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ON A PROBLEM OF BONSALL

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We say that a bounded linear operator T on a Hilbert space H is hyponormal if $\|Tx\| \geqq \|T^*x\|$ for all $x \in H$.

We shall consider the following problem due to Bonsall: If a hyponormal operator T is the sum of a compact operator and a generalized nilpotent operator, does it follow that T is normal?

An operator means a bounded linear operator on a Hilbert space. $\sigma(T)$, $\sigma_p(T)$, $\sigma_c(T)$ and $\sigma_r(T)$ denote the spectrum, the point spectrum, the continuous spectrum and the residual spectrum of an operator T, respectively.

Let $\mathfrak{N}_T(\lambda)$ be the λ-th proper subspace of a hyponormal operator T, that is $\mathfrak{N}_T(\lambda) = \{x \in H: Tx = \lambda x\}$, then by the properties of hyponormal operator, it is easy to verify that $\{\mathfrak{N}_T(\lambda): \lambda \in \sigma_p(T)\}$ is a family of mutually orthogonal reducing subspaces of T.

Therefore we have the following lemma.

LEMMA 1. *If T is hyponormal, then it can be expressed uniquely as a direct sum $T = T_1 \oplus T_2$ defined on a product space $H = H_1 \oplus H_2$ where H_1 is spanned by all the proper vectors of T such that*: (a) *T_1 is normal and $\sigma(T_1) =$ the closure of $\sigma_p(T)$,* (b) *T_2 is hyponormal and $\sigma_p(T_2) = \emptyset$,* (c) *$T$ is normal if and only if T_2 is normal.*

Next, we prepare the spectral properties of operators with some relation to a compact operator.

LEMMA 2. *If an operator T on H is the sum of a compact operator C and an operator S and if $T_{\mathfrak{M}}$ is the restriction of T on its invariant subspace $\mathfrak{M}$, then $\sigma_c(T_{\mathfrak{M}}) \subset \sigma(S)$.*

PROOF. Let $\mu \in \sigma_c(T_{\mathfrak{M}})$, then there exists a sequence $\{x_n\}$ of unit vectors in $\mathfrak{M}$ such that $\|T_{\mathfrak{M}}x_n - \mu x_n\| \to 0$ $(n \to \infty)$. Since C is compact, we may assume that (if necessary, by choosing a suitable subsequence) the sequence $\{Cx_n\}$ converges to a certain vector $x \in H$.

If $\mu \notin \sigma(S)$, then $S - \mu I$ is invertible. Let $x_0 = -(S - \mu I)^{-1}x$, then

$\|x_n - x_0\| \leqq \|(S-\mu I)^{-1}\| \cdot \|(S-\mu I)x_n + x\| = \|(S-\mu I)^{-1}\| \cdot \|(T_{\mathfrak{M}} - \mu I - C)x_n + x\|$ $\leqq \|(S-\mu I)^{-1}\| \{\|T_{\mathfrak{M}} x_n - \mu x_n\| + \|Cx_n - x\|\} \to 0$ $(n \to \infty)$. Thus $\mu \in \sigma_p(T_{\mathfrak{M}})$. This is a contradiction.

LEMMA 3. *If T is an operator on H such that $p(T)$ is compact for some polynomial $p(\cdot)$ and if $T_{\mathfrak{M}}$ is the restriction of T on its invariant subspace $\mathfrak{M}$, then $\sigma_c(T_{\mathfrak{M}}) \subset \{\lambda : p(\lambda) = 0\}$.*

PROOF. Let $\mu \in \sigma_c(T_{\mathfrak{M}})$, then there exists a sequence $\{x_n\}$ of unit vectors in $\mathfrak{M}$ such that $\|T_{\mathfrak{M}} x_n - \mu x_n\| \to 0$ and $\|p(T_{\mathfrak{M}})x_n - p(\mu)x_n\| \to 0$ $(n \to \infty)$. Since $p(T_{\mathfrak{M}})$ is compact, we may assume that (if necessary, by choosing a suitable subsequence) the sequence $\{p(T_{\mathfrak{M}})x_n\}$ converges to a certain vector $x \in \mathfrak{M}$. If $p(\mu) \neq 0$, then for $x_0 = x/p(\mu)$,

$$\|x_n - x_0\| \leqq \|x_n - p(T_{\mathfrak{M}})/p(\mu) \cdot x_n\| + \|p(T_{\mathfrak{M}})/p(\mu) \cdot x_n - x_0\| \to 0 \quad (n \to \infty).$$

Therefore $\mu \in \sigma_p(T_{\mathfrak{M}})$. This contradicts with $\mu \in \sigma_c(T_{\mathfrak{M}})$.

Now, we shall prove the following theorems.

THEOREM 1. *If a hyponormal operator T on H is the sum of a compact operator C and a generalized nilpotent operator N (i.e $\sigma(N) = \{0\}$), then T is normal.*

THEOREM 2. *Every hyponormal operator T on H with compact imaginary part is normal.*

THEOREM 3. *Every polynomially compact hyponormal operator T on H is normal.*

To prove our theorems, by Lemma 1, we have only to show that T_2 is normal or $H_2 = (0)$.

It is known that for any operator S, $\sigma_r(S)$ is open [2]. Since $\sigma(S)$ is closed, $\partial\sigma_r(S)$ (boundary of $\sigma_r(S)) \subset \sigma_p(S) \cup \sigma_c(S)$. Applying this fact for the operator T_2 in Lemma 1, we have $\partial\sigma_r(T_2) \subset \sigma_c(T_2)$. Therefore, in each case where T satisfies the condition of Theorem 1, Theorem 2 or Theorem 3, by Lemma 2 or Lemma 3, we have $\sigma(T_2) = \{0\}$, $\sigma(T_2) \subset$ real line or $\subset \{\lambda : p(\lambda) = 0\}$ respectively. Since it is known that every isolated point of the spectrum of a hyponormal operator is a proper value [1], in each case where T satisfies the conditions of Theorem 1 or Theorem 3, $H_2 = (0)$. And it is also known that every hyponormal operator with its spectrum on a real line is self-adjoint [3]. Therefore, in the case where T satisfies the conditions of Theorem 2, T_2 is self-adjoint.

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