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DIFFERENTIABLE RETRACTIONS IN BANACH SPACES

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1. Definitions. Let A and B be Banach spaces, let U be an open subset of A, and let f be a mapping of U into B. If $x_0 \varepsilon U$, then f is said to be differentiable at x_0 iff there is a continuous linear map $l: A \rightarrow B$ such that $\lim_{\|h\| \rightarrow 0} \frac{\|f(x_0+h)-f(x_0)-l(h)\|}{\|h\|}=0$. If such a map l exists, then it is unique and shall be denoted by $f'(x_0)$. If f is differentiable at each point of U and $f': U \rightarrow L(A, B)$ (where $L(A, B)$ is the space of all continuous linear maps on A into B) is continuous, then f is said to be of class C^1 on U. For the basic theorems concerning this type of differentiability, the reader is referred to [1; p.143] and [5; pp.188-200].

Let K be a subset of a Banach space A and let f be a continuous function mapping K into K. Then f is called a differentiable retraction of K if

(1) f is of class C^1 on $\operatorname{int}(K)$;

(2) f is a retraction of K; i.e., $f \circ f = f$

If f is a differentiable retraction of K and f is not the identity function on K, then f is called a proper differentiable retraction of K. The set $f(K)$ is said to be a differentiable retract of K.

2. Dimension lowering properties of differentiable retractions. Theorem 2.1, the principal theorem of this section, shows that the range of a proper differentiable retraction of certain types of sets is a nowhere dense subset of the Banach space. Before the statement and proof of Theorem 2.1, we make note of the following lemma whose proof may be found in [1; pp.156-157].

LEMMA. *Let E and F be two Banach spaces, f a differentiable mapping into F of an open neighborhood U of a segment S joining two points a and b in E. Then for each $x_0 \in U$, we have*

$$\|f(b) - f(a) - (f'(x_0))(b-a)\| \leqq \|b-a\| \cdot \sup_{x \in S} \|f'(x) - f'(x_0)\|.$$

THEOREM 2.1. *Let A be a Banach space and let K be a subset of A*

such that int(K) *is connected and* $\overline{\text{int}(K)} \supset K$. *If* $f: K \to K$ *is a proper differentiable retraction of* K, *then* int$(f(K))$ *is empty.*

PROOF. Suppose int $(f(K)) \neq \emptyset$. If $[\overline{\text{int}(f(K))} - \text{int}(f(K))] \cap \text{int}(K) = \emptyset$, then $\text{int}(K) = \text{int}(f(K)) \cup [\text{int}(K) - \overline{\text{int}(f(K))}]$. Since $\overline{\text{int}(K)} \supset K$ and since $f(K)$ is a closed subset of K, it follows that $\text{int}(K) - \overline{\text{int}(f(K))} \neq \emptyset$. By supposition, $\text{int}(f(K)) \neq \emptyset$. Hence, int$(K)$ is the union of two disjoint non-empty open sets which contradicts its connectedness. Therefore, $[\overline{\text{int}(f(K))} - \text{int}(f(K))] \cap \text{int}(K) \neq \emptyset$.

Let $p \in [\overline{\text{int}(f(K))} - \text{int}(f(K))] \cap \text{int}(K)$. Choose a neighborhood U of p. Because f is continuous at p and $f(p)=p$ there is a neighborhood V of p such that $f(V) \subset U$. Now $p \in [\overline{\text{int}(f(K))} - \text{int}(f(K))] \cap \text{int}(K)$, so that there is a point q such that $q \in [U \cap V \cap \text{int}(K) \cap (K - f(K))]$. It is easy to see that q and $f(q)$ are distinct points of U which are mapped to the same point under f. This proves that f is not locally one-to-one at p.

Since f is the identity on $f(K)$, $f'(x)$ is the identity linear mapping for $x \in \text{int}(f(K))$. Thus, f' continuous on int(K) and $p \in [\overline{\text{int}(f(K))} \cap \text{int}(K)]$ imply $f'(p)$ is the identity linear mapping. Let $B(p, \varepsilon)$ be an open ball about p of sufficiently small radius ε so that $B(p, \varepsilon) \subset \text{int}(K)$ and $\|f'(x) - f'(p)\| < 1$ for all $x \in B(p, \varepsilon)$. Choose distinct points a and b in $B(p, \varepsilon)$ such that $f(a) = f(b)$. Applying the Lemma we obtain

$$\|f(b) - f(a) - (f'(p))(b - a)\| \leqq \|b - a\| \cdot \sup_{x \in S} \|f'(x) - f'(p)\|,$$

where S is the line segment joining the two points a and b. Using that $f(a) = f(b)$ and that $f'(p)$ is the identity, the above equation reduces to

$$\|b - a\| \leqq \|b - a\| \cdot \sup_{x \in S} \|f'(x) - f'(p)\|$$

$$\text{or} \qquad 1 \leqq \sup_{x \in S} \|f'(x) - f'(p)\|.$$

This contradicts that $\|f'(x) - f'(p)\| < 1$ for all $x \in B(p, \varepsilon)$. Therefore, $\text{int}(f(K)) = \emptyset$.

The following corollary shows that proper differentiable retractions of certain subsets of n-dimensional Euclidean space R^n lower dimension.

COROLLARY. *Let* K *be a subset of* R^n *such that* int(K) *is connected and* $\overline{\text{int}(K)} \supset K$. *Let* f *be a proper differentiable retraction of* K. *If the dimension of* $f(K)$ *is* k, *then* $k \leqq n - 1$.

PROOF. Suppose $k = n$. Then there is a non-empty subset of $f(K)$ which is open in R^n[2; p.44]. Hence, $\text{int}(f(K)) \neq \emptyset$ which contradicts Theorem 2.1.

We now give several examples to illustrate that each of the restrictions on K in the previous Theorem was necessary.

EXAMPLE. Let $D_1, D_2 \subset R^2$ be given by $D_1 = \{(x, y) \in R^2 : (x+1)^2 + y^2 \leqq 1\}$ and let $D_2 = \{(x, y) \in R^2 : (x-1)^2 + y^2 \leqq 1\}$. Let $K = D_1 \cup D_2$ and define $f: K \to K$ by

$$f(x, y) = \begin{cases} (0, 0), & \text{if } (x, y) \in D_1, \\ (x, y), & \text{if } (x, y) \in D_2. \end{cases}$$

It is easy to verify that f is a proper differentiable retraction of K onto D_2. Notice that K is connected, $\overline{\text{int}(K)} = K$, $\text{int}(K)$ is not connected, and $\text{int}(f(K)) \neq \emptyset$.

EXAMPLE. Let $K = \{(x, y) \in R^2 : x^2 + y^2 \leqq 1\} \cup \{(x, y) \in R^2 : 1 \leqq x \leqq 2$ and $y = 0\}$ and let $f: K \to K$ be given by

$$f(x, y) = \begin{cases} (x, y), & \text{if } x^2 + y^2 \leqq 1, \\ (1, 0), & \text{if } 1 \leqq x \leqq 2 \text{ and } y = 0. \end{cases}$$

It is easy to see that f is a proper differentiable retraction of K such that $\text{int}(f(K)) \neq \emptyset$. Notice that K is connected, $\text{int}(K)$ is connected, but $\overline{\text{int}(K)} \not\supset K$.

There are continuous retractions, different from the identity, which are locally one-to-one at all points except possibly at boundary points relative to the range. For example, $f: R^1 \to R^1$ given by $f(x) = |x|$ is a proper retraction of R^1 which is locally one-to-one at every real number $x \neq 0$.

The final theorem of this section shows that proper differentiable retractions of certain subsets of R^n are nowhere locally one-to-one.

THEOREM 2.2. *Let K be a subset of R^n such that* $\text{int}(K)$ *is a non-empty connected set and* $\overline{\text{int}(K)} \supset K$. *If f is a proper differentiable retraction of K, then f is not locally one-to-one at any point $x \in K$.*

PROOF. Let $x \in \text{int}(K)$ and let U be a bounded set, open in R^n, such that $x \in U$ and $\overline{U} \subset \text{int}(K)$. By the Corollary to Theorem 2.1, $\dim(f(\overline{U})) \leqq n-1$. Since $\dim(\overline{U}) = n$, $g = f|_{\overline{U}}$ is a mapping of a compact set $\overline{U}$ into K which lowers dimension. By a theorem in [2; pp.91-93], there exists a point p such that $\dim(g^{-1}(p)) \geqq 1$. This clearly implies that f is not locally one-to-one at x. Since $\overline{\text{int}(K)} \supset K$, the result follows.

A stronger form of Theorem 2.1 can be obtained for R^1. Let D denote the set of all continuous functions on the closed unit interval $I=[0,1]$ into I which are differentiable on I. If $\circ$ denotes functional composition, then $(D,\circ)$ is a semigroup. In [4] it was shown that the only idempotents of $(D,\circ)$ are the identity function and the constant functions. Now being an idempotent in this semigroup is equivalent to being a retraction of I which is differentiable on I. From this it follows that Theorem 2.1 is valid for R^1 with only the assumption that the retraction f is differentiable (f need not be assumed continuous). It would be interesting to have some theorems about such retractions (differentiable but not necessarily of Class C^1) in arbitrary Banach spaces.

3. The unit sphere in l_2 a differentiable retract of the unit ball. Let l_2 be the (real) Hilbert space of all square summable sequences of real numbers, let B denote the closed unit ball in l_2 ($B=\{(x_1, x_2, \cdots)\in l_2 : \sum_{i=1}^{\infty} x_i^2 \leqq 1\}$), and let S denote the unit sphere in l_2($S=\{(x_1, x_2, \cdots)\in l_2 \sum_{i=1}^{\infty} x_i^2=1\}$). The purpose of this section is to prove that S is a differentiable retract of B. It is known that S is a retract of B.

If $x, y\in l_2$, then let (x,y) denote the inner product of x and y.

LEMMA. *The following are equivalent*:
(1) *There exists a continuous function $f{:}B\rightarrow B$ such that f has no fixed point and f is of class C^1 on the interior of B.*
(2) *There exists a differentiable retraction g of B onto S.*

PROOF. To see that (1) implies (2), let f be a function satisfying the conditions in (1) and define $g:B\rightarrow S$ by $g(x)=x+\dfrac{x-f(x)}{\|x-f(x)\|}\, a(x)$,

where $a(x)=\sqrt{1-(x,x)+\left[\left(x,\dfrac{x-f(x)}{\|x-f(x)\|}\right)\right]^2}-\left(x,\dfrac{x-f(x)}{\|x-f(x)\|}\right)$,

A routine calculation shows that, if $x\in S$, then $a(x)=0$. Hence, $g(x)=x$ for all $x\in S$, and g is a retraction of B onto S (that the range of g is exactly S follows from showing that $(g(x), g(x))=1$ for all $x\in B$). Since inner product is of class C^1 on l_2 [1; p.144], norm is of class C^1 on $l_2-\{0\}$, f is of class C^1 on the interior of B, and $1-(x,x)+\left[\left(x,\dfrac{x-f(x)}{\|x-f(x)\|}\right)\right]^2>0$ on the interior of B, it follows that g is of class C^1 on the interior of B. Therefore, g is the required differentiable retraction of B onto S.

It is easy to see that (2) implies (1); for if g is a differentiable retraction of B onto S, then $f=-g$ is a class C^1 mapping of B into B which leaves no point fixed.

The following example was invented by Mr. Christopher L. Lacher for another purpose.

EXAMPLE. Assign to each point $x=(x_1, x_2, \cdots) \in B$ the point $f(x)=(\sqrt{1-(x, x)}, x_1, x_2, \cdots)$. Since $\|f(x)\| = \sqrt{1-(x, x)+\sum_{i=1}^{\infty} x_i{}^2} = \sqrt{1-\|x\|^2+\|x\|^2} =1$ for all $x=(x_1, x_2, \cdots) \in B$, f is a mapping of B into B (actually into S) which can be shown to be a homeomorphism. The function f has on fixed point. Suppose, on the contrary, that $x^*=(x_1^*, x_2^*, \cdots)$ is a fixed point for f. Then $(x_1^*, x_2^*, \cdots)= (\sqrt{1-(x^*, x^*)}, x_1^*, x_2^*, \cdots)$ from which it follows that $x_1^*=x_2^*=x_3^* =\cdots$, i.e., x^* is a constant sequence. Since $\sum_{i=1}^{\infty} (x_1^*)^2 <\infty$, x^* must be the origin $(0, 0, \cdots)$ which is clearly not a fixed point for f. This establishes a contradiction, showing that f has no fixed point.

THEOREM 3.1. *The unit sphere S is a differentiable retract of the unit ball B.*

PROOF. Let $f: B \to B$ be the function described in the previous example (if $x=(x_1, x_2, \cdots)$, then $f(x)=\sqrt{(1-(x, x)}, x_1, x_2, \cdots)$. Consider the three functions $h: l_2 \to l_2$, given by $h(x_1, x_2, \cdots)=(0, x_1, x_2, \cdots)$, $j: B \to R^1$, given by $j(x)=\sqrt{1-(x, x)}$, and $k: R^1 \to l_2$, given by $k(r)=(r, 0, 0, \cdots)$. It is easily verified that $f=h|_B+k \circ j$. Since h and k are linear, each is of class C^1. Assume $x \in l_2$ and $\|x\|<1$. Then $1-(x, x)>0$ and it follows that j is of class C^1 on the interior of B. Therefore, f is of class C^1 on the interior of B. We have shown that f has no fixed point. This proves, by applying the Lemma, that there exists a differentiable retraction of B onto S. Hence, S is a differentiable retract of B.

REMARK 1. The proof of Theorem 3.1 is effective in the sense that one can in fact write down an algebraic formula for a differentiable retraction of B onto S.

REMARK 2. Since any real separable Hilbert space is isometrically isomorphic to l_2, the unit sphere in any real separable Hilbert space is a differentiable retract of the unit ball.

In the proof of (1) implies (2) in the Lemma a retraction g was constructed from a mapping $f: B \to B$ which was assumed to have no fixed point. The formula for g in terms of f appears in [3; pp.15-16] as part of the proof of the Brouwer Fixed-Point Theorem for the n-ball in Euclidean n-space. The author is indebted to Professor C.H. Edwards for pointing this out.

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