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# SPHERES AND CELLS IN NEGATIVELY CURVED SPACES

F. J. FLAHERTY

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**1. Introduction.** A riemannian manifold $M$ is said to be negatively curved if the sectional curvatures of $M$ are not all zero and lie in the interval $[-k,0]$ for some positive $k$. It is well known that a complete riemannian manifold with non-positive sectional curvature is covered by euclidean space; in particular, a complete simply connected negatively curved manifold is diffeomorphic to euclidean space. Thus a compact, connected, orientable hypersurface $N$ in a simply connected, complete negatively curved space separates $M-N$ topologically into two components. It is our purpose to give conditions under which the pair $(U, N)$ where $U$ is the closure of the bounded component of $M-N$ is homotopy equivalent to a pair $(D^{n+1}, S^n)$ where $D^{n+1}$ and $S^n$ are the standard disc and standard sphere respectively. All notation is the same as [1] and all references, unless otherwise noted, are to this paper.

**2. Preliminaries.** Since our methods use Morse theory, one needs an oscillation theorem.

PROPOSITION 1. *Let $M^{n+1}$ be negatively curved manifold with sectional curvature restricted to $[-k, 0]$, $k>0$. Let $S$ be the second fundamental form of a hypersurface $N^n$, at the point $p$ in $N$ and in the unit normal direction $V$, with eigenvalues $e_i$ restricted to $[a, b]$. Then the geodesic starting from $p$ in the direction $V$ has no focal points for $t<1/b$ and has at least $n$ focal points for $t>1/\sqrt{k}$ arc coth $a/\sqrt{k}$, where $t$ is the arc length parameter for $g$, and $\sqrt{k}<a$.*

PROOF. As in the proposition in section 3 in [1], one compares the zero of Jacobi fields in $M$ with those in flat spaces and hyperbolic spaces, respectively. Indeed, let $Y$ be a Jacobi field along $g$ satisfying the $S$ boundary condition in a flat space. Thus

$$Y(t) = \Sigma(y_i' t + y_i)U_i(t)$$

where the $U_i$ are parallel orthonormal along $g$ and $\Sigma y_i' y_i = -\Sigma e_i y_i$. Hence $Y$

cannot vanish before $t=1/b$.

On the other hand consider the Jacobi field $Y$ in the hyperbolic space of curvature $-k$:

$$Y(t) = \left((y_i'/\sqrt{k}) \sinh \sqrt{k}\, t + y_i \cosh \sqrt{k}\, t\right) U_i(t)$$

where $y_i \neq 0$ and $e_i y_i = -y_i'$. This field vanishes at $t=(1/\sqrt{k})$ arc coth $e_i/\sqrt{k}$. Thus index $g \geqq n$ for $t>(1/\sqrt{k})$ arc coth $a/\sqrt{k}$ and the proof is complete.

It is of interest to have a curvature condition which yields compactness of closed submanifolds of complete negatively curved manifolds.

PROPOSITION 2. *If $M$ is a complete negatively curved manifold with curvature in $[-k, 0]$ and $N$ is a closed hypersurface with eigenvalues of the second fundamental form $S$ in $(a, b]$ where $a^2 > k$ then $N$ is compact.*

PROOF. By a theorem of Myers, corollary 19.5 in [3], it suffices that $N$ have positive sectional curvature. Let $\sigma$ be a plane tangent to $N$ at $p$ then there exist orthonormal $u, w$ in $N_p$, the tangent space of $N$ at $p$, so that $u, w$ span $\sigma$, $S(u, w)=0$ and $u, w$ are eigenvectors for $S$. Thus by the classical formula of Gauss:

$$K_N(\sigma) = K(\sigma) + S(u, u)S(w, w)$$

where $K_N$ is the sectional curvature of $N$. A deriviation of this formula using the structural equations may be found in [2]. Thus

$$0 < -k + a^2 < K_N(\sigma)$$

and the proof is complete.

**3. Main theorem.** We approach the main theorem along an indirect course. Let $N$ be a closed riemannian submanifold of a complete riemannian manifold $M$. Further suppose that $g$ is a geodesic ray in a unit normal direction, $u$, to $N$. If there is a point $t_0$ in $[0, \infty)$ such that $d(g(t), N)=t$ for $t \leqq t_0$ and $d(g(t), N) < t$ for $t > t_0$ then we say that $g(t_0)$ is in the cut locus of the submanifold $N$. The cut locus of $N$, denoted by $C(N)$ is the set of all such points where $u$ varies in the normal sphere bundle of $N$. Clearly for any $u$ in the normal sphere bundle of $N$ there is at most one point in $C(N)$

along the geodesic in the direction $u$. Let $N^{\perp}$ denote the normal vector bundle of $N$ with respect to the induced riemannian structure and $T(a)$ the set of all vectors $v$ in $N^{\perp}$ such that $|v| < a$. The exponential map of the tangent bundle then restricts to $N^{\perp}$. Without changing notation we call this restriction the exponential map as well. If $C$ is a curve in $M$, $L(C)$ is the length of $C$. The space of curves from $p$ to $q$ whose length is no more than $\alpha$ is denoted by $\Omega_{\alpha}$.

LEMMA. *Let* $\exp : N^{\perp} \to M$ *and* $\exp | T(a)$ *have maximal rank. Let* $g_0, g_1$ *be geodesics defined in the following way*:

$$g_0(t) = \exp(tt_0 v) \qquad t \text{ in } [0, 1]$$

$$g_1(t) = \exp(2tw) \qquad t \text{ in } [0, 1/2]$$

$$\exp\,((1 - (2t - 1)(1 - t_0))v) \qquad t \text{ in } [1/2, 1].$$

*where* $v, w$ *are distinct vectors in* $N^{\perp}$ *and* $L(g_0) \leqq L(g_1)$. *Further let* $H_s$: $[0, 1] \to M$ *be a piecewise differentiable homotopy between* $g_0, g_1$ *with* $H_0 = g_0$, $H_1 = g_1$ *in the space of paths beginning in* $N$ *and ending at* $p = g_0(1)$. *Then there is a* $u$ *in* $[0, 1]$ *such that*

$$L(g_0) + L(H_u) \geqq 2a\,.$$

PROOF. This proof is too similar to that of the lemma in section 4 in [1] to bear repetition.

Let $M$ be a complete simply connected negatively curved space and hence diffeomorphic to euclidean space. Let $N$ be a compact, orientable, connected hypersurface of dimension at least 2 in $M$. As in the introduction $U$ will denote the closure of the bounded component of $M - N$. With this general situation as background we have:

PROPOSITION. *If the eigenvalues of the second fundamental form in the direction pointing into* $U$ *are restricted to* $[a, b]$ *where* $\sqrt{k} < a \leqq b < 2\sqrt{k}$ *and* $2/b > (1/\sqrt{k})$ arc coth $a/\sqrt{k}$ *then the distance from* $N$ *to* $C(N)$ *is at least* $1/b$.

PROOF. Consider two geodesics starting orthogonal to $N$ and meeting in the interior of $U$, that is $\exp(v) = \exp(w)$ for $v, w$ distinct in $T(1/b)$. For small $\varepsilon$, exp is a diffeomorphism on $T(\varepsilon)$. Choose $p = \exp(t_0 v)$ a regular value in $T(\varepsilon)$ and in the interior of $U$, where $0 < \varepsilon < 2/b - (1/\sqrt{k})$ arc coth $a/\sqrt{k}$. Consider then the following geodesics:

$$g_0(t) = \exp(tt_0v) \qquad t \text{ in } [0,1]$$

$$g_1(t) = \exp(2tw) \qquad t \text{ in } [0,1/2]$$

$$\exp((1-(2t-1)(1-t_0))v \qquad t \text{ in } [1/2,1].$$

Since the path space $\Omega(p, N)$ is connected there is a homotopy $H_s$ between $g_0, g_1$. Also notice that exp has maximal rank on $T(1/b)$. Thus there is a $u$ in $[0,1]$ so that $L(H_u) + L(g_0) \geqq 2/b$. Choose a number $\alpha$ such that max $(L(g_1)$, $(1/\sqrt{k})$ arc coth $a/\sqrt{k} \leqq \alpha < 2/b - L(g_0)$ and such that $\Omega_\alpha$ has no geodesic of length $\alpha$. Choose a number $\beta$ so that $\beta > \sup L(H_s)$ and $\Omega_\beta$ has no geodesic of length $\beta$. Thus if $g$ is a geodesic in $\Omega(p, N)$ of length greater than $\alpha$ the index of $g$ is at least 2. Since $g_0$ and $g_1$ can be connected by the homotopy $H_s$ in $\Omega_\beta$ and $\Omega_\beta$ is homotopy equivalent to $\Omega_\alpha$ with a cell of dimension at least two attached, one can connect $g_0$ to $g_1$ by a homotopy $G_s$ in $\Omega_\alpha$. Thus $L(G_s) \leqq \alpha$ for all $\alpha$ and this contradicts the lemma above with $g_1$ unbroken.

COROLLARY. *$N$ is a homotopy sphere.*

PROOF. As in the theorem in [1].

THEOREM. *Let $M$ be a complete simply connected negatively curved manifold with curvature restricted to $[-k, 0]$, $k > 0$. Let $N$ be a closed orientable connected hypersurface of dimension at least two in $M$. Suppose the eigenvalues of the second fundamental form that point into the bounded component lie in the interval $[a, b]$ where $\sqrt{k} < a \leqq b < 2\sqrt{k}$ and $2/b > (1/\sqrt{k})$* arc coth *$a/\sqrt{k}$ then the closure of the bounded component has the homology of a point.*

PROOF. First we observe that $N$ is compact by proposition 2. Let $U$ denote the closure of the bounded component of $M-N$. The homotopy type of $\Omega(U, N)$ is determined by the geodesics in $U$, beginning and ending in $N$. Let $g$ be such a geodesic. Clearly $g$ has a cut point, in fact the mid-point of $g$ is in $C(N)$. Thus $L(g) > 2/b > (1/\sqrt{k})$ arc coth $a/\sqrt{k}$ and the index of $g$ is at least $n$, by proposition 1. Thus $\pi_i(U, N) = 0$ for $0 \leqq i \leqq n$ and by the relative Hurewicz theorem $H_i(U, N) = 0$, $0 \leqq i \leqq n$ and $\pi_{n+1}(U, N) = H_{n+1}(U, N) = Z$. As a result $H_{n+1}(U) = 0$ and $U$ has the homology of a point. This completes the proof.

COROLLARY. *If dimension $M = n+1 \geqq 6$ then $U$ is diffeomorphic to $D^{n+1}$.*

PROOF. This follows from the $h$-cobordism theorem of Smale, see [4].

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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF SOUTHERN CALIFORNIA
LOS ANGELES, CALIFORNIA, U. S. A.