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CHARACTERIZATION OF LOCAL RINGS

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1. Introduction. A ring with identity is said to be a local ring if the sum of any two non-units is a non-unit or equivalently if the ring has a unique maximal right ideal. Most of the known important local rings are either integral domains or divisible rings. It is trivially true that Z (the set of all zero-divisors) is included in J (Jacobson radical). In addition Z turns out to be an ideal in many cases. Also the condition that Z is an ideal is necessary for a ring to have a local quotient ring. These observations necessiate one to study rings with Z as an ideal or with $Z \subseteq J$ and to investigate when such rings become local rings. In §2, we begin by proving that in commutative case divisible rings with uniform condition are local. But in the non-commutative case these are proved to be local with the added right Noetherian condition. This right uniform property exactly characterizes local rings among right injective rings. In §3 we consider rings with Z as an ideal and with $Z \subseteq J$ and find conditions for these rings to be local. Among right injective rings, we find local rings as well as semi-simple rings. These two classes of rings can be characterized by extreme properties. In §4, we study two properties which separate local and semi-simple rings among right injective rings.

Throughout this paper we assume that every ring R has identity and Z and J denote the set of all zero-divisors in R and Jacobson radical of R respectively.

2. Divisible Rings.

2.1 DEFINITION: A ring with identity is said to be divisible if every regular element (not a left or a right zero-divisor) is a right unit (having a right inverse). This implies that every regular element is a unit.

2.2 PROPOSITION: *Let R be a divisible ring. Then R is a local ring iff Z is included in a proper ideal. In this case Z becomes an ideal.*

PROOF: Because of the divisible property, every proper right ideal is included in Z. Now if Z is included in a proper ideal, then evidently Z is an ideal. Thus R is a local ring with Z as the unique maximal right ideal. The other part is trivially true.

2.3 REMARK: In a ring R with identity, if Z is included in a proper ideal, then R has no idempotents other than 0 and 1. Hence proper Von Neumann regular rings though divisible, do not have the above property.

2.4 DEFINITION: A ring is said to be right uniform if every proper right ideal is large, i.e., every non-zero right ideal has non-null intersection with every non-zero right ideal.

2.5 THEOREM: *If R is a commutative divisible and uniform ring, then R is a local ring.*

PROOF: Let a and b be any two non-units in R. Since R is divisible, a, $b \in Z$, i.e., $ax=0=by$, $x, y \neq 0$. Since R is uniform, $xl=ym\neq 0$ for some $l, m \in R$. Then $(a+b)xl = bxl = bym = 0$. Hence $a+b \in Z$. This implies that $a+b$ is a non-unit and so R is a local ring.

2.6 THEOREM: *Let R be a right Noetherian and two-sided uniform divisible ring. Then R is a local ring.*

PROOF: Let a and b be any two non-units in R. Then $a, b \in Z$. Let $ax=0=by$. Since $xl=ym\neq 0$ by right uniform property, $(a+b)xl=bxl=bym=0$. Hence $a+b \in Z$. Similarly if $xa=yb=0$, we have $a+b \in Z$, by left uniform property. Assume now $ax=0=yb$. Because R is right Noetherian, $a^n=0$, $a^{n-1}\neq 0$ by virtue of [2, Theorem 6.1]. Then by left uniform property, $la^{n-1}=my$. Hence $la^{n-1}(a+b)=la^{n-1}b=myb=0$, i.e., $a+b \in Z$. Thus $a+b$ is a non-unit and hence R is a local ring with Z as the unique maximal right ideal.

2.7 REMARK: In the case of right Noetherian rings R, the conclusion that R is a local ring fails if either one of the conditions of uniformity or divisibility is dropped out. Proper semi-simple rings and the ring of integers present counter examples. But there exist non-uniform divisible rings. However this uniform property classifies local rings among right injective rings (which are divisible by [5; Theorem 3.1]).

2.8 EXAMPLE: Let $R=\{a+bx+cy\colon a, b, c \in$ a division ring D; $x^2=y^2=xy=yx=0$ and $dx=xd$ and $dy=yd$ for every $d \in D\}$. Define $a+bx+cy=0$ iff $a=b=c=0$. Then R is a ring with the usual rules of multiplication and addition. Since $xR\cap yR=0$ and $Rx\cap Ry=0$, R is not right or left uniform but R is a divisible local ring with the unique maximal right ideal generated by x and y.

2.9 THEOREM: *Let R be a right injective ring with identity. Then the following are equivalent:*

i) *R is a local ring.*
ii) *R has no proper idempotents.*
iii) *No proper projective right ideal is injective.*

iv) *R is right uniform.*

PROOF: Evidently i) $\Rightarrow$ ii). Since every injective right ideal is generated by an idempotent and hence is projective, ii)$\Rightarrow$ iii).

iii) $\Rightarrow$ iv): We observe first that R has no proper injective right ideals. Hence R is the injective hull of every one of the right ideals and R is right uniform [1; Theorem 57.13].

iv)$\Rightarrow$ i): Since R has identity and R is right injective, $\mathrm{Hom}_R(R, R)$ is isomorphic to the ring of all left multiplications and hence to R. But $\mathrm{Hom}_R(R, R)$ is a local ring since R is right uniform [6; Proposition 2.2]. Thus R is a local ring.

3. Rings with Z as an ideal. In integral domains Z is trivially an ideal and $Z\subseteq J$. There exist rings with zero-divisors for which Z is an ideal and $Z\subseteq J$. It can be observed from [2; Theorem 6.1], that uniform and two sided Noetherian rings satisfy this property because Z is a nil ideal and hence $Z\subseteq J$. Now we shall find a criterion for these rings to be local rings.

3.1 PROPOSITION. *Let R be a ring such that Z is an ideal and $Z\subseteq J$. Then R is a local ring iff R/Z is a local ring.*

PROOF: Let R/Z be a local ring. Denote the elements of R/Z by $\bar{a}$, $a\in R$. If $\bar{a}\bar{x}=1$, then $ax=1+n$, $n\in Z$. Since $Z\subseteq J$, ax is a unit, i.e., $axy=1$, This implies $xya=1$ since $Z\subseteq J$. Hence a is a unit. It can easily be verified that a is a non-unit if $\bar{a}$ is a non-unit. Now, if a and b are non-units, evidently by the above characterization, $\bar{a}$ and $\bar{b}$ are non-units. Hence $\bar{a}+\bar{b}$ is a non-unit since R/Z is a local ring. This implies that $a+b$ is a non-unit. The other part is trivially true since onto homomorphic image of a local ring is a local ring.

3.2 THEOREM: *Let R be a ring satisfying the following conditions.*

i) *Right ideals containing J are principal.*

ii) *J is completely prime and is a principal left ideal.*

iii) *$Z\subseteq J$.*

iv) *$J\neq 0$.*

Then R is a local ring.

PROOF: It suffices to prove that J is composed of all non-units. Let $J=Rx$ and $a\notin J$. Then by i) $J+aR=bR$ and $b\notin J$. Hence $x=by$. Since J is completely prime, $y\in J$. Therefore $y=cx$, i.e., $x=by=bcx$. Thus $1-bc\in Z$. This implies by iii) $1-bc\in J$ and bc is a unit and hence b is a unit. Consequently $J+aR=R$. Therefore it follows that $1=j+ac$, $j\in J$ and $1-ac\in J$ and ac is a unit. Thus we conclude that a is a unit.

3.3 REMARK. In the above theorem the condition that $J\neq 0$ is necessary as can be seen from the example of the ring of integers.

As an immediate consequence of 3.2 we have the following.

3.4 COROLLARY. *Let R be a principal ideal ring (every right and left ideal is principal) and $Z \subseteq J$. Then R is a local ring iff $J \neq 0$ and J is a completely prime ideal.*

3.5 COROLLARY. *Let R be a principal ideal ring which is not a domain. Then, if $Z = J$, R is a local ring.*

We conclude this section with a discussion whether Z can be large.

3.6 PROPOSITION. *Let R be a ring with Z as an ideal. Then, if Z is not a large right ideal, R is an integral domain.*

PROOF: Let $Z \neq 0$. By hypothesis $Z \cap A = 0$ for a right ideal $A(\neq 0)$. Consider $x(\neq 0) \in Z$. For definiteness let $tx = 0$. If $y(\neq 0) \in A$, then $(t+y)x = yx \in A \cap Z$. This implies $yx = 0$. Hence $t + y \in Z$ and so $y \in Z$. Thus $y = 0$, a contradiction.

3.7 REMARK: If Z is a large ideal, R need not be an integral domain as can be noted in the following example. Let $R = \{a + bx : a, b \in$ a division ring D; $x^2 = 0$ and $dx = xd$ for every $d \in D\}$. Then R is a ring with the only ideal (x) and $Z = (x)$ is a nilpotent ideal and is a large right ideal.

4. Local and semi-simple rings. A trivial local ring as well as a semi-simple ring is a divison ring. In 2.9 we proved that the right injective rings which are local are characterized by right uniform property. Now we show in the following that semi-simple rings which are a subclass of right injective rings can be described exactly by the opposite property.

4.1 PROPOSITION: *Let R be a ring with identity such that no proper right ideal is large. Then R is semi-simple.*

PROOF: Let A be any proper right ideal. By hypothesis $A \cap B = 0$ for some right ideal $B(\neq 0)$. Then by Zorn's lemma, there exist a proper right ideal B^* such that $A \cap B^* = 0$, and $C \supset B^*$ and $C \cap A = 0 \Rightarrow C = B^*$. It can be verified easily that $A + B^*$ is a large right ideal. Hence by hypothesis $A + B^* = R$. Thus every proper right ideal is a direct summand, which implies that R is semi-simple.

It is also possible to separate local and semi-simple rings among right injective and right Noetherian rings by the property whether or not every proper principal right ideal is projective.

4.2 PROPOSITION: *Let R be a right injective and right Noetherian ring. Then R is semi-simple iff every principal right ideal is projective.*

PROOF: Let every principal right ideal be projective. Then by QF theorem of [3; page 80], every principal right ideal is injective, i.e., principally generated by an idempotent. This fact together with the finitely generated condition on right ideals, makes every right ideal principally generated by an idempotent. Thus R becomes semi-simple. The other part is trivially satisfied.

4.3 LEMMA. *Let R be a ring with identity satisfying the ascending chain condition on annhilator right ideals. Then a right unit is a two-sided unit.*

PROOF: Let $a\neq 0$ and $ax=1$. Assume $xa\neq 1$. Since $a(xa-1)=0$, $a^r=\{t\in R\mid at=0\}$ is a non-zero right ideal. Now $a^r\subseteq(a^2)^r\subseteq(a^3)^r\subseteq\cdots$ Because of the a.c.c condition, $(a^n)^r=(a^{n+1})^r$ for some positive integer n. Let $u(\neq 0)\in(a^n)^r$. Then $a^n\ u=0$. But $u=(ax)u$. Hence $a^n(ax)u=0$, i.e., $xu\in(a^{n+1})^r$, i.e., $xu\in(a^n)^r$. This implies $a^nxu=0$ and $a^{n-1}(axu)=0$, i.e., $a^{n-1}u=0$. Thus $u\in(a^{n-1})^r$ and so $(a^{n-1})^r=(a^n)^r$. Proceeding in this way we get $a^r=(a^2)^r$. Let $u\in a^r=(a^2)^r$. Then $a^2(u-xu)=0$. But $a(u-xu)=au-u\ =-u$. Hence $u-xu\in(a^2)^r\cap a^r$ only if $u=0$. This implies $a^r=0$, a contradiction.

4.4 THEOREM. *Let R be a right injective and right Noetherian ring. Then R is a local ring with the maximal right ideal as a nilpotent ideal iff no proper principal right ideal is projective.*

PROOF: Let no proper principal right ideal be projective. Then R has no idempotents and hence R is a local ring by 2.9. By the same theorem 2.9, R is also right uniform. So $P=\{a\in R\mid a^r\neq 0\}$ is a nilpotent ideal, [2; Theorem 3.1]. We will show now that P is the set of all non-units in R. Let a be non-unit and $a\notin P$. Then a is right regular. This implies aR is projective and hence by hypothesis $aR=R$, i.e., a is a right unit., By 4.3, a becomes a two-sided unit.

Conversely if the maximal right ideal of the local ring R is nilpotent, no proper principal right ideal is a free ideal. But in local rings projective ideals are free [4; Theorem. 2]. Hence Q.E.D.

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