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NON-NORMAL ABELIAN SUBALGEBRAS

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A W^*-subalgebra $\mathcal{A}_0$ of a W^*-algebra $\mathcal{A}$ is said to be *normal in* $\mathcal{A}$ if $(\mathcal{A}_0'\cap\mathcal{A})'\cap\mathcal{A}=\mathcal{A}_0$ (i.e. if $\mathcal{A}_0$ has the double commutant property relative to $\mathcal{A}$). A W^*-algebra $\mathcal{A}$ is called *normal* if every W^*-subalgebra $\mathcal{A}_0$ of $\mathcal{A}$ containing the center of $\mathcal{A}$ is normal in $\mathcal{A}$.

It is well known (cf. [3] and [4]) that all type I factors (in fact, all type I W^*-algebras) are normal. That no type II factor is normal is proved in [3]. Examples of non-normal type III factors are contained in [5]. There also exist a few examples of non-normal abelian subalgebras in factors of type II (cf. [2], [4], [6]).

In this paper we give a rather simple construction of an infinite sequence of abelian subalgebras which are non-normal in a hyperfinite type II_1 factor $\mathcal{A}$ and which are pairwise non-conjugate under *-automorphisms of $\mathcal{A}$.

In section 1 we shall construct, for each $n\geqq 4$, a hyperfinite factor $\mathcal{A}_n$ and an abelian subalgebra C_{n0} which is non-normal in $\mathcal{A}_n$. Since all hyperfinite factors are *-isomorphic, we can suppose that all these subalgebras exist in one hyperfinite factor. In section 2 we prove that the subalgebras C_{n0} are pairwise non-conjugate.

1. Construction of subalgebras. The factors employed here shall be constructed according to the following general scheme: Let G be a countable discrete group with identity e. Let $\mathfrak{H}$ be $L_2(G)$, the Hilbert space of square-summable complex valued functions on G. For each $g\in G$ there is a unitary operator U_g defined on $\mathfrak{H}$ by $U_g x(g')=x(g'g)$. These operators generate a W^*-algebra $\mathcal{A}$ which is a factor of type II_1 if all the non-trivial equivalency classes of G are infinite and which is, in addition, *hyperfinite* if G is the union of an increasing sequence of finite subgroups.

Let $\mathfrak{H}'$ be the set of those functions $y\in\mathfrak{H}$ possessing the following property: for every $x\in\mathfrak{H}$ the convolution product $x*y$ belongs to $\mathfrak{H}$. With each $y\in\mathfrak{H}'$ we associate the operator U_y defined by $U_y x=x*y$. Then $\mathcal{A}=\{U_y\,|\,y\in\mathfrak{H}'\}$ and we have:

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(i) $U_y{}^* = U_{\tilde{y}} \quad (\tilde{y}(g) = \overline{y(g^{-1})})$
(ii) $U_{y*z} = U_z U_y$ and $U_{y+z} = U_y + U_z \quad (y, z \in \mathfrak{H}')$
(iii) $U_{g^{-1}} = U_{\varepsilon_g}$ where ε_g is the characteristic function of $\{g\}$

Finally, if $\overline{G}$ is any subgroup of G, the operators U_g $(g \in \overline{G})$ generate a subalgebra $\mathcal{A}(\overline{G})$ of $\mathcal{A}$ and

$$\mathcal{A}(\overline{G}) = \{U_z | z \in \mathfrak{H}',\ z(g) = 0 \quad \text{if } g \notin \overline{G}\}\ .$$

The particular factors and subalgebras we shall work with shall be constructed as follows:

Let F denote an infinite commutative field which is the union of an increasing sequence of finite subfields. Then $F = \bigcup_{i=1}^{\infty} F_i$ where F_i are finite fields and $F_1 \subseteqq F_2 \subseteqq \cdots$.

For each $n \geqq 4$, let G_n be the group of $n \times n$ matrices over F of the form:

$$(1) \qquad \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1,n-1} & a_{1n} \\ 0 & 1 & a_{23} & \cdots & a_{2,n-1} & a_{2n} \\ 0 & 0 & 1 & \cdots & a_{3,n-1} & a_{3n} \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & 1 & a_{n-1,n} \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}$$

where $a_{11} \neq 0$.

Let $\mathcal{A}_n$ be the W^*-algebra $\mathcal{A}(G_n)$, the algebra generated by all operators U_g $(g \in G_n)$ on $L_2(G_n)$. It is proved in [1] that $\mathcal{A}_n$ is a hyperfinite factor of type II_1.

For each $n \geqq 4$, let G_{n0} be the subgroup of G_n consisting of all elements of the form:

$$(2) \qquad \begin{pmatrix} 1 & b_{12} & b_{13} & b_{14} & b_{15} & \cdots & b_{1,n-1} & b_{1n} \\ 0 & 1 & b_{12} & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}$$

and let H_{n0} be that subgroup of G_{n0} for which $b_{1n} = 0$. Let $\mathcal{A}_{n0} = \mathcal{A}(G_{n0})$

be the subalgebra of $\mathcal{A}_n$ generated by the operators U_g $(g \in G_{n0})$, and $C_{n0} = \mathcal{A}(H_{n0})$ be the subalgebra of $\mathcal{A}_n$ generated by the operators U_g $(g \in H_{n0})$.

Then, $\mathcal{A}_{n0}$ is a maximal abelian subalgebra of $\mathcal{A}_n$ (cf. [1]) and C_{n0} is abelian. The following lemmas in this section will show that C_{n0} is non-normal in $\mathcal{A}_n$.

LEMMA 1. *G_{n0} is the centralizer of H_{n0}. That is, an element $g \in G_n$ commutes with every $h \in H_{n0}$ if and only if $g \in G_{n0}$.*

PROOF. Clearly, if $g \in G_{n0}$, g commutes with all of H_{n0}.

Conversely, suppose $g \in G_n$ is of form (1) and $gh = hg$ for every $h \in H_{n0}$. Let h be of form (2) with $b_{1n} = 0$. Direct computation establishes the result for $n = 4$. Assume now that $n \geqq 5$. Partition g as

$$(3) \qquad \left(\begin{array}{c|c} g_{11} & g_{12} \\ \hline 0 & 1 \end{array}\right) \quad \text{so that } g_{11} \in G_{n-1}.$$

Partition h as

$$(4) \qquad \left(\begin{array}{c|c} h_{11} & 0 \\ \hline 0 & 1 \end{array}\right) \quad \text{so that } h_{11} \in G_{n-1,0}.$$

(Note that h_{11} need not belong to $H_{n-1,0}$).

Then:

$$(5) \qquad gh = \left(\begin{array}{c|c} g_{11}h_{11} & g_{12} \\ \hline 0 & 1 \end{array}\right) \text{ and } hg = \left(\begin{array}{c|c} h_{11}g_{11} & h_{11}g_{12} \\ \hline 0 & 1 \end{array}\right).$$

Now if $gh = hg$ for all $h \in H_{n0}$ then $g_{11}h_{11} = h_{11}g_{11}$ for all $h_{11} \in G_{n-1,0}$ so that $g_{11} \in G_{n-1,0}$ since $G_{n-1,0}$ is maximal abelian in G_{n-1} (cf. [1]).

Furthermore, $g_{12} = h_{11}g_{12}$, i.e.

$$(6)\qquad \begin{pmatrix} a_{1n} \\ a_{2n} \\ a_{3n} \\ \cdot \\ a_{n-1,n} \end{pmatrix} = \begin{pmatrix} a_{1n}+b_{12}a_{2n}+b_{13}a_{3n}+\cdots+b_{1,n-1}a_{n-1,n} \\ a_{2n}+b_{12}a_{3n} \\ a_{3n} \\ \cdot \\ a_{n-1,n} \end{pmatrix}$$

so that $a_{jn}=0$ for all j, $2\leqq j\leqq n-1$, i.e. $g\in G_{n0}$. Q. E. D.

The following property (β') is a slight variation of the property (β) of Dixmier [2].

DEFINITION 1. Let G be a group. Let $\overline{G}$ and H be subgroups such that $H\subseteqq\overline{G}\subseteqq G$. $\overline{G}$ is said to have property (β') relative to H if, given an arbitrary finite subset $B\subseteqq G$ and an arbitrary $g\in G\backslash\overline{G}$ there exists an element $h_0\in H$ such that (i) $g^{-1}h_0g \neq h_0$ and (ii) $u, v\in B$ and $u^{-1}h_0v=h_0$ together imply that $u=v$.

LEMMA 2. (Dixmier [2]) *Let y be a complex function on G vanishing outside a finite set B. Let g and h be elements of G such that the conditions $u\in Bg^{-1}$, $v\in Bg^{-1}$, $u^{-1}hv=h$ imply $u=v$. Then* $|y(g)|^2\leqq(\tilde{y}*\varepsilon_h*y)(g^{-1}hg)$.

The following lemma and its proof are adapted from [2].

LEMMA 3. *Suppose G is a countable discrete group with infinite equivalency classes and $H\subseteqq\overline{G}\subseteqq G$. Let $\mathcal{A}(G)$ be the algebra described previously. Let $\mathcal{A}(H)$ and $\mathcal{A}(\overline{G})$ be the subalgebras corresponding to H and $\overline{G}$, respectively. Suppose that $\overline{G}$ is the centralizer of H and that $\overline{G}$ has property (β') relative to H. Then $\mathcal{A}(H)'\cap\mathcal{A}(G)=\mathcal{A}(\overline{G})$.*

PROOF. It is clear that $\mathcal{A}(\overline{G})$ is contained in $\mathcal{A}(H)'\cap\mathcal{A}(G)$. To establish the reverse inclusion, suppose that $A=U_x\in\mathcal{A}(H)'\cap\mathcal{A}(G)$. Since the unitary operators form a generating set for the W^*-algebra $\mathcal{A}(H)'\cap\mathcal{A}(G)$, we may assume U_x is unitary. Therefore, $U_{h^{-1}}=U_xU_{h^{-1}}U_x^*$ for all $h\in H$. That is, using the terminology previously defined, $U_{\varepsilon_h}=U_{\tilde{x}*\varepsilon_h*x}$. Hence, $(\tilde{x}*\varepsilon_h*x)(g')=0$ unless $g'=h$. To show that $U_x\in\mathcal{A}(\overline{G})$ we will establish that $x(g)=0$ if $g\in G\backslash\overline{G}$.

Let $\varepsilon>0$ be given and $g\in G\backslash\overline{G}$. Then there exists a complex function y on G, vanishing outside a finite set B, such that

$$\|x-y\|_2\leqq\varepsilon,\quad \|y\|_2\leqq\|x\|_2,\quad \text{and}\quad y(g)=x(g).$$

Let, for $z \in \mathfrak{H}'$, $\|z\|_\infty = \text{l.u.b.}\ \{|z(g')|\,|\,g' \in G\}$. Then, for every $h \in H$,

$$\|\tilde{y} * \varepsilon_h * y - \tilde{x} * \varepsilon_h * x\|_\infty = \|(\widetilde{y-x}) * \varepsilon_h * y + \tilde{x} * \varepsilon_h * (y-x)\|_\infty$$
$$\leqq \|y-x\|_2\,\|y\|_2 + \|x\|_2\,\|y-x\|_2 \leqq 2\varepsilon\|x\|_2\,.$$

Using property (β'), choose $h_0 \in H$ such that $g^{-1}h_0g \neq h_0$ and such that $u, v \in Bg^{-1}$ and $u^{-1}h_0v = h_0$ imply $u=v$. Then, using Lemma 2,

$$|x(g)|^2 = |y(g)|^2 \leqq |(\tilde{y} * \varepsilon_{h_0} * y)(g^{-1}h_0g)|$$
$$\leqq |(\tilde{x} * \varepsilon_{h_0} * x)(g^{-1}h_0g)| + 2\varepsilon\|x\|_2 = 2\varepsilon\|x\|_2$$
$$\text{since}\quad g^{-1}h_0g \neq h_0\,.$$

Since ε is arbitrary, $x(g) = 0$. Q.E.D.

LEMMA 4. *G_{n0} has property (β') relative to H_{n0}.*

The proof of this lemma is presented in section 3.

THEOREM 1. *The subalgebras C_{n0} are non-normal in $\mathcal{A}_n$.*

PROOF. Lemmas 1 and 4 allow us to apply Lemma 3, putting $G=G_n$, $\overline{G}=G_{n0}$, and $H=H_{n0}$. We may then conclude that

$$C'_{n0} \cap \mathcal{A}_n = \mathcal{A}_{n0}\,. \tag{7}$$

Since $\mathcal{A}_{n0}$ is maximal abelian, $\mathcal{A}'_{n0} \cap \mathcal{A}_n = \mathcal{A}_{n0}$.
Therefore, $(C'_{n0} \cap \mathcal{A}_n)' \cap \mathcal{A}_n = \mathcal{A}'_{n0} \cap \mathcal{A}_n = \mathcal{A}_{n0}$.
Since C_{n0} is properly contained in $\mathcal{A}_{n0}$, C_{n0} is non-normal. Q.E.D.

Statement (7) leads immediately to:

COROLLARY 1. *$\mathcal{A}_{n0}$ is the unique maximal abelian subalgebra containing C_{n0}.*

2. The subalgebras C_{n0} are pairwise non-conjugate. Suppose, in general, that $\mathcal{A}_1$ is a W^*-subalgebra of the factor $\mathcal{A}$. Denote by $R(\mathcal{A}_1)$ the W^*-algebra generated by all unitaries $U \in \mathcal{A}$ such that $U\mathcal{A}_1U^* \subseteqq \mathcal{A}_1$. Then $R(\mathcal{A}_1)$ is a W^*-subalgebra of $\mathcal{A}$, and $\mathcal{A}_1 \subseteqq R(\mathcal{A}_1) \subseteqq \mathcal{A}$. Let $R^1(\mathcal{A}_1) = R(\mathcal{A}_1)$ and, for each $j \geqq 2$, define $R^j(\mathcal{A}_1)$ to be $R(R^{j-1}(\mathcal{A}_1))$.

DEFINITION 2. (cf. [7]) $\mathcal{A}_1$ is said to be of *length L in* $\mathcal{A}$ if there is a chain

$$\mathcal{A}_1 \subsetneqq R(\mathcal{A}_1) \subsetneqq R^2(\mathcal{A}_1) \subsetneqq \cdots \subsetneqq R^L(\mathcal{A}_1) = \mathcal{A}$$

It is proved in [1] that the length of a subalgebra $\mathcal{A}_1$ in $\mathcal{A}$ is a *-algebraic invariant.

LEMMA 5. *Let $\mathcal{A}_0$ be an abelian W*-subalgebra of the factor $\mathcal{A}$. Suppose $\mathcal{A}_0$ is contained in a unique maximal abelian subalgebra $M(\mathcal{A}_0)$. Let σ be a *-automorphism of $\mathcal{A}$. Then $\sigma(M(\mathcal{A}_0))$ is the unique maximal abelian subalgebra containing the abelian subalgebra $\sigma(\mathcal{A}_0)$.*

PROOF. Clearly, if $\sigma(M(\mathcal{A}_0))$ were not maximal abelian, so that there existed $\mathcal{B}_0 \supsetneqq \sigma(M(\mathcal{A}_0))$, then $\sigma^{-1}(\mathcal{B}_0) \supsetneqq M(\mathcal{A}_0)$.

And, if $\sigma(M(\mathcal{A}_0))$ were not unique, so that $\mathcal{D}_0$ were also maximal abelian, $\mathcal{D}_0 \supseteqq \sigma(\mathcal{A}_0)$, then $\sigma^{-1}(\mathcal{D}_0)$ would also be maximal abelian and contain $\mathcal{A}_0$. Q.E.D.

LEMMA 6. *Let $\mathcal{A}_0$ and $\mathcal{B}_0$ be abelian W*-subalgebras of the factor $\mathcal{A}$, contained in unique maximal abelian subalgebras $M(\mathcal{A}_0)$ and $M(\mathcal{B}_0)$, respectively. If σ is a *-automorphism of $\mathcal{A}$ such that $\sigma(\mathcal{A}_0) = \mathcal{B}_0$, then $\sigma(M(\mathcal{A}_0)) = M(\mathcal{B}_0)$.*

PROOF. Since $M(\mathcal{B}_0)$ is then the unique maximal abelian subalgebra containing $\sigma(\mathcal{A}_0)$, by the previous lemma, $\sigma(M(\mathcal{A}_0)) = M(\mathcal{B}_0)$. Q.E.D.

LEMMA 7. *Let $\mathcal{A}_0$ and $\mathcal{B}_0$ be abelian W*-subalgebras of the factor $\mathcal{A}$, contained in unique maximal abelian subalgebras $M(\mathcal{A}_0)$ and $M(\mathcal{B}_0)$, respectively. If the length of $M(\mathcal{A}_0)$ is not equal to the length of $M(\mathcal{B}_0)$, then $\mathcal{A}_0$ and $\mathcal{B}_0$ are not conjugate under *-automorphisms of $\mathcal{A}$.*

PROOF. Suppose $\mathcal{A}_0$ and $\mathcal{B}_0$ were conjugate under σ, so that $\sigma(\mathcal{A}_0)=\mathcal{B}_0$. Then, $\sigma(M(\mathcal{A}_0)) = M(\mathcal{B}_0)$. On the other hand, since the length of a subalgebra in $\mathcal{A}$ is a *-algebraic invariant, and since the length of $M(\mathcal{A}_0)$ is not equal to the length of $M(\mathcal{B}_0)$, we cannot have $\sigma(M(\mathcal{A}_0)) = M(\mathcal{B}_0)$. Q.E.D.

We may now suppose that all the subalgebras C_{n0} lie in one hyperfinite factor $\mathcal{A}$.

THEOREM 2. *The abelian subalgebras C_{n0} are pairwise non-conjugate under *-automorphisms of $\mathcal{A}$.*

PROOF. It is proved in [1] that, for each $n \geqq 4$, $\mathcal{A}_{n0}$ has length $n-2$ in $\mathcal{A}_n$.

Since $\mathcal{A}_{n0}$ is the unique maximal abelian subalgebra containing C_{n0}, the result follows from Lemma 7. Q.E.D.

3. Proof of Lemma 4. Let $g \in G_n \backslash G_{n0}$ be given, g of form (1). Let B be a finite subset of G_n, $B = \{u^{(1)}, u^{(2)}, \cdots, u^{(m)}\}$. We must produce an element $h_0 \in H_{n0}$ such that

(i) $gh_0 \neq h_0 g$

(ii) $u^{(p)}h_0 = h_0 u^{(q)}$ implies $u^{(p)} = u^{(q)}$, $1 \leqq p, q \leqq m$.

Let $u^{(p)}$ be of form (1) with entries $a_{ij}{}^{(p)}$.

Let $h \in H_{n0}$ be of form (2) with entries b_{ij}, the b_{ij} to be determined. Because of the nature of h, it is clear that, regardless of the choice of b_{ij}, the matrices gh and hg are identically equal to g, except for the first two rows. Also, $u^{(p)}h$ agrees with $u^{(p)}$ except for these rows and $hu^{(q)}$ agrees with $u^{(q)}$ except for these rows. Accordingly, we investigate rows 1 and 2 of these four matrices.

$$gh = \begin{bmatrix} a_{11} & a_{11}b_{12} + a_{12} & a_{11}b_{13} + a_{12}b_{12} + a_{13} & c_4 & c_5 \cdots c_n \\ 0 & 1 & b_{12} + a_{23} & a_{24} & a_{25} \cdots a_{2n} \end{bmatrix},$$

where $c_j = a_{11}b_{1j} + a_{1j}$, $4 \leqq j \leqq n$.

$$hg = \begin{bmatrix} a_{11} & a_{12} + b_{12} & a_{13} + b_{12}a_{23} + b_{13} & d_4 & d_5 \cdots d_n \\ 0 & 1 & b_{12} + a_{23} & e_4 & e_5 \cdots e_n \end{bmatrix},$$

where

$$d_j = a_{1j} + b_{1j} + \sum_{k=2}^{j-1} b_{1k}a_{kj}, \quad 4 \leqq j \leqq n,$$
$$e_j = a_{2j} + b_{12}a_{3j}, \quad 4 \leqq j \leqq n.$$

Clearly, if gh is to equal hg, we must have $a_{12}+b_{12} = a_{11}b_{12}+a_{12}$. Hence, if $b_{12} \neq 0$, a_{11} must equal 1. *We henceforth assume* $b_{12} \neq 0$ and $a_{11} = 1$. Taking this into consideration, (1, 3) (the entry in first row, third column) gives $b_{12}(a_{12}-a_{23}) = 0$, i.e. $a_{12} = a_{23}$. Next, we must have, for each j, $4 \leqq j \leqq n$, $e_j = a_{2j}$. That is, $b_{12}a_{3j} = 0$. Therefore, $a_{3j} = 0$ for $4 \leqq j \leqq n$. Finally, we must have, for each j, $4 \leqq j \leqq n$, $c_j = d_j$. That is:

$$(*) \qquad \sum_{k=2}^{j-1} b_{1k}a_{kj} = 0.$$

We leave this for the present to consider the first two rows of $u^{(p)}h$ and $hu^{(q)}$:

$$u^{(p)}h = \begin{pmatrix} a_{11}{}^{(p)} & a_{11}{}^{(p)}b_{12}+a_{12}{}^{(p)} & a_{11}{}^{(p)}b_{13}+a_{12}{}^{(p)}b_{12}+a_{13}{}^{(p)} & c_4{}^{(p)} & c_5{}^{(p)} & \cdots & c_n{}^{(p)} \\ 0 & 1 & b_{12}+a_{23}{}^{(p)} & a_{24}{}^{(p)} & a_{25}{}^{(p)} & \cdots & a_{2n}{}^{(p)} \end{pmatrix}$$

where $c_j{}^{(p)} = a_{11}{}^{(p)}b_{1j} + a_{1j}{}^{(p)}$

$$hu^{(q)} = \begin{pmatrix} a_{11}{}^{(q)} & a_{12}{}^{(q)}+b_{12} & a_{13}{}^{(q)}+b_{12}a_{23}{}^{(q)}+b_{13} & d_4{}^{(q)} & d_5{}^{(q)} & \cdots & d_n{}^{(q)} \\ 0 & 1 & b_{12}+a_{23}{}^{(q)} & e_4{}^{(q)} & e_5{}^{(q)} & \cdots & e_n{}^{(q)} \end{pmatrix}$$

where

$$d_j{}^{(q)} = a_{1j}{}^{(q)} + b_{1j} + \sum_{k=2}^{j-1} b_{1k}a_{kj}{}^{(q)}, \quad 4 \leqq j \leqq n$$

and
$$e_j{}^{(q)} = a_{2j}{}^{(q)} + b_{12}a_{3j}{}^{(q)}, \quad 4 \leqq j \leqq n.$$

Now, if $u^{(p)}h$ is equal to $hu^{(q)}$ we must have $a_{11}{}^{(p)} = a_{11}{}^{(q)}$. Further, considering (1, 2), we need $b_{12}(a_{11}{}^{(p)}-1) = a_{12}{}^{(q)}-a_{12}{}^{(p)}$. Let

$$A_1 = \left\{ \frac{a_{12}{}^{(q)} - a_{12}{}^{(p)}}{a_{11}{}^{(p)}-1} \;|\; p, q = 1, 2, \cdots \; m\,;\; a_{11}{}^{(p)} \neq 1 \right\}$$

A_1 is a *finite* set. *We now assume* $b_{12} \notin A_1$. Then, unless $a_{11}{}^{(p)} = 1$, $u^{(p)}h$ differs from $hu^{(q)}$ in (1, 2). We assume henceforth that $a_{11}{}^{(p)} = 1$ so that also $a_{12}{}^{(p)} = a_{12}{}^{(q)}$. Next, (2, 3) requires $a_{23}{}^{(p)} = a_{23}{}^{(q)}$. And (1, 3) requires $b_{12}(a_{12}{}^{(p)}-a_{23}{}^{(q)}) = a_{13}{}^{(q)}-a_{13}{}^{(q)}$. Let

$$A_2 = \left\{ \frac{a_{13}{}^{(q)} - a_{13}{}^{(p)}}{a_{12}{}^{(p)}-a_{23}{}^{(q)}} \;|\; p, q = 1, 2, \cdots, m\,;\; a_{12}{}^{(p)} \neq a_{23}{}^{(q)} \right\}.$$

We assume that b_{12} does not belong to the finite set A_2. Reasoning as before, this requires $a_{12}{}^{(p)}=a_{23}{}^{(q)}$ and $a_{13}{}^{(p)}=a_{13}{}^{(q)}$. Next, for each j, $4 \leqq j \leqq n$, we need $e_j{}^{(q)}=a_{2j}{}^{(p)}$, i.e. $b_{12}a_{3j}{}^{(q)}=a_{2j}{}^{(p)}-a_{2j}{}^{(q)}$. For each j, $4 \leqq j \leqq n$, let

$$A_3{}^{(j)} = \left\{ \frac{a_{2j}{}^{(p)} - a_{2j}{}^{(q)}}{a_{3j}{}^{(q)}} \;|\; p, q = 1, 2, \cdots, m\,;\; a_{3j}{}^{(q)} \neq 0 \right\}.$$

We now assume $b_{12} \notin A_3{}^{(j)}$. Thus, $a_{3j}{}^{(q)} = 0$ and $a_{2j}{}^{(p)} = a_{2j}{}^{(q)}$. Finally, we need, for each j, $4 \leqq j \leqq n$, $d_j{}^{(q)} = c_j{}^{(p)}$, i.e.

$$(**) \qquad \sum_{k=2}^{j-1} b_{1k} a_{kj}^{(q)} = a_{1j}^{(p)} - a_{1j}^{(q)} .$$

We now investigate equations (*) and (**), recalling that $b_{12} \notin A_1 \cup A_2 \cup A_3^{(4)} \cup \cdots \cup A_3^{(n)} \cup \{0\}$. (*) gives the following $(n-3)$ equations (recalling that $a_{3j}=0$):

$$\begin{aligned}
(*) \qquad & j = 4) \quad b_{12}a_{24} = 0 \\
(*) \qquad & j = 5) \quad b_{12}a_{25} + b_{14}a_{45} = 0 \\
(*) \qquad & j = 6) \quad b_{12}a_{26} + b_{14}a_{46} + b_{15}a_{56} = 0 \\
& \quad \vdots \\
(*) \qquad & j = n) \quad b_{12}a_{2n} + b_{14}a_{4n} + b_{15}a_{5n} + \cdots + b_{1,n-1}a_{n-1,n} = 0 .
\end{aligned}$$

(**) gives the following $m^2(n-3)$ equations (recalling that $a_{3j}^{(q)} = 0$ for all j, q):

$$\begin{aligned}
(**) \qquad & j = 4) \quad b_{12}a_{24}^{(q)} = a_{14}^{(p)} - a_{14}^{(q)} \\
(**) \qquad & j = 5) \quad b_{12}a_{25}^{(q)} + b_{14}a_{45}^{(q)} = a_{15}^{(p)} - a_{15}^{(q)} \\
(**) \qquad & j = 6) \quad b_{12}a_{26}^{(q)} + b_{14}a_{46}^{(q)} + b_{15}a_{56}^{(q)} = a_{16}^{(p)} - a_{16}^{(q)} \\
& \quad \vdots \\
(**) \qquad & j = n) \quad b_{12}a_{2n}^{(q)} + b_{14}a_{4n}^{(q)} + \cdots + b_{1,n-1}a_{n-1,n}^{(q)} = a_{1n}^{(p)} - a_{1n}^{(q)} .
\end{aligned}$$

Considering (*) $j = 4$) we see that $a_{24} = 0$. Now let

$$A_4^{(12)} = \left\{ \frac{a_{14}^{(p)} - a_{14}^{(q)}}{a_{24}^{(q)}} \,\middle|\, p, q = 1, 2, \cdots, m ;\ a_{24}^{(q)} \neq 0 \right\}$$

$$\vdots$$

$$A_n^{(12)} = \left\{ \frac{a_{1n}^{(p)} - a_{1n}^{(q)}}{a_{2n}^{(q)}} \,\middle|\, p, q = 1, 2, \cdots, m ;\ a_{2n}^{(q)} \neq 0 \right\}$$

(all these sets are finite)

Assume henceforth that $b_{12} \notin A_4^{(12)} \cup A_5^{(12)} \cup \cdots \cup A_n^{(12)}$. Then, equations (**) $j = 4$ cannot be satisfied unless $a_{24}^{(q)} = 0$, in which case $a_{14}^{(p)} = a_{14}^{(q)}$. We may assume, therefore, that $a_{24}^{(q)} = 0$ for all q and that $a_{14}^{(p)} = a_{14}^{(q)}$ for all p, q.

Now, *fix* b_{12} subject to all previous restrictions. We now institute the following procedure:

At step $k-3$, $k=4,5,\cdots,n-4$, define the following finite sets:

$$A_s^{(1k)} = \left\{ \frac{a_{1s}^{(p)} - a_{1s}^{(q)} - b_{12}a_{2s}^{(q)} - \sum_{r=4}^{k-1} b_{1r}a_{rs}^{(q)}}{a_{ks}^{(q)}} \right\}$$

where $s = k+1,\ k+2, \cdots, n$.

Then, restrict b_{1k} so that $b_{1k} \notin \bigcup_{s=k+1}^{n} A_s^{(1k)}$. Also, restrict b_{1k} so that:

$$b_{1k} \neq \frac{-b_{12}a_{2t} - \sum_{r=4}^{k-1} b_{1r}a_{rt}}{a_{kt}} \tag{†}$$

for any $t = k+1,\ k+2, \cdots, n$, whenever $a_{kt} \neq 0$. (For $k = 4$, the summation in (†) is understood to be zero.)

Then, because of (†), in order to satisfy equation (*) $j = k+1$), it will be necessary that $a_{k,k+1} = 0$, whence, by (†) for previous values of k,

$$a_{k-1,k+1} = a_{k-2,k+1} = \cdots = a_{2,k+1} = 0.$$

Next, because $b_{1k} \notin \bigcup_{s=k+1}^{n} A_s^{(1k)}$, it will not be possible to satisfy equations (**) $j = k+1$) unless $a_{k,k+1}^{(q)}=0$ whereupon, by a combined use of previous restrictions on $b_{12}, \cdots, b_{1,k-1}$ and (†) it will follow that $a_{k-1,k+1}^{(q)} = a_{k-2,k+1}^{(q)} = \cdots = a_{2,k+1}^{(q)} = 0$ and that $a_{1k}^{(p)} = a_{1k}^{(q)}$ for all p, q.

Finally, fix b_{1k} subject to all previous restrictions and proceed to the next step.

Therefore, at the end of the k^{th} step, we have established that the super-diagonal entries of g in the $(k+4)^{th}$ column, except for the entry in the first row, are zero. And, also at the end of the k^{th} step, we have established that the $(k+4)^{th}$ column of $u^{(p)}$ is identical to the $(k+4)^{th}$ column of $u^{(q)}$. (The work previous to step 1 took care of columns 1, 2, 3, and 4 both for g and $u^{(p)}$, $u^{(q)}$.)

Finally, let h_0 consist of the *fixed* entries b_{ij}. This element h_0 is the one required to guarantee that $gh_0 \neq h_0g$ and $u^{(p)}h_0 = h_0u^{(q)}$ only if $u^{(p)} = u^{(q)}$.

We exemplify the above procedure in the case $k=6$, $n \geqq 8$.

At step 3, we define

$$A_s^{(16)} = \left\{ \frac{a_{1s}^{(p)} - a_{1s}^{(q)} - b_{12}a_{2s}^{(q)} - b_{14}a_{4s}^{(q)} - b_{15}a_{5s}^{(q)}}{a_{6s}^{(q)}} \right\}.$$

We restrict b_{16} so that $b_{16} \notin \bigcup_{s=7}^{n} A_s^{(16)}$ and also

$$b_{16} \neq \frac{-b_{12}a_{27}-b_{14}a_{47}-b_{15}a_{57}}{a_{67}}$$

and

$$b_{16} \neq \frac{-b_{12}a_{28}-b_{14}a_{48}-b_{15}a_{58}}{a_{68}}$$

$$\vdots$$

$$b_{16} \neq \frac{-b_{12}a_{2n}-b_{14}a_{4n}-b_{15}a_{5n}}{a_{6n}}.$$

Consider equation (*) $j = 7$):

$$b_{12}a_{27} + b_{14}a_{47} + b_{15}a_{57} + b_{16}a_{67} = 0.$$

By (†), since $b_{16} \neq \frac{-b_{12}a_{27}-b_{14}a_{47}-b_{15}a_{57}}{a_{67}}$, we must have $a_{67} = 0$ This means that $b_{12}a_{27}+b_{14}a_{47}+b_{15}a_{57}=0$. But, by (†) for step 2, $b_{15} \neq \frac{-b_{12}a_{27}-b_{14}a_{47}}{a_{57}}$, so that $a_{57} = 0$. Therefore, $b_{12}a_{27} + b_{14}a_{47} = 0$. But, by (†) for step 1, $b_{14} \neq \frac{-b_{12}a_{27}}{a_{47}}$, so that $a_{47} = 0$. And, since $b_{12} \neq 0$, $a_{27} = 0$.

Consider equations (**) $j = 7$):

$$b_{12}a_{27}^{(q)} + b_{14}a_{47}^{(q)} + b_{15}a_{57}^{(q)} + b_{16}a_{67}^{(q)} = a_{17}^{(p)} - a_{17}^{(q)}.$$

Since $b_{16} \notin A_7^{(16)}$, we must have $a_{67}^{(q)} = 0$, whereupon

$$b_{12}a_{27}^{(q)} + b_{14}a_{47}^{(q)} + b_{15}a_{57}^{(q)} = a_{17}^{(p)} - a_{17}^{(q)}.$$

Since $b_{15} \notin A_7^{(15)}$, we must have $a_{57}^{(q)} = 0$, whereupon

$$b_{12}a_{27}^{(q)} + b_{14}a_{47}^{(q)} = a_{17}^{(p)} - a_{17}^{(q)}.$$

Finally, we get

$$a_{27}^{(q)} = 0 \quad \text{and} \quad a_{17}^{(p)} = a_{17}^{(q)}.$$

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