ on $[a, b]$, then

$$L_n(f; x) - f(x) = A\frac{\psi(x)}{n} + O\left(\frac{1}{n}\right), \text{ uniformly over } [a, b],$$

where the weight function $\psi(x)$ is a bounded, twice continuously differentiable and not equal to zero on (a, b).

P_3: There exists a positive integer $m(>1)$ such that

$$L_n\{(t-x)^{2m}; x\} = O\left(\frac{1}{n}\right), \text{ uniformly over } [a, b].$$

THEOREM A. (P. P. Korovkin [2]). *Let* $\{L_n(f;x)\}$ *be an infinite sequence of linear positive operators, which satisfies the three conditions*

$$L_n(1; x) = 1 + \alpha_n(x),$$
$$L_n(t; x) = x + \beta_n(x)$$

and

$$L_n(t^2; x) = x^2 + \gamma_n(x),$$

$\alpha_n(x)$, $\beta_n(x)$ *and* $\gamma_n(x)$ *being any functions uniformly tending to zero in* $[a, b]$ *as* $n \to \infty$. *Then* $L_n(f)$ *converges uniformly in* $[a, b]$ *to* $f(x)$, *if* $f(x)$ *is continuous in* $[a, b]$.

THEOREM B. (R. G. Mamedov [5] and F. Schurer [6]). *Assume that the sequence of linear positive operators* $\{L_n(f; x)\}$ *has the property that*

$$L_n(1; x) = 1, \quad x \in [a, b],$$

$$L_n(t; x) = x + \frac{\psi_1(x)}{\varphi(n)} + o\left(\frac{1}{\varphi(n)}\right), \text{ uniformly over } [a, b],$$

$$L_n(t^2; x) = x^2 + \frac{\psi_2(x)}{\varphi(n)} + o\left(\frac{1}{\varphi(n)}\right), \text{ uniformly over } [a, b].$$

If there exists a positive integer $m(>1)$ *such that*

$$L_n\{(t-x)^{2m}; x\} = o\left(\frac{1}{\varphi(n)}\right), \text{ uniformly over } [a, b],$$

then for each function $f(x) \in C^{(2)}[a, b]$, *we have*

$$L_n(f; x) - f(x) = \frac{2f'(x)\psi_1(x) + f''(x)\{\psi_2(x) - 2x\psi_1(x)\}}{2\varphi(n)} + o\left(\frac{1}{\varphi(n)}\right),$$

$$\text{uniformly on } [a_1, b_1],\ a < a_1 < b_1 < b,$$

where $C^{(2)}[a,b]$ *is the set of all real functions* $f(x)$ *of which the second derivatives* $f''(x)$ *exist in* $[a,b]$ *and are bounded.*

PROPOSITION 1. *If the three conditions*

$$L_n(1;x) = 1, \quad x \in [a,b],$$

$$L_n(t;x) = x, \quad x \in [a,b],$$

$$L_n(t^2;x) = x^2 + \frac{\psi(x)}{n} + o\left(\frac{1}{n}\right), \text{ uniformly on } [a,b]$$

are satisfied for a sequence of linear positive operators $\{L_n(f;x)\}$, *which have the property*

$$L_n\{(t-x)^4;x\} = o\left(\frac{1}{n}\right), \text{ uniformly on } [a,b],$$

then for each function $f(x) \in C^{(2)}[a,b]$, *we get*

$$L_n(f;x) - f(x) = \frac{\psi(x)f''(x)}{2n} + o\left(\frac{1}{n}\right), \text{ uniformly on } [a_1,b_1],$$

where $a < a_1 < b_1 < b$.

PROOF. In the above theorem B, we have only to set

$$m = 2, \psi_1(x) \equiv 0, \varphi(n) = n \text{ and } \psi_2(x) = \psi(x).$$

Analogously we have

PROPOSITION 2. *Let a sequence of linear positive operators* $\{L_n(f;x)\}$ *have the same properties as the assumption of proposition* 1, *then for each function* $f(x) \in C^{(2)}[a,b]$, *we get*

$$L_n(f;x) - f(x) = \frac{\psi(x)f''(x)}{2n} + o\left(\frac{1}{n}\right), \quad x \in [a,b].$$

3. Asymptotic formula of a special linear positive operator.

PROPOSITION 3. *The sequence of operators* $\{M_n(f;x)\}$ *converges in* $[0,R]$ *uniformly to the function* $f(x)$, *if* $f(x)$ *is continuous in* $[0,R]$ *and equal to zero on* (R,∞).

RROOF. We show that the sequence of operators $\{M_n(f;x)\}$ converges uniformly in $[0, R]$ to $f(x)$ if $n \to \infty$ in the cases $f(x) \equiv 1$, $f(x) = x$, and $f(x)=x^2$. The uniform convergence of $M_n(f; x)$ to 1 in the case $f(x) \equiv 1$ follows from (3). When $f(x)=x$, we have to consider the series

$$M_n(t; x) = \sum_{k=0}^{\infty} (-1)^k \frac{\varphi_n^{(k)}(x)}{k!} x^k \frac{k}{n}.$$

Using the identity 4), the right hand is equal to

$$x \sum_{k=1}^{\infty} (-1)^{k-1} \frac{\varphi_{n+c}^{(k-1)}(x)}{(k-1)!} x^{k-1} \tag{5}$$

$$= x\varphi_{n+c}(0) = x \cdot 1 = x .$$

When $f(x)=x^2$ we arrive at the series

$$\begin{aligned} M_n(t^2; x) &= \sum_{k=0}^{\infty} (-1)^k \frac{\varphi_n^{(k)}(x)}{k!} x^k \left(\frac{k}{n}\right)^2 \\ &= x \sum_{k=0}^{\infty} (-1)^{k-1} \frac{\varphi_{n+c}^{(k-1)}(x)}{(k-1)!} x^{k-1} \frac{k}{n} \\ &= x^2 \frac{n+c}{n} \sum_{k=2}^{\infty} (-1)^{k-2} \frac{\varphi_{n+2c}^{(k-2)}(x)}{(k-2)!} x^{k-2} \\ &\qquad + \frac{k}{n} \sum_{k=1}^{\infty} (-1)^{k-1} \frac{\varphi_{n+c}^{(k-1)}(x)}{(k-1)!} x^{k-1}. \end{aligned}$$

Since

$$\sum_{k=2}^{\infty} (-1)^{k-2} \frac{\varphi_{n+2c}^{(k-2)}(x)}{(k-2)!} x^{k-2} = \varphi_{n+2c}(0) = 1$$

and

$$\sum_{k=1}^{\infty} (-1)^{k-1} \frac{\varphi_{n+c}^{(k-1)}(x)}{(k-1)!} x^{k-1} = \varphi_{n+c}(0) = 1,$$

we have

$$M_n(t^2; x) = x^2 + \frac{x(1+cx)}{n} . \tag{6}$$

From (3), (5) and (6), applying theorem A here, we obtain the proposition 3.

PROPOSITION 4. *For each function* $f(x) \in C^{(2)}[a, b]$, *we have*

$$M_n(f; x) = f(x) + \frac{f''(x)}{2} \frac{x(1+cx)}{n} + o\left(\frac{1}{n}\right), \text{ uniformly over } [a_1, b_1],$$

where $0 \leqq a < a_1 < b_1 < b \leqq R$.

PROOF. We have only to verify the fact that

$$M_n\{(t-x)^4; x\} = o\left(\frac{1}{n}\right), \text{ uniformly on } [a, b].$$

From (3), (5) and (6),

$$\text{(7)} \quad M_n\{(t-x)^4; x\} = M_n(t^4 - 4t^3x + 6t^2x^2 - 4tx^3 + x^4; x)$$

$$= M_n(t^4; x) - 4xM_n(t^3; x) + 3x^4 + \frac{6x^2(cx^2+x)}{n}$$

$$= M^{(1)} - 4xM^{(2)} + 3x^4 + \frac{6x^2(cx^2+x)}{n}, \text{ say.}$$

$$\text{(8)} \quad M^{(2)} = \sum_{k=0}^{\infty} (-1)^k \frac{\varphi_n^{(k)}(x)}{k!} x^k \left(\frac{k}{n}\right)^3 = x\sum_{k=1}^{\infty} (-1)^{k-1} \frac{\varphi_{n+c}^{(k-1)}(x)}{(k-1)!} x^{k-1} \left(\frac{k}{n}\right)^2$$

$$= \frac{n+c}{n} x^2 \sum_{k=2}^{\infty} (-1)^{k-2} \frac{\varphi_{n+2c}^{(k-2)}(x)}{(k-2)!} x^{k-2} \frac{k}{n} + x\sum_{k=1}^{\infty} (-1)^{k-1} \frac{\varphi_{n+c}^{(k-1)}(x)}{(k-1)!} x^{k-1} \frac{k}{n^2}$$

$$= \frac{(n+c)(n+2c)}{n^2} x^3 \sum_{k=3}^{\infty} (-1)^{k-3} \frac{\varphi_{n+3c}^{(k-3)}(x)}{(k-3)!} x^{k-3} + \frac{2(n+c)x^2}{n^2} \sum_{k=2}^{\infty} (-1)^{k-2} \frac{\varphi_{n+2c}^{(k-2)}(x)}{(k-2)!} x^{k-2}$$

$$+ \frac{(n+c)x^2}{n^2} \sum_{k=2}^{\infty} (-1)^{k-2} \frac{\varphi_{n+2c}^{(k-2)}(x)}{(k-2)!} x^{k-2} + \frac{x}{n^2} \sum_{k=1}^{\infty} (-1)^{k-1} \frac{\varphi_{n+c}^{(k-1)}(x)}{(k-1)!} x^{k-1}$$

$$= \frac{(n+c)(n+2c)x^3}{n^2} + \frac{2(n+c)x^2}{n^2} + \frac{(n+c)x^2}{n^2} + \frac{x}{n^2}$$

$$= \left(1 + \frac{3c}{n} + \frac{2c^2}{n^2}\right)x^3 + \left(\frac{3}{n} + \frac{3c}{n^2}\right)x^2 + \frac{x}{n^2}.$$

On the other hand

$$M^{(1)}=\sum_{k=0}^{\infty}(-1)^k\frac{\varphi_n^{(k)}(x)}{k!}x^k\left(\frac{k}{n}\right)^4=x\sum_{k=1}^{\infty}(-1)^{k-1}\frac{\varphi_{n+c}^{(k-1)}(x)}{(k-1)!}x^{k-1}\left(\frac{k}{n}\right)^3$$

$$=\frac{n+c}{n}x^2\sum_{k=2}^{\infty}(-1)^{k-2}\frac{\varphi_{n+2c}^{(k-2)}(x)}{(k-2)!}x^{k-2}\left(\frac{k}{n}\right)^2+x\sum_{k=1}^{\infty}(-1)^{k-1}\frac{\varphi_{n+c}^{(k-1)}(x)}{(k-1)!}x^{k-1}\frac{k^2}{n^3}$$

$=M_{(1)}^{(1)}+M_{(2)}^{(1)}$, say.

(9) $$M_{(1)}^{(1)}=\frac{(n+c)(n+2c)}{n^2}x^3\sum_{k=3}^{\infty}(-1)^{k-3}\frac{\varphi_{n+3c}^{(k-3)}(x)}{(k-3)!}x^{k-3}\frac{k}{n}$$

$$+\frac{2(n+c)}{n^2}x^2\sum_{k=2}^{\infty}(-1)^{k-2}\frac{\varphi_{n+2c}^{(k-2)}(x)}{(k-2)!}x^{k-2}\frac{k}{n}$$

$$=\frac{(n+c)(n+2c)(n+3c)}{n^2}x^4\sum_{k=4}^{\infty}(-1)^{k-4}\frac{\varphi_{n+4c}^{(k-4)}(x)}{(k-4)!}x^{k-4}$$

$$+\frac{3(n+c)(n+2c)}{n^3}x^3\sum_{k=3}^{\infty}(-1)^{k-3}\frac{\varphi_{n+3c}^{(k-3)}(x)}{(k-3)!}x^{k-3}$$

$$+\frac{2(n+c)(n+2c)}{n^3}x^3\sum_{k=3}^{\infty}(-1)^{k-3}\frac{\varphi_{n+3c}^{(k-3)}(x)}{(k-3)!}x^{k-3}$$

$$+\frac{4(n+c)}{n^3}x^2\sum_{k=2}^{\infty}(-1)^{k-2}\frac{\varphi_{n+2c}^{(k-2)}(x)}{(k-2)!}x^{k-2}$$

$$=\frac{(n+c)(n+2c)(n+3c)}{n^3}x^4+\frac{5(n+c)(n+2c)}{n^3}x^3+\frac{4(n+c)}{n^3}x^2.$$

(10) $$M_{(2)}^{(1)}=\frac{x^2(x+c)}{n^2}\sum_{k=2}^{\infty}(-1)^{k-2}\frac{\varphi_{n+2c}^{(k-2)}(x)}{(k-2)!}x^{k-2}\frac{k}{n}+\frac{x}{n^2}\sum_{k=1}^{\infty}(-1)^{k-1}\frac{\varphi_{n+c}^{(k-1)}(x)}{(k-1)!}\frac{k}{n}$$

$$=\frac{(n+c)(n+2c)}{n^3}x^3\sum_{k=3}^{\infty}(-1)^{k-3}\frac{\varphi_{n+3c}^{(k-3)}(x)}{(k-3)!}x^{k-3}$$

$$+\frac{2(n+c)x^2}{n^3}\sum_{k=2}^{\infty}(-1)^{k-2}\frac{\varphi_{n+2c}^{(k-2)}(x)}{(k-2)!}x^{k-2}+\frac{n+c}{n^3}x^2\sum_{k=2}^{\infty}(-1)^{k-2}\frac{\varphi_{n+c}^{(k-2)}(x)}{(k-1)!}x^{k-2}$$

$$+\frac{x}{n^3}\sum_{k=1}^{\infty}(-1)^{k-1}\frac{\varphi_{n+c}^{(k-1)}(x)}{(k-1)!}x^{k-1}$$

$$=\frac{(n+c)(n+2c)}{n^3}x^3+\frac{3(n+c)}{n^3}x^2+\frac{x}{n^3}.$$

By (9) and (10)

$$(11)\quad M^{(1)}=\frac{(n+c)(n+2c)(n+3c)}{n^3}x^4+\frac{6(n+c)(n+2c)}{n^3}x^3+\frac{7(n+c)}{n^3}x+\frac{x}{n^3}$$

$$=\left(1+\frac{6c}{n}+\frac{11c^2}{n^2}+\frac{6c^3}{n^3}\right)x^4+\left(\frac{6}{n}+\frac{18c}{n^2}+\frac{12c^2}{n^3}\right)x^3$$

$$+\left(\frac{7}{n^2}+\frac{7c}{n^3}\right)x^2+\frac{x}{n^3}.$$

By (7), (8) and (11)

$$(12)\quad M_n\{(t-x)^4;x\}=x^4+\frac{6cx^4+6x^3}{n}+\frac{11c^2x^4+18cx^3+7x^2}{n^2}$$

$$+\frac{6c^3x^4+12c^2x^3+7cx^2+x}{n^3}+3x^4+\frac{6cx^4+6x^3}{n}-4x^4$$

$$-\frac{12cx^4+12x^3}{n}-\frac{8c^2x^4+12cx^3+4x^2}{n^2}$$

$$=\frac{3c^2x^4+6cx^3+3x^2}{n^2}+\frac{6c^3x^4+12c^2x^3+7cx^2+x}{n^3}$$

$$=o\left(\frac{1}{n}\right),\ \text{uniformly over } [a,b].$$

Consequently, using (12) and proposition 1 we have for each function $f(x)\in C^{(2)}[a,b]$,

$$M_n(f;x)=f(x)+\frac{f''(x)}{2}\frac{x(1+cx)}{n}+o\left(\frac{1}{n}\right),\ \text{uniformly on } [a_1,b_1].$$

Analogously, using (12) and proposition 2 we obtain

PROPOSITION 5. *For any function* $f(x)\in C^{(2)}[a,b]$, *we have*

$$M_n(f;x)=f(x)+\frac{f''(x)}{2}\frac{x+cx^2}{n}+o\left(\frac{1}{n}\right),\ x\in[a,b],$$

where $0\leqq a<b\leqq R$.

COROLLARY 1. (E.V.Voronovskaja [8]). *If $f(x)$ belongs to $C^{(2)}[a,b]$, then*

$$B_n(f;x)=f(x)+\frac{f''(x)}{2}\frac{x(1-x)}{n}+o\left(\frac{1}{n}\right),\ x\in[a,b].$$

This holds uniformly on the interval $[a_1,b_1]$ if $0\leqq a<a_1<b_1<b\leqq 1$.

PROOF. In the proposition 4 and proposition 5, we set

$$[0,R]=[0,1],\ \varphi_n(y)=(1-y)^n \text{ and } C=-1.$$

COROLLARY 2 (O. Szász [7]). *If $f(x)$ belongs to $C^{(2)}[a,b]$, then*

$$S_n(f;x)=f(x)+\frac{f''(x)}{2}\frac{x}{n}+o\left(\frac{1}{n}\right),\ x\in[a,b].$$

This holds uniformly on the interval $[a_1,b_1]$ if $0\leqq a<a_1<b_1<b\leqq R$.

PROOF. We have only to take

$$\varphi_n(y)=e^{-ny} \text{ and } C=0,$$

in the proposition 4 and proposition 5.

4. Local saturation theorem. Let $0\leqq a<b\leqq R$ be given. We consider the following class U of functions $u(x), x\in[0,R]: u(x)=\psi(x)q(x)$, where $q(x)$ is twice continuously differentiable, vanishes outside of an interval (α,β) with $a<\alpha<\beta<b$. Auxiliary numbers a_i and b_i $(i=1,2)$ are chosen to satisfy $a<a_1<a_2<\alpha<\beta<b_2<b_1<b$. For the linear positive operators and $f(x)\in C[a,b]$, let us define the linear functionals A_n by

$$A_n(f)=2\sum_{na<k<bn}\frac{L_n\left(f;\frac{k}{n}\right)-f\left(\frac{k}{n}\right)}{\psi\left(\frac{k}{n}\right)}u\left(\frac{k}{n}\right) \tag{13}$$

$$=2\sum_{na_1<k<nb_1}\left[L_n\left(f;\frac{k}{n}\right)-f\left(\frac{k}{n}\right)\right]q\left(\frac{k}{n}\right).$$

We assume that for each $u(x)\in U$, there is a constant K such that

$$|A_n(f)|\leqq k\|f\|,\qquad \|f\|\equiv\max_{x\in[a,b]}|f(x)|,$$

where the constant K is independent of $f(x)$.

THEOREM 1. *For the linear positive operators which have the properties:* P_1, P_2, P_3 *and* $f(x) \in C[a, b]$, *we get*

(i) *If there is a constant* K *such that*

$$|A_n(g)| \leqq K\|g\|, \qquad \textit{for any } g(x) \in C[a, b],$$

and

$$(14) \qquad |L_n(f; x) - f(x)| < \frac{M\psi(x)}{2n}, \ x \in [a, b] \ (n=1, 2, \cdots),$$

then $f(x)$ *has a derivative which belongs to* $\mathrm{Lip}_M 1$ *on* $[a, b]$.

(ii) *If* $f'(x)$ *exists and belongs to* $\mathrm{Lip}_M 1$ *on* $[a_1, b_1]$, *then*

$$|L_n(f; x) - f(x)| < \frac{M\psi(x)}{2n} + o\left(\frac{1}{n}\right), \textit{ uniformly in } x \in [a_2, b_2].$$

(iii) *If in addition to the assumption of* (i), *the relation*

$$L_n(f; x) - f(x) = o\left(\frac{1}{n}\right)$$

holds a.e. on $[a_1, b_1]$, *then* $f(x)$ *is linear on* $[a_1, b_1]$.

We write this result by the notation

$$\text{L.Sat.}\,[L_n] = [f' \in \mathrm{Lip}_M 1, n^{-1}, \text{linear futction}, \psi(x)].$$

4.1. Proof of (i) **of theorem 1.** We prove the relation

$$(15) \qquad \lim_{n\to\infty} A_n(g) = \int_a^b g(x)u''(x)dx, \text{ for } g(x) \in C[a, b] \text{ and } u(x) \in U.$$

Assume first that $g(x) \in C^{(2)}[a, b]$. From the properties P_1, P_2, P_3 and proposition 1, we have

$$(16) \qquad L_n(g; x) - g(x) = \frac{\psi(x)g''(x)}{2n} + o\left(\frac{1}{n}\right), \text{ uniformly in } x \in [a_1, b_1],$$

for any $g(x) \in C^{(2)}[a, b]$.

From (13) and (16) it follows that

$$A_n(g)=\frac{1}{n}\sum_{na_1<k<nb_1} g''\left(\frac{k}{n}\right)u\left(\frac{k}{n}\right)+o\,(1)$$

$$\longrightarrow\int_a^b g''(x)u(x)dx \quad (n\to\infty),$$

which is equivalent to (15). Since $C^{(2)}[a,b]$ is dense in $C[a,b]$ and there is a constant K such that

$$|A_n(g)|\leqq k\|g\|, \text{ for any } g(x)\in C[a,b],$$

the relation (15) is established for all $g(x)\in C[a,b]$. In the other hand, we can write

$$A_n(f)=\int_a^b u(x)d\lambda_n(x)\,, \tag{17}$$

with the step function

$$\lambda_n(x)=2\sum\frac{L_n\left(f;\frac{k}{n}\right)-f\left(\frac{k}{n}\right)}{\psi\left(\frac{k}{n}\right)},$$

where the summation run over k such that $a<\dfrac{k}{n}<x$. We assume that $f(x)$ satisfies (14) for $x\in[a,b]$. Then the function $\lambda_n(x)$ have a total variation not greater than MR, and an increment $|\lambda_n(x)-\lambda_n(y)|$ does not exceed the number of points $\dfrac{k}{n}$ in $[x,y]$ multiplied with $\dfrac{MR}{n}$. By Helly's theorem [9], we can extract a subsequence $\{\lambda_{n_p}(x)\}$ which converges on $[a,b]$ to a function $\lambda(x)$ of bounded variation and we have

$$\lim_{p\to\infty} A_{n_p}(f)=\int_a^b u(x)d\lambda(x). \tag{18}$$

From (15) and (18)

$$\int_a^b f(x)u''(x)dx=\int_a^b \Lambda(x)u''(x)dx,$$

where $\Lambda(x)$ is an indefinite integral of $\lambda(x)$. Since this is true for all $u(x) \in U$, we have

$$f(x) = \Lambda(x) + g(x) + h, \quad x \in [a, b],$$

with some constants g and h.
Hence

$$f'(x) = \lambda(x) + g, \; x \in [a, b].$$

For the completion of (i), we have only to verify that $\lambda(x)$ belongs to $\mathrm{Lip}_M 1$, which is trivial by the definition of $\lambda(x)$.

4.2. Proof of (ii) **of theorem 1.** From the hypothesis, for $t, x \in [a_2, b_2]$,

$$\begin{aligned} |f(t)-f(x)-(t-x)f'(x)| &= \left| \int_x^t [f'(y)-f'(x)]dy \right| \\ &\leqq M \left| \int_x^t (y-x) \; dy \right| \\ &= \frac{1}{2} M(t-x)^2 . \end{aligned}$$

Since

$$L_n\{(t-x)^2; x\} = L_n(t^2; x) - x^2 = \frac{\psi(x)}{n} + o\left(\frac{1}{n}\right), \text{ uniformly in } x,$$

we get

$$\begin{aligned} |L_n(f; x) - f(x)| &= |L_n\{f(t) - f(x) - (t-x)f'(x); x\}| \\ &\leqq L_n\{|f(t) - f(x) - (t-x)f'(x)|; x\} \\ &\leqq \frac{M}{2} L_n\{(t-x)^2; x\} \\ &= \frac{M}{2} L_n(t^2 - 2xt + x^2; x) \\ &= \frac{M}{2} \frac{\psi(x)}{n} + o\left(\frac{1}{n}\right), \text{ uniformly in } x \in [a_2, b_2], \end{aligned}$$

which concludes the proof of (ii).

4.3. Proof of (iii) **of theorem 1.** Since (14) is satisfied, hence, since $f'(x)$ is absolutely continuous, $f''(x)$ exists a.e. on $[a_1, b_1]$. By proposition 2, for almost all $x \in [a_1, b_1]$

$$\lim_{n\to\infty} n\{L_n(f; x) - f(x)\} = \frac{\psi(x)f''(x)}{2}. \tag{19}$$

From (19) and the assumption of (iii), we have

$$\frac{\psi(x)f''(x)}{2} = 0 \qquad \text{a.e. on } [a_1, b_1].$$

Since $\psi(x) \neq 0$ over $[a_1, b_1]$, we get

$$f''(x) = 0 \qquad \text{a.e. on } [a_1, b_1].$$

Consequently, we have that $f(x)$ is linear on $[a_1, b_1]$ by the continuity of $f'(x)$. Thus, we complete the proof of theorem 1.

5. Determination of the class of local saturation by some linear positive operators on $C[a, b]$, $O \leqq a < b \leqq R$ $(R \geqq 1)$.

5.1. We consider V.A. Baskakov's operator $M_n(f; x)$ which satisfies the following conditions:

$$\frac{np_{n+c,l}}{p_{nl}} = \frac{lc+n}{cx+1}, \tag{20}$$

$$\int_0^{E(c)} p_{nl}(x)dx = \frac{1}{n-c}, \quad \int_0^{E(c)} xp_{nl}(x)dx = \frac{l+1}{(n-c)(n-2c)}, \tag{21}$$

$$\sum_{\alpha \leqq \frac{l}{n} \leqq \beta} \int_R^{E(c)} x^i p_{nl}(x)dx = O\left(\frac{1}{n}\right) \ (i = 0, 1), \tag{22}$$

where

$$P_{nl}(x) = (-1)^l \frac{\varphi_n^{(l)}(x)}{l!} x^l, \ E(c) = \frac{5c^2 - c - 2}{2c(c-1)} \ (\text{if } c \neq 0, 1),$$

and $E(1) = E(0) = \infty$.

THEOREM 2. *For space* $C[a, b]$ $(0 \leqq a < b \leqq R)$, *V.A. Baskakov's operator* $M_n(f; x)$ *which satisfies the properties* (20), (21) *and* (22), *is saturated locally with the weight function* $\psi(x) = x(1+cx)$. *That is,*

$$\text{L.Sat.}[M_n] = [f'(x) \in \text{Lip}_M 1,\ n^{-1},\ \textit{linear function},\ x(1+cx)].$$

Applying theorem 1, for the proof of theorem 2, we have only to verify the existence of a constant K such that

$$|A_n(f)| \leqq K\|f\|, \text{ for any } f(x) \in C[a, b].$$

In order to prove it, we need the following lemmas.

LEMMA 1. *Let us write*

$$T_{nr} = \sum_{l=0}^{\infty} (nx-l)^r p_{nl}(x) \textit{ and } T_n^* = \sum_{l=0}^{\infty} |nx-l|\, p_{nl}(x)\,.$$

Then, we have

$$T_{n0} = 1,\ T_{n1} = 0,\ T_{n2} = nx(1+cx) \tag{23}$$

and

$$T_n^* \leqq \sqrt{nx(1+cx)}\,. \tag{23$'$}$$

PROOF.

$$T_{n0} = \sum_{l=0}^{\infty} p_{nl}(x) = 1,\ T_{n1} = \sum_{l=0}^{\infty} (nx-l) p_{nl}(x) = nx - nx = 0,$$

$$T_{n2} = \sum_{l=0}^{\infty} (nx-l)^2 p_{nl}(x) = n^2x^2 - 2n^2x^2 + n^2 \sum_{l=0}^{\infty} \frac{l^2}{n^2} p_{nl}(x)$$

$$= n^2 \sum_{l=0}^{\infty} \frac{l^2}{n^2} p_{nl}(x) - n^2x^2 = n^2\left(x^2 + \frac{cx^2+x}{n}\right) - n^2x^2$$

$$= nx(1+cx)\,,$$

$$T_n^* = \sum_{l=0}^{\infty} |nx-l|\, p_{nl}(x) \leqq \left\{\sum_{l=0}^{\infty} p_{nl}(x)|nx-l|^2\right\}^{\frac{1}{2}} \left\{\sum_{l=0}^{\infty} p_{nl}(x)\right\}^{\frac{1}{2}}$$

$$= \{T_{n2}\}^{\frac{1}{2}} = \sqrt{nxc(1+cx)}\,.$$

LEMMA 2. *For given* $0 < a < b < R$, *there exist constants* C_r $(r = 1, 2)$ *such that for each polynomial*

$$R_n(x) = \sum_{l=0}^{\infty} a_l p_{nl}(x),\ |a_l| \leqq L,$$

one has

$$|R_n^{(r)}(x)| \leqq C_r L n^{\frac{r}{2}},\ \textit{if } a \leqq x \leqq b \textit{ and } cx + 1 \neq 0. \tag{24}$$

PROOF. Let us set $X = \{x(1+cx)\}^{-1}$. From the property (20), we have

(25) $p'_{nl}(x) = (l - nx)Xp_{nl}(x)$

(26) $p''_{nl}(x) = -nXp_{nl}(x) + (l - nx)^2X^2p_{nl}(x) - (2cx + 1)(l - nx)X^2p_{nl}(x)$.

From (25), we get

$$\begin{aligned} |R'_n(f;\ x)| &\leqq L\, X \sum_{l=0}^{\infty} |l - nx|\, p_{nl}(x) \\ &\leqq L\, X\, X^{-\frac{1}{2}} n^{\frac{1}{2}} \\ &= \frac{L\sqrt{n}}{\sqrt{x(1+cx)}} \leqq C_1\, L\, n^{\frac{1}{2}}. \end{aligned} \tag{27}$$

By (26), we have

$$\begin{aligned} R''_n(f; x) &= \sum_{l=0}^{\infty} a_l\, p''_{ln}(x) \\ &= -nX \sum_{l=0}^{\infty} a_l p_{nl}(x) + X^2 \sum_{l=0}^{\infty} a_l (l - nx)^2 p_{nl}(x) \\ &\quad - (2cx + 1)X^2 \sum_{l=0}^{\infty} a_l (l - nx) p_{nl}(x). \end{aligned}$$

Since

$$\sum_{l=0}^{\infty} (l - nx) p_{nl}(x) = -nx \sum_{l=0}^{\infty} a_l p_{nl}(x) + n \sum_{l=0}^{\infty} a_l \frac{l}{n}\, p_{nl}(x),$$

we get

$$(28) \qquad |R''(f;x)| \leqq L\{nX + nx(1+cx)X^2 + 2nx(2cx+1)X^2\}$$
$$= Ln\left\{\frac{x+3}{x(1+cx)} + \frac{2c}{(cx+1)^2}\right\}$$
$$= 2Ln\frac{(3cx+2)}{x(cx+1)^2}$$
$$\leqq C_2 Ln.$$

From (27) and (28), we obtain lemma 2.

LEMMA 3. *For arbitrary* $\delta > 0$, *there is a constant* $C(\delta)$ *such that*

$$(29) \qquad \sum_{|\frac{l}{n}-x|\geqq\delta} p_{nl}(x) \leqq C(\delta)n^{-1}, x \in [a, b].$$

PROOF.

$$\sum_{|\frac{l}{n}-x|\geqq\delta} p_{nl}(x) \leqq \delta^{-2} \sum_{|\frac{l}{n}-x|\geqq\delta} \left(\frac{l}{n}-x\right)^2 p_{nl}(x)$$
$$\leqq n^{-2}\delta^{-2}T_{n2}$$
$$= n^{-1}\delta^{-2}x(1+cx)$$
$$\leqq C(\delta)\frac{1}{n}, \ x \in [a, b].$$

LEMMA 4. *Let* $Q_n(x)$ *be a sequence of twice continuously differentiable functions on* $[0, R]$; *let* (A) *the maximum* μ_n *of* $|Q_n(x)|$ *on the intervals* $(0, a_2)$ *anb* (b_2, R) *be* $\mu_n = O(n^{-1})$, *and* (B) *the maximum* M_n *of* $|Q_n''(x)|$ *on* (a_1, b_1) *be* $M_n = O(n)$. *Then*

$$(30) \qquad \sum_{na_1<k<nb_1} Q_n\left(\frac{k}{n}\right) - n\int_0^R Q_n(x)dx = O(1).$$

REMARK 2. The lemma 4 is a slight modification of a lemma by G.G. Lorentz [4]; we omit the proof.

5.2. Proof of theorem 2. We can rewrite (13) in the form

$$A_n(f) = 2\sum_{l=0}^{\infty} f\Big(\frac{l}{n}\Big)\Big\{\sum_{na_1<k<nb_1} q\Big(\frac{k}{n}\Big)p_{nl}\Big(\frac{k}{n}\Big) - q\Big(\frac{l}{n}\Big)\Big\}$$

$$= 2\sum_{l=0}^{\infty} f\Big(\frac{l}{n}\Big)\Big[q\Big(\frac{l}{n}\Big)\Big\{\sum_{na_1<k<nb_1} p_{nl}\Big(\frac{k}{n}\Big) - 1\Big\}$$

$$+ q'\Big(\frac{l}{n}\Big)\sum_{na_1<k<nb_1}\Big(\frac{k}{n} - \frac{l}{n}\Big)p_{nl}\Big(\frac{k}{n}\Big)$$

$$+ \frac{1}{2}\sum_{na_1<k<nb_1}\Big(\frac{k}{n} - \frac{l}{n}\Big)^2 p_{nl}\Big(\frac{k}{n}\Big)q''(\xi_{kl})\Big],$$

where the ξ_{kl} are between $\frac{k}{n}$ and $\frac{l}{n}$. Since $q''(x)$ is bounded, the statement will follow if we can prove that the three sums

$$S_n^{(1)} = \sum_{l=0}^{\infty}\Big|\sum_{na_1<k<nb_1} p_{nl}\Big(\frac{k}{n}\Big) - 1\Big|\,\Big|q\Big(\frac{l}{n}\Big)\Big|,$$

$$S_n^{(2)} = \sum_{l=0}^{\infty}\Big|\sum_{na_1<k<nb_1}\Big(\frac{k}{n} - \frac{l}{n}\Big)p_{nl}\Big(\frac{k}{n}\Big)\Big|\,\Big|q'\Big(\frac{l}{n}\Big)\Big|,$$

$$S_n^{(3)} = \sum_{l=0}^{\infty}\sum_{na_1<k<nb_1}\Big(\frac{k}{n} - \frac{l}{n}\Big)^2 p_{nl}\Big(\frac{k}{n}\Big),$$

are bounded. For the third sum we have, using the estimation (23) for T_{n2} in lemma 1,

$$\text{(31)} \qquad S_n^{(3)} = \sum_{na_1<k<nb_1}\sum_{l=0}^{\infty}\Big(\frac{k}{n} - \frac{l}{n}\Big)^2 p_{nl}\Big(\frac{k}{n}\Big)$$

$$= \frac{1}{n^2}\sum_{na_1<k<nb_1} T_{n2}\Big(\frac{k}{n}\Big)$$

$$= \frac{1}{n^2}\sum_{na_1<k<nb_1} n\frac{k}{n}\Big(1 + c\frac{k}{n}\Big) = O(1).$$

To estimate $S_n^{(1)}$, we can rewrite it in the form

$$S_n^{(1)} = \sum_{l=0}^{\infty} q_{nl}\Big[\sum_{na_1<k<nb_1} p_{nl}\Big(\frac{k}{n}\Big) - 1\Big],$$

where $q_{nl}=\pm q\left(\frac{l}{n}\right)$ and $q_{nl}=0$ for $\frac{l}{n}\leqq\alpha$ *or* $\frac{l}{n}\geqq\beta$. Thus the q_{nl} are bounded. If we put

$$Q_n(x)=\sum_{l=0}^{\infty}q_{nl}p_{nl}(x)=\sum_{n\alpha<l<n\beta}q_{nl}p_{nl}(x)\,, \tag{32}$$

then, since $\int_0^{E(c)}p_{nl}(x)dx=\frac{1}{n-c}$, we have

$$S_n^{(1)}=\sum_{na_1<k<nb_1}Q_n\left(\frac{k}{n}\right)-(n-c)\int_0^{E(c)}Q_n(x)dx.$$

Since $E(c)\geqq 1$, from the assumption (22)

$$S_n^{(1)}=\sum_{na_1<k<nb_1}Q_n\left(\frac{k}{n}\right)-n\int_0^{R}Q_n(x)dx+O(1).$$

For the functions (32), the condition (A) in lemma 4 is checked by means of lemma 3 and (B) by means of lemma 2. Hence, using lemma 4 we get

$$S_n^{(1)}=O(1). \tag{33}$$

To estimate $S_n^{(2)}$, we write it in the form

$$S_n^{(2)}=\sum_{na_1<k<nb_1}\left\{\frac{k}{n}\overline{Q}_n\left(\frac{k}{n}\right)-\overline{\overline{Q}}_n\left(\frac{k}{n}\right)\right\}, \tag{34}$$

where

$$\overline{Q}_n(x)=\sum_{l=0}^{\infty}q'_{nl}p_{nl}(x),\quad \overline{\overline{Q}}_n(x)=\sum_{l=0}^{\infty}\frac{l}{n}q'_{nl}p_{nl}(x),$$

$$q'_{nl}=\pm q'\left(\frac{l}{n}\right)\text{ and } q_{nl}=0 \text{ for } \frac{l}{n}<\alpha \text{ or } \frac{l}{n}>\beta.$$

Since from (21)

$$\int_0^{E(c)}xp_{nl}(x)dx=\frac{l+1}{(n-c)(n-2c)},$$

we have

$$(35)\qquad \left|\int_0^{E(c)} x\overline{Q}_n(x)dx-\int_0^{E(c)}\overline{\overline{Q}}_n(x)dx\right|=\frac{1}{n}\left|\sum_{l=0}^{\infty} q'_{nl}\frac{n-2lc}{(n-2c)(n-c)}\right|$$

$$\leqq \text{Const.}\sum_{na_2<l<nb_2}\frac{1}{n^2}=O\left(\frac{1}{n}\right).$$

Like the functions (32), also the functions $\overline{Q}_n(x)$ *and* $\overline{\overline{Q}}_n(x)$ satisfy the conditions (A) and (B) of lemma 4. For functions $x\overline{Q}_n(x)$ this follows from the fact

$$|(x\overline{Q}_n(x))''|\leqq 2|\overline{Q}'_n|+|\overline{Q}_n''|,\ 0\leqq x\leqq R$$

and lemma 2 with $r=1,2$. Applying (22), (35) and lemma 4 to the sum (34), we obtain

$$|S_n^{(2)}|=\left|\sum_{na_1<k<nb_1}\frac{k}{n}\overline{Q}_n\left(\frac{k}{n}\right)-n\int_0^{E(c)}x\overline{Q}_n(x)dx\right.$$

$$\left.+n\int_0^{E(c)}x\overline{Q}_n(x)dx-n\int_0^{E(c)}\overline{\overline{Q}}_n(x)dx+n\int_0^{E(c)}\overline{\overline{Q}}_n(x)dx-\sum_{na_1<k<nb_1}\overline{\overline{Q}}_n\left(\frac{k}{n}\right)\right|$$

$$\leqq\left|\sum_{na_1<k<nb_1}\frac{k}{n}\overline{Q}_n\left(\frac{k}{n}\right)-n\int_0^{R}x\overline{Q}_n(x)dx\right|$$

$$+n\left|\int_0^{E(c)}x\overline{Q}_n(x)dx-\int_0^{E(c)}\overline{\overline{Q}}_n(x)dx\right|+\left|\sum_{na_1<k<nb_1}\overline{\overline{Q}}_n\left(\frac{k}{n}\right)-n\int_0^{R}\overline{\overline{Q}}_n(x)dx\right|$$

$$+n\left|\int_R^{E(c)}x\overline{Q}_n(x)dx\right|+n\left|\int_R^{E(c)}\overline{\overline{Q}}_n(x)dx\right|=O(1).$$

Consequently, from (31), (33) and the estimation for $S_n^{(2)}$, we complete the proof of theorem 2.

THEOREM 3 (G. G. Lorentz [4]).
L.Sat.$[B_n]=[f'(x)\in \text{Lip}_M 1, n^{-1}$, *linear function*, $x(1-x)]$.

PROOF. Since $f(-1)=1$, it is trivial.

REMARK 3. The whole interval case of theorem 3 has been investigated by K. de Leeuw [3].

THEOREM 4.
L.Sat.$[S_n]=[f'(x)\in \text{Lip}_M 1,\ n^{-1}$, *linear function*, $x]$.

PROOF. By theorem 2, we have only to verify

$$\sum_{n\alpha\leq l\leq n\beta}\int_R^\infty x^i p_{nl}(x)dx=O\left(\frac{1}{n}\right)\quad(i=0,1).$$

$$\int_R^\infty p_{nl}(x)dx=R\int_1^\infty p_{nl}(Rt)dt=\frac{n^lR^{l+1}}{l!}\int_1^\infty t^le^{-nRt}dt$$

$$=\frac{n^lR^{l+1}}{l!}\left[\frac{e^{-nR}}{nR}+\frac{le^{-nR}}{(nR)^2}+\cdots+\frac{l!e^{-nR}}{(nR)^{l+1}}\right]$$

$$=e^{-nR}\sum_{j=0}^{l}\frac{R^j}{j!n^{-j+1}}.$$

Let us set $\alpha=\varepsilon$ and $\beta=R-\varepsilon\,(>0)$, where ε is an arbitrarily fixed positive number and sufficiently small.

$$S\equiv n\sum_{n\alpha\leq l\leq n\beta}\int_R^\infty p_{nl}(x)dx=n\sum_{\varepsilon n\leq l\leq(R-\varepsilon)n}e^{-nR}\sum_{j=0}^{l}\frac{R^j}{n^{-j+1}j!}$$

$$=e^{-nR}\sum_{\varepsilon n\leq l\leq(R-\varepsilon)n}\ \sum_{0\leq j\leq\varepsilon n}\frac{n^jR^j}{j!}+e^{-nR}\sum_{\varepsilon n\leq l\leq(R-\varepsilon)n}\ \sum_{\varepsilon n+1\leq j\leq l}\frac{n^jR^j}{j!}$$

$$=S_1+S_2,\ \text{say}.$$

$$S_1\leq\text{const. }e^{-nR}n\sum_{0\leq j\leq\varepsilon n}\frac{(nR)^j}{j!}=\text{const.}\sum_{0\leq j\leq\varepsilon n}\frac{n^{j+1}e^{-nR}R^j}{j!}$$

$$\equiv\text{const.}\sum_{j=0}^{\infty}\frac{u_j(n)}{j!},$$

where

$$u_j(n)\equiv\begin{cases}n^{j+1}R^je^{-nR}, & \text{if } 0\leq j\leq\varepsilon n.\\ 0, & \text{if } j>\varepsilon n.\end{cases}$$

Let us write

$$U_j\equiv\sup_n u_j(n).$$

Since

$$u_j'(x) = R^j\{(j+1)x^j e^{-Rx} - Rx^{j+1}e^{-Rx}\}$$
$$= R^j x^j e^{-Rx}\{j+1-Rx\},$$

we have

$$\begin{cases} u_j'(x) \geqq 0, & \text{if } 0 \leqq x \leqq \dfrac{j+1}{R}. \\ u_j'(x) < 0, & \text{if } \dfrac{j+1}{R} < x. \end{cases}$$

On the other hand, the functions $u_j(x)$ are zero if $x < \frac{j}{\varepsilon}$ and we can take j such that $\frac{j}{\varepsilon} > \frac{j+1}{R}$. Then, we have

$$\sup_x u_j(x) = u_j\left(\frac{j}{\varepsilon}\right) = e^{-\frac{R}{\varepsilon}j}\left(\frac{j}{\varepsilon}\right)^{j+1} R^j.$$

Using Stirling's formula, we get

$$\frac{U_j}{j!} \sim \frac{R^j e^{-\frac{R\varepsilon}{j}} \varepsilon^{-j-1} j^{j+1}}{j^j e^{-j}\sqrt{2\pi j}}$$
$$= \sqrt{2\pi}\, j^{\frac{1}{2}}\, e^{-j\left(\frac{R}{\varepsilon}-1\right)} \frac{1}{R}\left(\frac{R}{\varepsilon}\right)^{j+1}.$$

Hence

$$\log \frac{U_j}{j!} \sim -j\left(\frac{R}{\varepsilon}-1\right) + (j+1)\log\frac{R}{\varepsilon} + \frac{1}{2}\log j - \frac{1}{2}\log 2\pi - \log R$$
$$= -j\left(\frac{R}{\varepsilon} - \log\frac{R}{\varepsilon} - 1\right) + \frac{1}{2}\log j + \log\frac{1}{\varepsilon} - \frac{1}{2}\log 2\pi.$$

For the function $f(\varepsilon) = \frac{R}{\varepsilon} - \log\frac{R}{\varepsilon} - 1$, we have

$$\lim_{\varepsilon \to +0} f(\varepsilon) = \infty, \quad f(R) = 0,$$

and $f(\varepsilon)$ is strictly decreasing over $(0, R)$.

Since $0<\varepsilon<\frac{R}{2}$ and $\log 2=0.69\cdots$,

$$\frac{R}{\varepsilon}-\log\frac{R}{\varepsilon}-1\geqq 2-\log 2-1>0.3.$$

Hence, there is an integer $j_0(\varepsilon, R)$ such that

$$\log\frac{U_j}{j!}\leqq-\frac{j}{5}+\frac{1}{2}\log j+\text{const.}$$

$$<-\frac{j}{10},\ \text{for } j\geqq j_0(\varepsilon, R).$$

and we get

$$\sum_{j=j_0}^{\infty}\frac{U_j}{j!}\leqq\sum_{j=j_0}^{\infty}e^{-\frac{1}{10}j}<\infty;$$

$$\lim_{n\to\infty}u_j(n)=0,\ \text{for a fixed } j.$$

Hence

$$\lim_{n\to\infty}\sum_{j=0}^{\infty}\frac{u_j(n)}{j!}=0.$$

That is

$$S_1=o(1). \tag{36}$$

Analogously we shall estimate S_2 in the following.

$$S_2=e^{-nR}\sum_{\varepsilon n\leqq l\leqq(R-\varepsilon)n}\ \sum_{j=\varepsilon n+1}^{l}\frac{n^jR^j}{j!}$$

$$\leqq\text{const. } e^{-nR}n\sum_{\varepsilon n+1\leqq j\leqq(R-\varepsilon)n}\frac{n^jR^j}{j!}=\text{const.}\sum_{\varepsilon n+1\leqq j\leqq(R-\varepsilon)n}\frac{n^{j+1}e^{-nR}R^j}{j!}$$

$$=\text{const.}\sum_{j=0}^{\infty}\frac{v_j(n)}{j!},$$

where

$$v_j(n) \equiv \begin{cases} n^{j+1}R^j e^{-nR}, & \text{if } \varepsilon n < j \leqq (R-\varepsilon)n. \\ 0 & , \text{ if } j \geqq \varepsilon n. \end{cases}$$

Let us set

$$v_j \equiv \sup_n v_j(n).$$

Since the functions $v_j(x)$ are zero if $x < \frac{j}{R-\varepsilon}$ and we can take j such that $\frac{j+1}{R} < \frac{j}{R-\varepsilon}$,

$$\sup_x v_j(x) = v_j\left(\frac{j}{R-\varepsilon}\right) = e^{-\frac{R}{R-\varepsilon}j} R^j \left(\frac{j}{R-\varepsilon}\right)^{j+1}.$$

Using Stirling's formula, we obtain

$$\frac{V_j}{j!} \sim \frac{e^{-\frac{R}{R-\varepsilon}j} j^{j+1}(R-\varepsilon)^{-j-1}R^j}{j^j e^{-j}\sqrt{2\pi j}}$$

$$= \sqrt{2\pi}\, j^{\frac{1}{2}} e^{-j\left(\frac{R}{R-\varepsilon}-1\right)} \frac{1}{R} \left(\frac{R}{R-\varepsilon}\right)^{j+1}.$$

Hence

$$\log \frac{V_j}{j!} \sim \frac{1}{2} \log j - j\left(\frac{R}{R-\varepsilon} - 1\right) + (j+1) \log \frac{R}{R-\varepsilon} - \frac{1}{2} \log 2\pi - \log R$$

$$= \frac{1}{2} \log j - j\left(\frac{1}{1-\frac{\varepsilon}{\mathrm{R}}} - 1\right) + (j+1) \log \frac{1}{1-\frac{\varepsilon}{\mathrm{R}}} - \frac{1}{2} \log 2\pi - \log \mathrm{R}.$$

On the other hand, we know the following formulae

$$\text{(37)} \qquad \frac{1}{1-\frac{\varepsilon}{R}} = 1 + \left(\frac{\varepsilon}{R}\right) + \left(\frac{\varepsilon}{R}\right)^2 + \cdots,$$

$$\text{(38)} \qquad \log \frac{1}{1-\frac{\varepsilon}{R}} = \frac{\varepsilon}{R} + \frac{1}{2}\left(\frac{\varepsilon}{R}\right)^2 + \cdots.$$

From (37) and (38), we get

$$\log \frac{V_j}{j!} \sim \frac{1}{2}\log j - j\left\{\frac{1}{2}\left(\frac{\varepsilon}{R}\right)^2 + O\left[\left(\frac{\varepsilon}{R}\right)^2\right]\right\} + \log \frac{1}{1-\frac{\varepsilon}{R}} - \frac{1}{2}\log 2\pi R^2.$$

Hence, there is an integer $j_1(\varepsilon, R)$ such that

$$\log \frac{V_j}{j!} \leqq -\frac{1}{4}\left(\frac{\varepsilon}{R}\right)^2 j, \text{ for } j \geqq j_1(\varepsilon, R),$$

and we have

$$\sum_{j=j_1}^{\infty} \frac{V_j}{j!} \leqq \sum_{j=j_1}^{\infty} e^{-\frac{1}{4}\left(\frac{\varepsilon}{R}\right)^2 j} < \infty;$$

$$\lim_{n\to\infty} v_j(n) = 0, \quad \text{for a fixed } j.$$

Hence

$$\lim_{n\to\infty} \sum_{j=0}^{\infty} \frac{v_j(n)}{j!} = 0.$$

That is

(39) $$S_2 = o(1).$$

Combining (36) with (39), we get

(40) $$\sum_{\alpha \leqq \frac{l}{n} \leqq \beta} \int_R^{\infty} p_{nl}(x)dx = o\left(\frac{1}{n}\right).$$

Using (40), we can verify easily the fact

(41) $$\sum_{\alpha \leqq \frac{l}{n} \leqq \beta} \int_R^{\infty} x p_{nl}(x)dx = O\left(\frac{1}{n}\right).$$

Consequently, from (40) and (41) we can prove theorem 4.

REFERENCES

[1] V. A. BASKAKOV, An example of a sequence of linear positive operators in the space of continuous function, Doklady Akad. Nauk, 113(1957), 249-251.
[2] P. P. KOROVKIN, Linear operators and approximation theory, Delhi, 1960.
[3] K. de LEEUW, On the degree of approximation by Bernstein polynomials, Journ. d'Analyse Math., 7(1959), 89-104.
[4] G. G. LORENTZ, Inequalities and the saturation classes of Bernstein polynomials, Proc. of the Conference at Oberwolfach, 1963, 200-207.
[5] R. G. MAMEDOV, Asymptotic approximation of differentiable functions with linear positive operators, Doklady Akad. Nauk, 128(1959), 471-474.
[6] F. SCHURER, On the approximation of functions of many variables with linear positive operators, Indagationes Math., 25(1963), 313-327.
[7] O. SZÁSZ, Generalization of S. Bernstein's polynomials to the infinite interval, J. Res. Nat. Bur. Standards, 45(1950), 239-245: Collected Mathematical Works, Cincinnati (1955), 1401-1407.
[8] E. VORONOVSKAJA, Détermination de la forme asymptotique d'approximation des fonctions par les polynômes de S. Bernstein, C. R. Acad. Sci. U. S. S. R., (1932), 79-85.
[9] D. V. WIDDER, The Laplace transform, Princeton University Press, 1946.

COLLEGE OF GENERAL EDUCATION
TÔHOKU UNIVERSITY
SENDAI, JAPAN.