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GENERATORS OF CERTAIN VON NEUMANN ALGEBRAS

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1. Recently, a few results on generation of von Neumann algebras were obtained by some authors. C. Davis [1] proved that a factor of type I on a separable Hilbert space is generated by three projections and is also generated by two unitary operators. It is known [4] that every von Neumann algebra of type I on a separable Hilbert space possesses a single generator, and some examples of each type II_1, II_∞ and III having a single generator are given [5], [6]. The purpose of this paper is to prove some results on generation of certain von Neumann algebras, parallel to the theorems in [1]. A part of this note has been published in Japanese [Sûgaku, 19(1967), 172-173].

2. In this paper, an operator means a bounded linear operator on a Hilbert space. A von Neumann algebra $\boldsymbol{M}$ is said to be generated by a family $\{A, B, \cdots\}$ of operators, if $\boldsymbol{M}$ is the smallest von Neumann algebra containing each member of $\{A, B, \cdots\}$, and it is denoted by $R(A, B, \cdots)$. This terminology is used for a family of von Neumann algebras. For a von Neumann algebra $\boldsymbol{M}$ on a Hilbert space $\boldsymbol{H}$, $\boldsymbol{M}_2$ means the algebra consisting of all 2×2 matrices over $\boldsymbol{M}$ acting on $\boldsymbol{H}\oplus\boldsymbol{H}$.

The following theorem parallels [1: Theorem 1].

THEOREM 1. *If a von Neumann algebra $\boldsymbol{M}$ on a Hilbert space $\boldsymbol{H}$ has a single generator A, then there exist three projections E_1, E_2 and E_3 on $\boldsymbol{H}\oplus\boldsymbol{H}$ such that $R(E_1, E_2, E_3)=\boldsymbol{M}_2$.*

PROOF. We can assume that A is invertible and $\|A\| < 1$. Then, let S and T be the positive square roots of $I-A^*A$ and A^*A respectively. Using A, S and Y we define

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$$E_1 = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}, \quad E_2 = \begin{pmatrix} AA^* & AS \\ SA^* & I-A^*A \end{pmatrix}, \quad E_3 = \begin{pmatrix} I-A^*A & ST \\ ST & A^*A \end{pmatrix}.$$

If we observe that S, T and A^*A are mutually commuting, we can easily prove that E_1, E_2 and E_3 are projections. Direct computation shows that

$$E_2 + E_1E_2E_1 - E_1E_2 - E_2E_1 = \begin{pmatrix} 0 & 0 \\ 0 & I-A^*A \end{pmatrix}.$$

Thus $R(E_1, E_2)$ contains all matrices of the form $\begin{pmatrix} 0 & 0 \\ 0 & X \end{pmatrix}$ with $X \in R(I-A^*A)$. In particular, $\begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 \\ 0 & S^{-1} \end{pmatrix}$ are contained in $R(E_1, E_2)$. Now,

$$\begin{pmatrix} 0 & 0 \\ 0 & S^{-1} \end{pmatrix} E_2 \left\{ E_1 + \begin{pmatrix} 0 & 0 \\ 0 & S^{-1} \end{pmatrix} \right\} = \begin{pmatrix} 0 & 0 \\ A^* & I \end{pmatrix},$$

and so

$$\begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix} = \left\{ \begin{pmatrix} 0 & 0 \\ A^* & I \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix} \right\}^*$$

belongs to $R(E_1, E_2)$. Since we have

$$E_1 - E_1 E_3 E_1 = \begin{pmatrix} A^*A & 0 \\ 0 & 0 \end{pmatrix},$$

$R(E_1, E_3)$ contains all matrices of the form $\begin{pmatrix} X & 0 \\ 0 & 0 \end{pmatrix}$ with $X \in R(A^*A)$. Thus, in particular, $\begin{pmatrix} T^{-1} & 0 \\ 0 & 0 \end{pmatrix}$ is in $R(E_1, E_3)$. Hence $\begin{pmatrix} 0 & I \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} T^{-1} & 0 \\ 0 & 0 \end{pmatrix} E_3 \begin{pmatrix} 0 & 0 \\ 0 & S^{-1} \end{pmatrix}$, $\begin{pmatrix} 0 & 0 \\ I & 0 \end{pmatrix} = \begin{pmatrix} 0 & I \\ 0 & 0 \end{pmatrix}^*$ are contained in $R(E_1, E_2, E_3)$. As $\begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix}$ is in $R(E_1, E_2)$,

$$\begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ I & 0 \end{pmatrix}$$

is contained in $R(E_1, E_2, E_3)$. Hence $R(E_1, E_2, E_3)$ contains every matrix

$\begin{pmatrix} X & 0 \\ 0 & 0 \end{pmatrix}$ with $X \in R(A) = \boldsymbol{M}$. Therefore it follows that $R(E_1, E_2, E_3)$ contains every matrix $\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$ with $A_{ij} \in \boldsymbol{M}$ $(i, j=1, 2)$, because

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = \begin{pmatrix} A_{11} & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} A_{12} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & I \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ I & 0 \end{pmatrix} \begin{pmatrix} A_{21} & 0 \\ 0 & 0 \end{pmatrix}$$
$$+ \begin{pmatrix} 0 & 0 \\ I & 0 \end{pmatrix} \begin{pmatrix} A_{22} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & I \\ 0 & 0 \end{pmatrix}.$$

REMARK 1. The number three in Theorem 1 cannot be reduced in general. In fact, the ring of all operators on a separable Hilbert space has a single generator and in this case the number three cannot be reduced [1: Theorem 1].

According to [2], a factor $\boldsymbol{M}$ is said to be hyperfinite if (i) $\boldsymbol{M}$ is of type I_n $(n < +\infty)$, or (ii) $\boldsymbol{M}$ is of finite type and is generated by an increasing sequence of factors of type $\mathrm{I}_1, \mathrm{I}_2, \mathrm{I}_4, \cdots \mathrm{I}_{2^n}, \cdots$. It is well known that two hyperfinite continuous factors are isomorphic. In [6], N. Suzuki and the author proved that a hyperfinite factor on a separable Hilbert space is generated by a single operator. Thus we have

COROLLARY 1. *There exist three projections which generate a hyperfinite factor on a separable Hilbert space.*

PROOF. In the case of type I_n $(n < +\infty)$, the assertion is clear by [1]. Suppose that $\boldsymbol{M}$ is a hyperfinite continuous factor on a separable Hilbert space. Then $\boldsymbol{M}_2$ is also a hyperfinite continuous factor, and so $\boldsymbol{M}_2$ is isomorphic to $\boldsymbol{M}$. Since $\boldsymbol{M}$ has a single generator as noted above, $\boldsymbol{M}_2$ is generated by three projections and thus $\boldsymbol{M}$ is generated by three projections.

The following theorem is parallel to [1: Theorem 2].

THEOREM 2. *Under the same assumption as in Theorem* 1, *there exist two unitary operators which genetate* $\boldsymbol{M}_2$. *They may be chosen so one of them is a symmetry.*

PROOF. As in the proof of Theorem 1, we can assume that A is invertible and is a strict contraction. Let

$$E=\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}, \quad U=\begin{pmatrix} A & Z \\ -S & A^* \end{pmatrix}$$

where S and Z are the positive sequare roots of $I-A^*A$ and $I-AA^*$ respectively. Then E is a projection and U is a unitary operator (cf. [3]). Since $I-2E$ is a symmetry, it suffices to show that $R(E,U)=\boldsymbol{M}_2$. From the last equality of the proof of Theorem 1, it is also sufficient to prove that the matrices $\begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & I \\ 0 & 0 \end{pmatrix}$ are contained in $R(E,U)$. Since $EUE=\begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}$ belongs to $R(E,U)$. Thus all matrices $\begin{pmatrix} X & 0 \\ 0 & 0 \end{pmatrix}$ with $X \in R(A)$ are contained in $R(E,U)$. In particular, $\begin{pmatrix} A^{-1} & 0 \\ 0 & 0 \end{pmatrix}$ belongs to $R(E,U)$. Now we have

$$UEU^*=\begin{pmatrix} AA^* & -AS \\ -SA^* & I-A^*A \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix} UEU^* \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & I-A^*A \end{pmatrix}.$$

Because of $\begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix} = U^*U-E$, $\begin{pmatrix} 0 & 0 \\ 0 & I-A^*A \end{pmatrix}$ is contained in $R(E,U)$, and so $\begin{pmatrix} 0 & 0 \\ 0 & X \end{pmatrix}$ belongs to $R(E,U)$, for each $X \in R(I-A^*A)$. Of course, $\begin{pmatrix} 0 & 0 \\ 0 & S^{-1} \end{pmatrix}$ is contained in $R(E,U)$. Thus

$$\begin{pmatrix} 0 & 0 \\ 0 & S^{-1} \end{pmatrix} UEU^* \left\{ E + \begin{pmatrix} 0 & 0 \\ 0 & S^{-1} \end{pmatrix} \right\} = \begin{pmatrix} 0 & 0 \\ -A^* & I \end{pmatrix}$$

belongs to $R(E,U)$, and hence

$$\begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix} = -\left\{ \begin{pmatrix} 0 & 0 \\ -A^* & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix} \right\}^*$$

is contained in $R(E,U)$. Therefore

$$\begin{pmatrix} A^{-1} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & I \\ 0 & 0 \end{pmatrix}$$

is in $R(E, U)$ and the proof is completed.

Corresponding to Corollary 1, we have

COROLLARY 2. *There exist two unitary operators which generate a hyperfinite factor on a separable Hilbert space. They may be chosen so one of them is a symmetry.*

REMARK 2. As proved in [5] and [6], there exists an operator which generates a von Neumann algebra of type II_∞ (resp. III). Thus we obtain an example of von Neumann algebra of type II_∞ (resp. III) which is generated by three projections and is also generated by two unitary operators.

ADDED IN PROOF. Recently, C. Pearcy and D. Topping obtained in the paper [Sums of small numbers of certain operators, Michigan Math. Journ., 14 (1967), 453-465] results similar to Theorem 1 above.

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