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EXISTENCE OF A GLOBALLY UNIFORM-ASYMPTOTICALLY STABLE PERIODIC AND ALMOST PERIODIC SOLUTION

TARO YOSHIZAWA[1)]

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Many authors have discussed the existence of periodic and almost periodic solutions under the assumption that the system has a bounded solution which is uniform-asymptotically stable in the large. Reuter [1] considered the second order differential equation

$$\ddot{x} + kf(x)\dot{x} + g(x) = kp(t), \quad k > 0 \quad \left(\cdot = \frac{d}{dt}\right),$$

and assuming that there exists an $A > 0$ such that $|x(t)| < A$ and $|\dot{x}(t)| < A$ ultimately, he proved that any pair of solutions $x(t), u(t)$ satisfies

$$x(t) - u(t) \to 0, \ \dot{x}(t) - \dot{u}(t) \to 0 \text{ as } t \to \infty$$

under additional conditions on $f(x), g(x)$ and k. The author [2] has proved similar results for general systems by assuming the existence of some Liapunov function. As will be seen later, actually we can conclude for the above equation that there exists a periodic solution or an almost periodic solution which is uniform-asymptotically stable in the large, if $p(t)$ is periodic or almost periodic. Reissig [3] also considered periodic systems of n-degrees of freedom and proved that a system is extremely stable if and only if each motion converges to a unique periodic motion, and his result does not depend upon the phase space being of even dimension. LaSalle [4] discussed a more general case by considering an asymptotic stability property of motions and obtained Reissig's result as a special case.

For functional-differential equations, Hale [5] and the author [6] have discussed the existence of a periodic and almost periodic solution which is uniform-asymptotically stable in the large. Hale has discussed a linear system

$$\text{(L)} \qquad \dot{x}(t) = f(t, x_t) + h(t)$$

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and proved that, if the zero solution of

$$\text{(L*)} \qquad \dot{x}(t) = f(t, x_t)$$

is uniform-asymptotically stable in the large and if $f(t, \varphi)$ and $h(t)$ are almost periodic in t, then there exists an almost periodic solution of (L) which is uniform-asymptotically stable in the large. Hale's condition that the zero solution of (L*) is uniform-asymptotically stable in the large is the best condition, because it is also a necessary condition in order that there exists an almost periodic solution of (L) which is uniform-asymptotically stable in the large.

In this paper, we shall discuss a necessary and sufficient condition in order that there exists a periodic and almost periodic solution which is uniform-asymptotically stable in the large, by connecting with a Liapunov function. To simplify the statements, we shall consider a system of ordinary differential equations.

Consider a system of differential equations

$$\text{(1)} \qquad \frac{dx}{dt} = F(t, x),$$

where x and $F(t, x)$ are n-vectors. Denoting by I the interval $0 \leqq t < \infty$ and by R^n the Euclidean n space, we assume that $F(t, x)$ is defined and continuous on $I \times R^n$, and moreover, we assume that for any $\alpha > 0$ there exists an $L(\alpha) > 0$ such that

$$\text{(2)} \qquad \|F(t, x) - F(t, y)\| \leqq L(\alpha)\|x - y\|$$

if $\|x\| \leqq \alpha$ and $\|y\| \leqq \alpha$, and we shall denote by $F(t, x) \in \bar{C}_0(x)$ this fact. We shall denote by $x(t; x_0, t_0)$ the solution of (1) through the point (t_0, x_0).

First of all, under the assumptions above we shall consider a necessary and sufficient condition in order that the system (1) has a bounded solution which is uniform-asymptotically stable in the large. For convenience, we shall state a necessary condition and a sufficient condition separately.

THEOREM 1. *Suppose that there exists a bounded solution* $\varphi(t)$ *of* (1) *which is uniform-asymptotically stable in the large and for which* $\|\varphi(t)\| \leqq B' < B$ *for all* $t \geqq 0$. *Then, the solutions of* (1) *are uniform-bounded and uniform-ultimately bounded for bound* B, *and there exists a continuous Liapunov function* $V(t, x, y)$ *defined on* $0 \leqq t < \infty$, $\|x\| \leqq B$, $\|y\| \leqq B$, *which* *satisfies the following conditions*;

(i) $a(\|x-y\|) \leqq V(t, x, y) \leqq b(\|x-y\|)$, *where* $a(r)$ *and* $b(r)$ *are continuous increasing positive definite functions for* $0 \leqq r \leqq 2B$,

(ii) $|V(t, x_1, y_1) - V(t, x_2, y_2)| \leqq K\{\|x_1 - x_2\| + \|y_1 - y_2\|\}$, *where* $K > 0$ *is a constant*,

(iii) *in the domain* $0 \leqq t < \infty$, $\|x\| < B$, $\|y\| < B$, *we have*

$$(3) \quad \dot{V}(t, x, y) = \overline{\lim_{h \to 0+}} \frac{1}{h} \{V(t+h, x+hF(t, x), y+hF(t, y)) - V(t, x, y)\} \leqq -cV(t, x, y),$$

where $c > 0$ *is a constant.*

PROOF. Since the solution $\varphi(t)$ is uniform-asymptotically stable in the large and we have $\|\varphi(t)\| \leqq B' < B$, it can be proved without difficulty that the solutions of (1) are uniform-bounded and uniform-ultimately bounded for bound B.

Let $B^* > 0$ be a constant such that if $t_0 \in I$ and $\|x_0\| \leqq B$, then $\|x(t; x_0, t_0)\| \leqq B^*$ for all $t \geqq t_0$. Then, the system (1) is uniform-asymptotically stable with respect to (B, B^*) (for the definition, see [7]). Therefore, by Theorem 21.2 in [7], there exists a continuous Liapunov function $V(t, x, y)$ defined on $0 \leqq t < \infty$, $\|x\| \leqq B$, $\|y\| \leqq B$, which satisfies the conditions (i), (ii) and (iii).

THEOREM 2. *Suppose that the solutions of* (1) *are uniform-bounded and uniform-ultimately bounded for bound* B *and there exists a continuous Liapunov function* $V(t, x, y)$ *defined on* $0 \leqq t < \infty$, $\|x\| \leqq B$, $\|y\| \leqq B$ *which satisfies the conditions* (i), (ii) *and* (iii) *in Theorem* 1. *Then, there exists a bounded solution* $\varphi(t)$ *of* (1) *which is uniform-asymptotically stable in the large and satisfies* $\|\varphi(t)\| < B$ *for* t *large. Actually, every solution of* (1) *is bounded and uniform-asymptotically stable in the large.*

PROOF. Let $\varphi(t)$ be a solution of (1). Then, there exists a $T^* \geqq 0$ such that $\|\varphi(t)\| < B$ for all $t \geqq T^*$. Let S be the set in R^n such that $\|x\| < B$. By using the Liapunov function $V(t, x, y)$, we can prove that any pair of solutions of (1) $x(t; x_0, t_0)$ and $x(t; y_0, t_0)$ such that $x(t; x_0, t_0) \in S$ and $x(t; y_0, t_0) \in S$ for all $t \geqq t_0$ approach uniformly each other. Thus, we can prove that the solution $\varphi(t)$ is uniform-asymptotically stable in the large, because the solutions of (1) are uniform-ultimately bounded for bound B.

REMARK 1. A necessary and sufficient condition in order that the solutions of (1) are uniform-bounded and uniform-ultimately bounded is given by the existence of a suitable Liapunov function, see Theorem 20.4 in [7].

As a consequence of the results above, we have the following theorems for periodic and almost periodic systems. Let

$$\text{(4)} \qquad \frac{dx}{dt} = F(t, x), \quad F(t + \omega, x) = F(t, x)$$

be a periodic system, where $F(t, x)$ is defined and continuous on $(-\infty, \infty) \times R^n$, is periodic in t of period ω and satisfies locally a Lipschitz condition with respect to x.

THEOREM 3. *In order that there exists a periodic solution $p(t)$ of period ω of* (4) *such that $\|p(t)\| < B$ which is uniform-asymptotically stable in the large, it is necessary and sufficient that the solutions of* (4) *are uniform-bounded and uniform-ultimately bounded for bound B and there exists a continuous Liapunov function $V(t, x, y)$ defined on $0 \leqq t < \infty$, $\|x\| \leqq B$, $\|y\| \leqq B$ which satisfies the conditions* (i), (ii) *and* (iii) *in Theorem* 1.

PROOF. Assume that there exists a periodic solution $p(t)$ such that $\|p(t)\| < B$. Then, by the periodicity of $p(t)$, there exists a constant B' such that $\|p(t)\| \leqq B' < B$, and by the periodicity of $F(t, x)$, we see that $F(t, x) \in \overline{C}_0(x)$. Therefore, it follows immediately from Theorem 1 that the condition is necessary.

Next, we shall prove that the condition is sufficient. Since the system (4) is periodic in t of period ω and the solutions of (4) are uniform-bounded and uniform-ultimately bounded for bound B, it follows from Theorem 29.3 in [7] that there exists a periodic solution $p(t)$ of period ω such that $\|p(0)\| \leqq B$. By the assumption that the solutions are uniform-ultimately bounded for bound B, we have $\|p(t)\| < B$ for sufficiently large t, and hence, $\|p(t)\| < B$ for all t, because of the periodicity of $p(t)$. By Theorem 2, this periodic solution $p(t)$ is uniform-asymptotically stable in the large.

Next, we shall consider an almost periodic system

$$\text{(5)} \qquad \frac{dx}{dt} = F(t, x),$$

where $F(t, x)$ is defined and continuous on $(-\infty, \infty) \times R^n$. We assume that $F(t, x)$ is almost periodic in t uniformly with respect to $x \in S$ for any compact set S in R^n and that $F(t, x) \in \overline{C}_0(x)$.

THEOREM 4. *In order that there exists an almost periodic solution $\psi(t)$ of* (5) *which is uniform-asymptotically stable in the large, it is necessary and sufficient that there exists a positive constant B such that the solutions of* (5) *are uniform-bounded and uniform-ultimately bounded for bound B and that there exists a continuous Liapunov function $V(t, x, y)$*

defined on $0 \leqq t < \infty$, $\|x\| < B + \varepsilon_0$, $\|y\| < B + \varepsilon_0$ *which satisfies the conditions* (i), (ii) *and* (iii) *in Theorem* 1 *for* $B+\varepsilon_0$ *instead of* B, *where* $\varepsilon_0 > 0$ *is arbitrarily small but fixed.*

PROOF. Assume that there exists an almost periodic solution $\psi(t)$. Then, there exists a $B' > 0$ such that $\|\psi(t)\| \leqq B'$. Thus, clearly the solutions of (5) are uniform-ultimately bounded for any B and consequently $B+\varepsilon_0$ such that $B > B'$, $\varepsilon_0 > 0$. Terefore, the necessity follows immediately from Theorem 1.

Now, assume that there exists a $B > 0$ which satisfies the conditions. Since the solutions of (5) are uniform-ultimately bounded for bound B, for the solution $\varphi(t)$ of (5) such that $\varphi(0) = 0$ there exists a $T \geqq 0$ such that if $t \geqq T$, then $\|\varphi(t)\| < B$. Since $\|\varphi(t)\| \leqq B$ for all $t \geqq T$ and $B + \varepsilon_0 > B$, considering the Liapunov function $V(t, x, y)$ on the domain $T \leqq t < \infty$, $\|x\| < B+\varepsilon_0$, $\|y\| < B+\varepsilon_0$ and applying Theorem 31.1 in [7], we see that there exists an almost periodic solution $\psi(t)$ such that $\|\psi(t)\| \leqq B$. By Theorem 2, this almost periodic solution is uniform-asymptotically stable in the large.

REMARK 2. In Theorems 1, 2, 3 and 4 the condition (iii) can be replaced by $\dot{V}(t, x, y) \leqq -c(\|x-y\|)$, where $c(r)$ is continuous and positive definite.

The same consideration as the above can be done for a system of functional-differential equations. For $x \in R^n$, $|x|$ is any norm of x, and for a given $h > 0$, C denotes the space of continuous mapping of the interval $[-h, 0]$ into R^n and for $\varphi \in C$, $\|\varphi\| = \sup_{-h \leqq \theta \leqq 0} |\varphi(\theta)|$. For any continuous function $x(u)$, the symbol x_t will denote the restriction of $x(u)$ to the interval $[t-h, t]$. Let $\dot{x}(t)$ denote the right-hand derivative of $x(u)$ at $u=t$, and consider a system of functional-differential equations

$$\dot{x}(t) = F(t, x_t),$$

where $F(t, \varphi)$ is defined and continuous on $I \times C$ or on $(-\infty, \infty) \times C$. We assume that for any $\alpha > 0$ there exists an $M(\alpha) > 0$ such that

$$|F(t, \varphi)| \leqq M(\alpha) \qquad \text{if } \|\varphi\| \leqq \alpha, \tag{6}$$

and moreover, we assume that for any $\alpha > 0$ there exists an $L(\alpha) > 0$ such that

$$|F(t, \varphi) - F(t, \psi)| \leqq L(\alpha)\|\varphi - \psi\| \quad \text{if } \|\varphi\| \leqq \alpha \text{ and } \|\psi\| \leqq \alpha. \tag{7}$$

For periodic and almost periodic systems, the condition (7) implies (6). Under the assumptions above, we can prove analogous theorems to Theorems 1, 2, 3 and 4. Here, it is noted that Theorem 37.1 in [7] can be proved without the restriction $\omega \geqq h$ by applying an asymptotic fixed point theorem due to Jones [8].

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MATHEMATICAL INSTITUTE
TÔHOKU UNIVERSITY
SENDAI, JAPAN.