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ON THE MEAN OF AN ENTIRE FUNCTION AND THE MEAN OF THE PRODUCT OF TWO ENTIRE FUNCTIONS

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Let $f(z)$ be an entire function, which is not a polynomial in general, of order ρ and lower order λ. Let $I_\delta(r,f)$ be defined by

$$I_\delta(r,f)=\left(\frac{1}{2\pi}\int^{2\pi}|f(re^{i\cdot})|^\delta d\theta\right)^{1/\delta},\ 0<\delta<\infty .$$

Then we have a theorem which was proved recently ([3], Theorem).

THEOREM. *Provided* $\delta\geqq 1$ *we have*

(1) $$\limsup_{r\to\infty}\frac{\log[rI_\delta(r,f')/I_\delta(r,f)]}{\log r}=\rho,\quad 0\leqq\rho\leqq\infty .$$

When ρ is finite it would appear that (1) holds for $0<\delta<1$ as well, as the example $f(z)=\exp z$ would show.

Our main aim at present is to show that we are able to prove the following theorem in the case of functions of finite order when $0<\delta<1$.

THEOREM 1. *When* $f(z)$ *is a function of finite order* ρ *and* $0<\delta<1$, *then*

$$\limsup_{r\to\infty}\frac{\log[rI_\delta(r,f')/I_\delta(r,f)]}{\log r}\leqq\rho.$$

We need the following lemma for the proof.

LEMMA 1. *If* $0<\delta<1$, *then for* $r<R$,

(2) $$I_\delta(r,f')<C(\delta)(R-r)^{-1}I_\delta(R,f),$$

where $C(\delta)$ *is a constant depending on* δ *alone.*

PROOF. It can be shown ([2], Lemma 1) that

(3) $$I(r, f') \leqq (R-r)^{-1} I(R, f)$$

where $I(t, u) \equiv I_1(t, u)$. Now let us assume that f has no zeros in $|z| < R$. Choose $\psi = f^{\delta}$ so that

$$|f'| = \delta^{-1} |\psi|^{(1-\delta)/\delta} |\psi'|, \quad \psi \equiv \psi(re^{i\theta}), \quad f \equiv f(re^{i\theta}),$$

where the accents indicate the respective derivatives. Hence by Hölder's inequality

$$\int_0^{2\pi} |f'|^{\delta} d\theta \leqq \delta^{-\delta} \int_0^{2\pi} |\psi|^{1-\delta} |\psi'|^{\delta} d\theta$$

$$\leqq \delta^{-\delta} \left(\int_0^{2\pi} |\psi| d\theta \right)^{1-\delta} \left(\int_0^{2\pi} |\psi'| d\theta \right)^{\delta}.$$

If we use (3) with ψ in the place of f, we get from the above inequality,

$$\left(\int_0^{2\pi} |f'|^{\delta} d\theta \right)^{1/\delta} \leqq \delta^{-1} \left[\int_0^{2\pi} |\psi(Re^{i\theta})| d\theta \right]^{(1-\delta)/\delta} \times \int_0^{2\pi} |\psi'| d\theta$$

$$\leqq \delta^{-1} \left[\int_0^{2\pi} |\psi(Re^{i\theta})| d\theta \right]^{(1-\delta)/\delta} \times (R-r)^{-1} \int_0^{2\pi} |\psi(Re^{i\theta})| d\theta$$

$$\leqq \delta^{-1} (R-r)^{-1} (2\pi)^{1/\delta} I_{\delta}(R, f)$$

which is equivalent to (2) when $C(\delta) = \delta^{-1}$. Next let us suppose that f has zeros in $|z| < R$. Then it is known ([1], p.207) that $f(z) = f_1(z) + f_2(z)$, where f_1 and f_2 have no zeros in $|z| < R$ and $|f_p(z)| < 2|f(z)|$, $p = 1, 2$. Hence we have from the previous result on using the familiar inequality that $|a+b|^p \leqq |a|^p + |b|^p$ for $0 \leqq p \leqq 1$,

$$2\pi [I_{\delta}(r, f')]^{\delta} = \int^{2\pi} |f'(re^{i\theta})|^{\delta} d\theta$$

$$\leqq \int_0^{2} |f'_1(re^{i\theta})|^{\delta} d\theta + \int^{2\pi} |f_2'(re^{i\theta})|^{\delta} d\theta$$

$$\leqq \left[\delta(R-r) \right]^{-\delta} \left[\int_0^{2\pi} |f_1(Re^{i\theta})|^{\delta} d\theta + \int_0^{2\pi} |f_2(Re^{i\theta})|^{\delta} d\theta \right]$$

$$\leqq 2^{\delta+1}[\delta(R-r)]^{-\delta}\int_0^{2\pi} |f(Re^{i\theta})|^{\delta} d\theta.$$

Finally

$$I_\delta(r, f') \leqq C(\delta)(R-r)^{-1} I_\delta(R, f)$$

where $C(\delta)=2[2^{1/\delta}/\delta]$.

PROOF OF THEOREM 1. Lemma 1 leads to the following inequality by a method which is available already ([3], pp.307-308).

$$I_\delta(r, f') \leqq r^{\rho-1+\varepsilon} I_\delta(r, f), \quad 0 < \delta < 1, \quad r \geqq r_0(\varepsilon),$$

where ε is an arbitrarily small positive quantity.

The theorem follows from this inequality.

The above theorem also holds with λ, the lower order in the place of ρ (Cf. [3], Lemma 4) and lim sup replaced by lim inf.

Let $f(z)$ and $g(z)$ be two entire functions and let $\alpha > 0$, $\beta > 0$,

$$I_{\alpha,\beta}(r) \equiv I(r, |f|^\alpha |g|^\beta) = \left(\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^\alpha |g(re^{i\theta})|^\beta d\theta\right)^{1/(\alpha+\beta)}$$

It is well known that $|f|^\alpha |g|^\beta$ is of class PL([5], p.9) and so $\log I_{\alpha,\beta}(r)$ is a convex function of $\log r$ ([5]). We will prove the following theorem which extends to two functions a result proved earlier for one function f ([6], Theorem 1).

THEOREM 2. *If $f(z)$ and $g(z)$ are two functions, which are not polynomials, of orders ρ_f and ρ_g respectively, then*

$$(\alpha+\beta)\log I_{\alpha,\beta}(r) \sim \log\,[\max_{|z|=r} |f^\alpha g^\beta|], \qquad r\to\infty,$$

If $f=g$ and $\alpha=\beta=\gamma$ we get the result for one function f as mentioned above.

For the proof we need the following lemma.

LEMMA 2. *For the entire functions f and g*

$$\rho = \limsup_{r\to\infty} \frac{\log\log I_{\alpha,\beta}(r)}{\log r} \leqq \rho_f + \rho_g \quad (0 \leqq \rho_f, \rho_g \leqq \infty).$$

PROOF. Since $|f|^{\alpha}|g|^{\beta}$ is of class PL, it follows that it is subharmonic in $|z| \leqq r < R$. Hence on $|z| = r$, we have by Poisson's formula for subharmonic functions

$$I_{\alpha,\beta}(r) \leqq [\max_{|z|=r} |f^{\alpha}g^{\beta}|]^{1/(\alpha+\beta)} \leqq [(R+r)/(R-r)]^{1/(\alpha+\beta)} I_{\alpha,\beta}(R). \tag{4}$$

The lemma now follows from the left hand inequality in (4).

PROOF OF THEOREM 2. Since $\log I_{\alpha,\beta}(r)$ is a convex function of $\log r$

$$\log I_{\alpha,\beta}(r) = \log I_{\alpha,\beta}(r_0) + \int_{r_0}^{r} \frac{m_{\alpha,\beta}(x)}{x} dx \tag{5}$$

where $m_{\alpha,\beta}(x)$ is a non decreasing function of x. By Lemma 2, since $\rho_f + \rho_g < \infty$,

$$\log I_{\alpha,\beta}(r) < r^{\rho+\varepsilon}, \quad r \geqq r_0(\varepsilon), \tag{6}$$

where ε is defined as before.

Also we get from (5) and (6)

$$\int_{r}^{2r} \frac{m_{\alpha,\beta}(x)}{x} dx < (2r)^{\rho+\varepsilon}$$

or

$$(\log 2) m_{\alpha,\beta}(r) < (2r)^{\rho+\varepsilon}$$

and ε being arbitrary

$$m_{\alpha,\beta}(r) < r^{\rho+\varepsilon}.$$

Hence for $r < R$

$$\int_{r}^{R} \frac{m_{\alpha,\beta}(x)}{x} dx < m_{\alpha,\beta}(R)\log(R/r)$$

$$= m_{\alpha,\beta}(R)\log\left(1 + \frac{R-r}{r}\right)$$

$$< R^{\rho+\varepsilon}[(R-r)/r].$$

Choosing R such that

$$R = r\left[1 + \frac{1}{r^{\rho+\varepsilon}}\right]$$

we get

$$\int_r^R \frac{m_{\alpha,\beta}(x)}{x}\,dx < 2^{\rho+\varepsilon} \tag{7}$$

This choice of R is correct since in (5) we can take r in the place of r_0 and $R > r$ in the place of r.

Hence from (4), (5), and (7)

$$\begin{aligned}\log[\max_{|z|=r}|f^\alpha g^\beta|] &\leqq \log\left(\frac{R+r}{R-r}\right) + (\alpha+\beta)\log I_{\alpha,\beta}(R) \\ &< \log(1+2r^{\rho+\varepsilon}) \\ &\qquad + (\alpha+\beta)[2^{\rho+\varepsilon} + \log I_{\alpha,\beta}(r)] \\ &\leqq (1+\varepsilon)(\alpha+\beta)\log I_{\alpha,\beta}(r)\end{aligned} \tag{8}$$

for all $r \geqq r_0(\varepsilon)$, ε being defined as in previous cases. The theorem now follows from the inequalities (4) and (8).

Finally if we define the new mean $I^{(k)}_{\alpha,\beta}(r)$ by

$$I^{(k)}_{\alpha,\beta} = r^{-k-1}\int_0^r x^k I_{\alpha,\beta}(x)dx, \quad k+1>0,$$

the following theorem can be proved.

THEOREM 3. *If ρ_f and ρ_g are finite,*

$$\limsup_{r\to\infty}\ [I_{\alpha,\beta}(r)/I^{(k)}_{\alpha,\beta}(r)]^{1/\log r} \leqq e^{\rho_f+\rho_g}.$$

The proof depends on Lemma 2 and the fact that $\log I^{(k)}_{\alpha,\beta}(r)$ is a convex function of $\log r$ (Cf.[4],pp.1277-79). We omit this for conciseness.

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