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FUNCTION OF EXPONENTIAL TYPE BELONGING TO L^p ON THE REAL LINE

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The following result or some special cases of it might have occured to a specialist on entire functions but we have never seen anything like this in print. To us it appears to be of sufficient interest to merit publication.

THEOREM. *Let $f(z)$ be an entire function of order* 1 *and type $\tau\,(0 \leqq \tau < \infty)$. Suppose $f(z) \in L^p (0 < p < \infty)$ on the real line and is real for real z. If*

$$\phi_p(y) = \left(\int_{-\infty}^{\infty} |f(x+iy)|^p dx\right)^{1/p},$$

then

$$\limsup_{\pm y \to \infty} |y|^{-1} \log \phi_p(y) = \tau .$$

With

$$\phi_\infty(y) = \sup_{-\infty < x < \infty} |f(x+iy)|$$

the conclusion holds also for $p = \infty$.

We deduce the theorem from certain well known results which we quote as lemmas.

LEMMA 1. *If $f(z)$ is regular and of exponential type in the upper half plane, $h(\pi/2) = \limsup_{y\to\infty} y^{-1} \log |f(iy)| \leqq c$ and $|f(x)| \leqq M$, $-\infty < x < \infty$, then*

$$|f(x+iy)| \leqq Me^{cy}, \; -\infty < x < \infty, 0 \leqq y < \infty .$$

For a proof of this lemma see [1,pp.82-84]. Lemma 2 is a theorem of Plancherel and Pólya [3] and its proof can also be found in [1,pp.98-101].

LEMMA 2. *If $f(z)$ is an entire function of exponential type τ, and if* *for some positive number* p, $f(x)\in L^p(-\infty,\infty)$ *then* $\phi_p(y)\leqq e^{\tau|y|}\phi_p(0)$ *and* $f(x)$ *is bounded on the real line.*

Lemma 3 is due to R. Nevanlinna. For a proof, see [1,pp.92-95].

LEMMA 3. *If* $f(z)$ *is regular and of exponential type* τ *in the upper half plane, and*

$$\int_{-\infty}^{\infty}\frac{\log^+|f(x)|}{1+x^2}\,dx<\infty,$$

then

$$\log|f(z)|\leqq\frac{y}{\pi}\int_{-\infty}^{\infty}\frac{\log|f(t)|}{(t-x)^2+y^2}\,dt+\tau y. \tag{1}$$

PROOF OF THE THEOREM. By Lemma 2,

$$\limsup_{\pm y\to\infty}\ |y|^{-1}\log\phi_p(y)\leqq\tau,\qquad(0<p<\infty). \tag{2}$$

For $p=\infty$ the same conclusion follows from Lemma 1.

By Lemma 2, $f(z)$ is bounded on the real line. Since $f(z)$ is real for real z it follows from Lemma 1 that

$$\limsup_{\pm y\to\infty}\ |y|^{-1}\log|f(iy)|=\tau, \tag{3}$$

otherwise $f(z)$ cannot be of type τ.

Now let us write (1) in the form

$$\log\{|f(x+iy)|e^{-\tau|y|}\}\leqq\pi^{-1}|y|\int_{-\infty}^{\infty}\log|f(t)|\,\frac{1}{(t-x)^2+y^2}\,dt.$$

Then if $0<p<\infty$, by Jensen's inequality [2,p.46] we have

$$\begin{aligned}\{|f(x+iy)|e^{-\tau|y|}\}^p&\leqq\pi^{-1}|y|\int_{-\infty}^{\infty}|f(t)|^p\,\frac{1}{(t-x)^2+y^2}\,dt\\&\leqq\pi^{-1}|y|\left(\int_{-\infty}^{\infty}|f(t)|^{2p}dt\right)^{\frac{1}{2}}\left(\int_{-\infty}^{\infty}\frac{dt}{\{(t-x)^2+y^2\}^2}\right)^{\frac{1}{2}}\\&=\pi^{-1}|y|\left(\int_{-\infty}^{\infty}|f(t)|^{2p}dt\right)^{\frac{1}{2}}\left(\frac{\pi}{2|y|^3}\right)^{\frac{1}{2}},\end{aligned}$$

or

$$\{|f(x+iy)|e^{-\tau|y|}\}^{2p} \leqq \frac{1}{2\pi|y|}\int_{-\infty}^{\infty}|f(t)|^{2p}dt.$$

Hence for every $p>0$

$$\max_{-\infty<x<\infty}|f(x+iy)|^{p} \leqq \frac{e^{\tau p|y|}}{2\pi|y|}\int_{-\infty}^{\infty}|f(x)|^{p}dx.$$

By Lemma 1

$$\max_{-\infty<x<\infty}|f(x)| \leqq e^{\tau|y|}\max_{-\infty<x<\infty}|f(x+iy)|.$$

Consequently

$$\max_{-\infty<x<\infty}|f(x)|^{p} \leqq \frac{e^{2\tau p|y|}}{2\pi|y|}\int_{-\infty}^{\infty}|f(x)|^{p}dx$$

for every y. The right hand side is minimum for $|y|=(2\tau p)^{-1}$. With this choice of y we get

$$\max_{-\infty<x<\infty}|f(x)|^{p} \leqq \frac{e\tau p}{\pi}\int_{-\infty}^{\infty}|f(x)|^{p}dx, \tag{4}$$

which is a result of independent interest. It is true for entire functions of exponential type τ belonging to $L^{p}(0<p<\infty)$ on the real line.

Inequality (4) implies that

$$\int_{-\infty}^{\infty}|f(x+iy)|^{p}dx \geqq \frac{\pi}{e\tau p}|f(iy)|^{p}$$

for every y. In conjunction with (3) this gives

$$\limsup_{\pm y\to\infty}|y|\log\phi_{p}(y) \geqq \tau \tag{5}$$

for $0<p<\infty$. For $p=\infty$ this follows from (3) alone.

The desired result follows from (2) and (5).

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